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ON THE UNIQUENESS OF MAXIMAL FUNCTIONS

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ABSTRACT. The uniqueness theorem for the one-sided maximal operator has been proved.

Let L be the class of real 2π-periodic integrable functions and let M be the one-sided maximal operator

$$M(f)(x) = \sup_{b>x} \frac{1}{b-x} \int_x^b f dm, \quad f \in L, \quad x \in \mathbb{R}$$

(m denotes the Lebesgue measure on the line $\mathbb{R}$).

In this paper we shall prove the following uniqueness

Theorem 1. *Let $f, g \in L$ and $M(f) = M(g)$. Then $f = g$ a.e. on $\mathbb{R}$.*

Sets of the type $\{x \in \mathbb{R} : M(f)(x) > t\} = \{x \in \mathbb{R} : M(g)(x) > t\}$ will be briefly denoted by $(M > t)$. Obviously $(M > t)_{t\in\mathbb{R}}$ is a class of bundled open sets continuous from the right, i.e.,

$$\bigcup_{t>\tau} (M > t) = (M > \tau).$$

Let

$$t_0 = \inf\{M(f) : x \in \mathbb{R}\} = \inf\{M(g)(x) : x \in \mathbb{R}\}.$$

For an arbitrary integrable function f if $t = \frac{1}{2\pi}\int_0^{2\pi} f dm$, then $M(f) \geq t$ on the whole line and $M(f)(x_0) = t$ for x_0 being the point of maximum of the function $x \mapsto \int_0^x f dm - tx$. Thus we can conclude that

$$\int_0^{2\pi} f dm = \int_0^{2\pi} g dm = t_0.$$

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Because of the Lebesgue differentiation theorem $f, g \le t_0$ a.e. on $\mathbb{R}\backslash(M > t_0)$. On the other hand, applying the Riesz rising sun lemma (see [1]), we have

$$\int\limits_{(M>t_0)} f dm = \int\limits_{(M>t_0)} g dm = t_0 \cdot m(M > t_0) \tag{1}$$

(see also [2], p. 58). Consequently $f = g = t_0$ a.e. on $\mathbb{R}\backslash(M > t_0)$ and to prove the theorem it suffices to show the validity of

Lemma 1. *Let* (a, b) *be a (finite) connected component of* $(M > t_0)$. *Then*

$$\int\limits_x^b f dm = \int\limits_x^b g dm \tag{2}$$

for each $x \in (a, b)$.

Proof. Assume x fixed and let $t_x = M(f)(x) = M(g)(x)$. For each $t \in [t_0, t_x)$ suppose (a_t, b_t) to be the connected component of $(M > t)$ which contains x and assume that $b_t = x$ whenever $t = t_x$ (note that $b_{t_0} = b$, by assumption). Obviously

$$\bigcup_{t>\tau} (a_t, b_t) \subset (a_\tau, b_\tau)$$

and it is easy to show that $t \mapsto b_t$ is a non-increasing function on $[t_0, t_x]$ continuous from the right.

Let D be the set of points of discontinuity of this function and let

$$D_c = \{t : b_\tau = b_t \quad \text{for some} \quad \tau > t\}.$$

If $t \in [t_0, t_x)\backslash(D \cup D_c)$ and b_t is a Lebesgue point of both functions f and g, then

$$f(b_t), g(b_t) \le t$$

(since $b_t \notin (M > t)$). On the other hand, for each $\tau \in (t, t_x)$ we have

$$\frac{1}{b_t - b_\tau} \int\limits_{b_\tau}^{b_t} f dm, \qquad \frac{1}{b_t - b_\tau} \int\limits_{b_\tau}^{b_t} g dm > t$$

(since (a_t, b_t) is a connected component of $(M > t)$ and $b_\tau \in (a_t, b_t)$; see Lemma 1 in [3]). Hence we can conclude that

$$f(b_t) = g(b_t) = t.$$

For $t \in D$ let

$$b'_t = \lim_{\tau \to t-} b_\tau.$$

Then

$$\frac{1}{b'_t - b_t}\int_{b_t}^{b'_t} f dm, \quad \frac{1}{b'_t - b_t}\int_{b_t}^{b'_t} g dm \leq t$$

(since $b_t \notin (M > t)$) and for each $\tau \in [t_0, t)$ we have

$$\frac{1}{b_\tau - b_t}\int_{b_t}^{b_\tau} f dm, \quad \frac{1}{b_\tau - b_t}\int_{b_t}^{b_\tau} g dm > \tau$$

(since (a_τ, b_τ) is a connected component of $(M > \tau)$ and $b_t \in (a_\tau, b_\tau)$). Hence, letting τ converge to t from the left, we get

$$\int_{b_t}^{b'_t} f dm = \int_{b_t}^{b'_t} g dm = t(b'_t - b_t).$$

Since $[x, b] = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= \{b_t : t \in [t_0, t_x] \backslash (D \cup D_c)\}, \\ A_2 &= \bigcup_{t \in D} [b_t, b'_t], \\ A_3 &= \{b_t : t \in D_c\}, \end{aligned}$$

and since $f = g$ a.e. on A_1,

$$\int_{A_2} f dm = \int_{A_2} g dm$$

and A_3 is a denumerable set, we can conclude that (2) holds. $\square$

Note that the lemma remains true if f and g are locally integrable functions on $\mathbb{R}$. Hence if we use the balancing ergodic equality (see [4]) instead of the equality (1), then we get the uniqueness theorem for the ergodic maximal operator.

Theorem 2. *Let $(T_\lambda)_{\lambda \geq 0}$ be an ergodic semiflow of measure-preserving transformations on a finite measure space $(X, \mathbb{S}, \mu)$ and let M be the ergodic maximal operator*

$$M(f)(x) = \sup_{a>0} \frac{1}{a} \int_0^a f(T_\lambda x) d\lambda, \quad f \in L(X).$$

Then $M(f) = M(g)$ implies that $f = g$ a.e. (in the sense of measure μ) on X.

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