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CHARACTERIZATION OF A REGULAR FAMILY OF SEMIMARTINGALES BY LINE INTEGRALS

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Abstract. A characterization of a regular family of semimartingales as a maximal family of processes with respect to which one can define a stochastic line integral with natural continuity properties is given.

According to the characterization of semimartingales by the theorem of Bichteler–Dellacherie–Mokobodzki (see [1],[2]) the class of semimartingales is the maximal class of processes with respect to which it is possible to define stochastic integrals of predictable processes with sufficiently natural properties, more exactly: an adapted cadlag process $X = (X_t, t \geq 0)$ is a semimartingale if and only if for every sequence of elementary predictable processes $(H^n, n \geq 1)$, which converges uniformly to 0, the corresponding integral sums

$$J_t(H^n) = \sum_{i \leq n-1} H_i^n (X_{t\wedge t_{i+1}} - X_{t\wedge t_i})$$

converge to 0 in probability for each $t \geq 0$.

The aim of this paper is to give an analogous characterization for a regular family of semimartingales $X^A = (X^a, a \in A)$ (see Definition 1 below) as a maximal family of processes relative to which one can define a stochastic line integral (along predictable curves $u : \Omega \times [0, \infty[\to A$) with natural continuity properties.

Let on probability space $(\Omega, \mathcal{F}, P)$ with filtration $F = (\mathcal{F}_t, t \geq 0)$, satisfying the usual conditions, a family $X^A = ((X(t, a), t \geq 0), a \in A)$ of adapted cadlag processes be given, where $(A, \mathcal{A})$ is a compact metric space with the metric r and Borel σ-algebra $\mathcal{A}$. Denote by $S, \mathcal{M}_{\rm loc}, \mathcal{M}^2_{\rm loc}, \mathcal{A}_{\rm loc}, \mathcal{V}$ classes of semimartingales, local martingales, locally square integrable martingales, processes of locally integrable variations and processes of finite variations (on every compact), respectively.

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For some $\lambda > 0$ let

$$\tilde{X}_t^{a,\lambda} = \sum_{s \le t} \Delta X_s^a I_{(|\Delta X_s^a| > \lambda)}.$$

If X^a is a semimartingale then the process $X^a - \tilde{X}^{a,\lambda}$ is a special semimartingale with the canonical decomposition

$$X^a - \tilde{X}^{a,\lambda} = M^{a,\lambda} + B^{a,\lambda},$$

where $B^{a,\lambda}$ is a predictable process of finite variation and $M^{a,\lambda}$ is a local martingale. Moreover, $|\Delta B^{a,\lambda}| \le \lambda, |\Delta M^{a,\lambda}| \le 2\lambda$ and hence $M^{a,\lambda} \in \mathcal{M}^2_{\text{loc}}$ (see, e.g., [2]).

Definition 1. We say that a family $X^A = (X^a, a \in A)$ is a regular family of semimartingales if it satisfies the following conditions:

(A) the process $X^a = (X(t,a), t \ge 0)$ is a semimartingale for every $a \in A$;

(B) there exists $\lambda > 0$, a predictable, increasing process $K = (K_t, t \ge 0)$ which dominates the characteristics of semimartingales $(X^a - \tilde{X}^{a,\lambda}$, $a \in A)$, i.e.,

$$\langle M^{a,\lambda}, M^{a,\lambda} \rangle \ll K, \operatorname{Var}(B^{a,\lambda}) \ll K$$

on every $[0,t]$, for each $a \in A$;

(C) the Radon–Nicodým derivative $\varphi(a,b) = d\langle M^{a,\lambda} - M^{b,\lambda}\rangle / dK$ (respectively, $g(a) = dB^{a,\lambda}/dK$) is a continuous function of (a,b) (respectively, of a) for almost every couple (ω,t) with respect to the measure μ^K, where μ^K is the Doleans measure of K; a function ΔX_t^a is continuous in probability uniformly on every compact;

(D) for each $t \in R_+$ a.s.

$$\int_0^t \sup_{a \in A} \psi_s(a,a) dK_s + \int_0^t \sup_{a \in A} |g_s(a)| dK_s + \\ + \sum_{s \le t} \sup_{a \in A} |\Delta X^a| I_{(\sup_{a \in A} |\Delta X^a| > \lambda)} < \infty.$$

For $m \in \mathcal{M}^2_{loc}$ denote by $H(a,m)$, $f(a,m)$ and $\psi(a,b)$ the derivatives

$$\frac{d\langle M^{a,\lambda}, m\rangle}{dK}, \quad \frac{d\langle M^{a,\lambda}, m\rangle}{d\langle m, m\rangle}, \quad \frac{d\langle M^{a,\lambda}, M^{b,\lambda}\rangle}{dK},$$

respectively. It was proved in [3] that condition (C) implies the existence of μ^K-a.e.($\mu^{\langle m\rangle}$-a.e., respectively) continuous in a modification of the function $H(a,m)$ (respectively, $f(a,m)$) and such a version will be considered.

Let us introduce the classes U and $\widehat{U}$ of predictable processes taking values in A and in $Q(A)$, respectively, where $Q(A)$ is the space of all probabilities on A. Assume that $Q(A)$ is provided with the Levy–Prokhorov metric.

Denote by U^0 the class consisting of elements of U taking values in some finite subset A_0 of A and by $\widehat{U}^0$ the class of elements of $\widehat{U}$ taking values in the set of probability measures concentrated in some finite subset of A, i.e., for which there exists a finite set $A_0 \subset A$ such that $u_t(\omega, A_0) = 1$ for each (ω, t).

Let U_d (respectively, $\widehat{U}_d$) be the class of piecewise constant functions consisting of elements $u \in U$ (respectively, $\widehat{U}$) such that for some finite sequence of (deterministic) moments $t_1 < t_2 < ... < t_N, u_t = u_{t_i}$ for $t_i < t \le t_{i+1}$ and u_{t_i} is a $\mathcal{F}_{t_i}$-measurable random variable taking values in A (respectively, in $Q(A)$).

Let $U_d^0 = U^0 \cap U_d$ and $\widehat{U}_d^0 = \widehat{U}^0 \cap \widehat{U}_d$.

It is convenient to denote the elements of U and $\widehat{U}$ by the same symbols, since every element $u \in U$ can be considered as a degenerate element of $\widehat{U}$. So we shall use the shortened notation: $u = (u_t(\omega), t \ge 0)$ for $u \in U$, $u = u(C) = (u_t(\omega), t \ge 0) = (u_t(\omega, C), t \ge 0)$, $C \in \mathcal{A}$ for $u \in \widehat{U}$, while the symbol $H(u, m)$ for $u \in U$ will be used instead of the exact notation $(H_t(\omega, u_t, m), t \ge 0)$; the symbols $f(u, m), \varphi(u, v), g(u)$ are understood in a similar way. For $u \in \widehat{U}$, we sometimes use the expressions $g(s, u_s), H_s(u_s, m)$ instead of $\int_A g_s(a) u_s(da), \int_A H_s(a, m) u_s(da)$, respectively.

If the family of semimartingales X^A satisfies conditions (A), (B), (C), (D) then for each $u \in U$ (respectively, for each $u \in \widehat{U}$) one can define a stochastic line integral (respectively, a generalized stochastic line integral) with respect to the family of semimartingales $(X^a, a \in A)$ along the curve $u = (u_t,\, t \ge 0)$ as

$$\int_0^t X(ds, u_s) = \int_0^t M(ds, u_s) + \int_0^t g_s(u_s) dK_s + \sum_{s \le t} (X(s, u(s)) - \\ - X(s-, u(s)) I_{(|X(s,u(s)) - X(s-,u(s))| > \lambda)}, \quad t \ge 0 \qquad (1)$$

$$\Bigg(\text{resp.,} \int_0^t X(ds, u_s) = \int_0^t M(ds, u_s) + \int_0^t \int_A g_s(a) u_s(da) dK_s + \\ + \sum_{s \le t} \int_A \Delta X_s^a I_{(|\Delta X_s^a| > \lambda)} u_s(da), \quad t \ge 0\Bigg), \qquad (2)$$

where $(\int_0^t M(ds, u_s),\, t \ge 0)$ (see [3],[4]) is a unique element of the stable space of martingales $\mathcal{L}^2(M^A)$ generated by the family $M^A = (M^{a,\lambda}, a \in A)$

such that

$$\Big\langle \int_0^{\cdot} M(ds,u_s),m\Big\rangle_t = \int_0^t H_s(u_s,m)dK_s \tag{3}$$

$$\bigg(\text{resp., } \Big\langle \int_0^{\cdot} M(ds,u_s),m\Big\rangle_t = \int_0^t \int_A H_s(a,m)u_s(da)dK_s\bigg) \tag{4}$$

for every $m \in \mathcal{L}^2(M^A)$ (we recall that $H_s(a,m) = d\langle M^{a,\lambda},m\rangle_s/dK_s$).

The stochastic line integral defined above does not depend on the choice of λ, on the choice of a dominating process K (if for them conditions (A), (B), (C), (D) are fulfilled) and posseses the following properties:

(1) for each $u \in U^0$ (respectively, for each $u \in \widehat{U}^0$)

$$\int_0^t X(ds,u_s) = \sum_{a\in A^0} \int_0^t I_{(u_s=a)}X(ds,a)$$

$$\bigg(\text{resp., } \int_0^t X(ds,u_s) = \sum_{a\in A^0} \int_0^t u_s(a)X(ds,a)\bigg)$$

and for every $u \in U_d$ (respectively, for every $u \in \widehat{U}_d$)

$$\int_0^t X(ds,u_s) = \sum_i [X(t_{i+1}\wedge t,u_{t_i}) - X(t_i\wedge t,u_{t_i})],$$

$$\bigg(\text{resp., } \int_0^t X(ds,u_s) = \sum_i \int_A [X(t_{i+1}\wedge t,a) - X(t_i\wedge t,a)]u_{t_i}(da)\bigg);$$

(2) for $u_t = u_t^1 I_{(t<\tau)} + u_t^2 I_{(t\geq\tau)}$

$$\int_0^t X(ds,u_s) = \int_0^{t\wedge\tau} X(ds,u_s^1) + \int_{t\wedge\tau}^t X(ds,u_s^2);$$

(3) for each $u \in U$ (respectively, for each $u \in \widehat{U}$)

$$\Delta \int_0^t (X(ds, u(s)) = X(t, u(t)) - X(t-, u(t))$$

$$\left(\text{resp., } \Delta \int_0^t (X(ds, u(s)) = \int_A (X(t, a) - X(t-, a)) u_t(da)\right);$$

(4) for each $u \in U$ (respectively, for each $u \in \widehat{U}$)

$$\left\langle \int M(ds, u(s)), \int M(ds, u(s)) \right\rangle_t = \int_0^t \psi_s(u_s, u_s) dK_s$$

$$\left(\text{resp., } \left\langle \int M(ds, u(s)), \int M(ds, u(s)) \right\rangle_t = \int_0^t \int_A \psi_s(a, a) u_s(da) dK_s\right);$$

(5) if $u^n \to u$ μ^K-a.e. (in the sense of weak convergence of values in the space of probabilities $Q(A)$ if $u^n \in \widehat{U}$), then for each $t \in R_+$

$$\sup_{s \le t} \left| \int_0^t X(ds, u_s^n) - \int_0^t X(ds, u_s) \right| \to 0, n \to \infty$$

in probability.

The stochastic line integral was introduced by Gikhman and Skorokhod in [5] for continuous process K and for the derivative $d\langle M^a - M^b \rangle / dK = \varphi(a, b)$ which is continuous with respect to a, b uniformly in (ω, t).

Let $\mathcal{H}$ be a space of predictable processes H bounded by unity of the form

$$H = \sum_{i \le n-1} H_i I_{]t_i, t_{i+1}]},$$

where $t_1 < t_2 < \cdots < t_n$ and $H_i \in \mathcal{F}_{t_i}$.

Definition 2 ([6]). A family of processes $((X^a, t \ge 0), a \in A)$ satisfies the U.T. (uniform tightness) condition if for each $t > 0$ the set

$$\left(\int_0^t H_s dX_s^a, H \in \mathcal{H}, a \in A \right)$$

is stochastically bounded.

We shall use the following statements proved by Mémin and Słomiński [7].

Proposition 1 ([7]). *A family of semimartingales* $((X_t^a, t \geq 0), a \in A)$ *satisfies the condition U.T. if and only if there exists* $\lambda > 0$ *such that for each* $t \in R_+$

1. *the family* $\mathrm{Var}(\tilde{X}^{a,\lambda})_t, a \in A)$ *is stochastically bounded;*
2. *the family* $([M^{a,\lambda}, M^{a,\lambda}]_t, a \in A)$ *is stochastically bounded;*
3. *the family* $\mathrm{Var}(B^{a,\lambda})_t, a \in A)$ *is stochastically bounded.*

Remark. This proposition was proved in [7] for $A = \{1, 2, \dots\}$. For an arbitrary set A this statement is proved without any changess. Note that since $M^{a,\lambda}$ is a local martingale with bounded jumps, the Lenglart inequality implies that condition 2 is equivalent to the stochastic boundedness of the family $(\langle M^{a,\lambda}, M^{a,\lambda}\rangle_t, a \in A)$ (see [8]).

Proposition 2 ([7]). *Let* $((X_t^n, t \geq 0), n \geq 1)$ *be a sequence of semimartingales satisfying the condition U.T. and let* X^n *converge to some process* X *in probability uniformly on every compact. Then the limiting process* X *is also a semimartingale and, moreover, for each* $t > 0$

$$\sup_{s \leq t} |M_s^{n,\lambda} - M_s^\lambda| \to 0, n \to \infty, \tag{5}$$

$$\sup_{s \leq t} |B_s^{n,\lambda} - B_s^\lambda|, n \to \infty \tag{6}$$

in probability, where M^λ (*respectively,* B^λ) *is a martingale part* (*respectively, the part of finite variation*) *of the semimartingale* $X - \sum \Delta X I_{(\Delta X > \lambda)}$.

Let us consider now a family $X^A = (X^a, a \in A)$ of cadlag adapted processes and for each piecewise constant function u from $\widehat{U}_d^0$, associated with a subdivision $0 = t_0 < t_1 < \cdots < t_n < \infty$, define a new stochastic process

$$J_t(u, X^A) = \sum_i \int_A [X(t_{i+1} \wedge t, a) - X(t_i \wedge t, a)] u_{t_i}(da) \tag{7}$$

which takes the form

$$J_t(u, X^A) = \sum_i [X(t_{i+1} \wedge t, u_{t_i}) - X(t_i \wedge t, u_{t_i})] \tag{8}$$

for functions u from the class U_d.

It is easy to see that if X^A is a family of semimartingales, the process $(J_t(u, X^A), t \geq 0)$ defined by expressions (8),(7) coincides with the stochastic line integral and with the generalized stochastic line integral, respectively.

Theorem 1. *Let $X^A = (X^a, a \in A)$ be a family of adapted cadlag processes. If for each sequence $(u^n, n \geq 1) \in \widehat{U}_d^0$ converging uniformly to some $u \in \widehat{U}$ (in the sense of weak convergence of values in the space of probabilities $Q(A)$) the corresponding integral sums $J_t(u^n, X^A)$ are fundamental in probability for every t, then the difference $X^a - X^b$ is a semimartingale for each pair $a \in A, b \in A$.*

Proof. Let, for each sequence $(u^n, n \geq 1) \in \widehat{U}_d^0$ converging uniformly to some $u \in \widehat{U}$,

$$|J_t(u^n, X^A) - J_t(u^m, X^A)| \to 0,\ n \to \infty,\ m \to \infty,$$

in probability (for each $t \in R_+$). We must show that $X^a - X^b \in S$ for each $a, b \in A$. To prove this, we use the same idea as in the proof of the theorem of Bichteler–Dellacherie–Mokobodzki which consists in the following: the continuity of the functional J and the semimartingale property are invariant under an equivalent change of measure and it is sufficient to construct a law Q equivalent to P such that the process $X^a - X^b$ be a quasimartingale relative to the measure Q.

The possibility of constructing such a measure Q is based on the following lemma whose proof one can find in [1].

Lemma 1. *Let G be a bounded convex set of $L^0(P)$. Then there exists a measure Q equivalent to P, with bounded density, such that*

$$\sup_{\gamma \in G} E^Q \gamma \leq C < \infty,$$

where $L^0(P)$ is the space (of classes of equivalence) of finite random variables provided with the topology of convergence in probability.

Let us take as G the image of the set $\widehat{U}_d^0$ under the mapping J_t. Evidently, G is a convex (as an image of a convex set) and bounded subset of $L^0(P)$. Therefore, applying Lemma 1 for $G = J_t(\widehat{U}_d^0)$ and using expression (8) for $J_t(u, X^A)$, we obtain

$$E^Q \sum_i [X(t_{i+1} \wedge t, u_{t_i}) - X(t_i \wedge t, u_{t_i})] \leq C \tag{9}$$

for each $u \in U_d^0$.

Let $\widetilde{u}$ be a function from U^0, associated with the subdivision $0 = t_0 < t_1 < \cdots < t_n < \infty$, such that

$$\widetilde{u}_i = \begin{cases} a & \text{if } E^Q[X(t_{i+1} \wedge t, a) - X(t_i \wedge t, a)/\mathcal{F}_{t_i}] \geq \\ & \geq E^Q[X(t_{i+1} \wedge t, b) - X(t_i \wedge t, b)/\mathcal{F}_{t_i}] \\ b & \text{otherwise} \end{cases} .$$

From (9) we have

$$E^Q J_t(\widetilde{u}, X^A) = E^Q \sum_i E^Q[X(t_{i+1} \wedge t, \widetilde{u}_i) - X(t_i \wedge t, \widetilde{u}_i)/\mathcal{F}_{t_i}] =$$

$$= E^Q \sum_i [I_{(E^Q[X(t_{i+1},a)-X(t_i,a)/\mathcal{F}_{t_i}] \geq E^Q[X(t_{i+1},b)-X(t_i,b)/\mathcal{F}_{t_i}])} E^Q \times$$

$$\times [X(t_{i+1} \wedge t, a) - X(t_i \wedge t, a)/\mathcal{F}_{t_i}] +$$

$$+ I_{(E^Q[X(t_{i+1},a)-X(t_i,a)/\mathcal{F}_{t_i}] < E^Q[X(t_{i+1},b)-X(t_i,b)/\mathcal{F}_{t_i}])} E^Q \times$$

$$\times [X(t_{i+1} \wedge t, b) - X(t_i \wedge t, b)/\mathcal{F}_{t_i}] =$$

$$= E^Q \sum_i E^Q [X(t_{i+1} \wedge t, a) - X(t_i \wedge t, a)/\mathcal{F}_{t_i}] \vee E^Q \times$$

$$\times [X(t_{i+1} \wedge t, b) - X(t_i \wedge t, b)/\mathcal{F}_{t_i}] \leq C. \tag{10}$$

On the other hand, since the family $Y^A = (-X^a,\, a \in A)$ satisfies the same continuity property as the family X^A, using for the family Y^A an inequality analogous to (9), we have, for each fixed a and $b \in A$,

$$E^Q \sum_i [-X(t_{i+1} \wedge t, a) + X(t_i \wedge t, a)] =$$

$$= -E^Q \sum_i E^Q [X(t_{i+1} \wedge t, a) - X(t_i \wedge t, a)/\mathcal{F}_{t_i}] \leq C, \tag{11}$$

and

$$-E^Q \sum_i E^Q [X(t_{i+1} \wedge t, b) - X(t_i \wedge t, b)/\mathcal{F}_{t_i}] \leq C. \tag{12}$$

Therefore it follows from (10), (11), (12) and from the equality $|x-y| = 2\max(x,y) - x - y$ that

$$E^Q \sum_i |E^Q[X(t_{i+1} \wedge t, a) - X(t_i \wedge t, a) - (X(t_{i+1} \wedge t, b) - X(t_i \wedge t, b))/\mathcal{F}_{t_i}]| \leq C.$$

Thus

$$\sup_{t_1 < \cdots < t_n} E^Q \sum_i |E^Q[X(t_{i+1} \wedge t, a) - X(t_i \wedge t, a) -$$

$$-(X(t_{i+1} \wedge t, b) - X(t_i \wedge t, b))/\mathcal{F}_{t_i}]| \leq C$$

for each $t \in R_+$, hence $X^a - X^b$ is a quasimartingale with respect to the measure Q, and according to Girsanov's theorem the process $X^a - X^b$ will be a semimartingale relative to the measure P. □

The following theorem gives a characterization of a dominated family of semimartingales:

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied. Then for each* $b \in A$ *the family* $(X^a - X^b, a \in A)$ *is a dominated family of semimartingales, i.e., for each fixed* $b \in A$ *the family* $(X^a - X^b, a \in A)$ *satisfies condition* (B) *of Definition* 1.

Proof. According to Theorem 1 the process $X^a - X^b$ is a semimartingale for each $a, b \in A$. Let us prove that for every fixed $b \in A$ the family of semimartingales $(X^a - X^b, a \in A)$ is dominated.

Evidently, for every $u \in \widehat{U}_d^0$ and $b \in A$ the process $(J_t(u, X^A) - X_t^b, t \geq 0)$ is a semimartingale. One can show that the family of semimartingales $(J(u, X^A) - X^b, u \in \widehat{U}_d^0)$ satisfies the condition U.T. (for each fixed $b \in A$), i.e., the set

$$\Big(\int_0^t H_s d(J_s(u, X^A) - X_s^b), u \in \widehat{U}_d^0, H \in \mathcal{H} \Big) \tag{13}$$

is stochastically bounded for every $t \in R_+$.

It follows from the continuity property of the functional J that the set $(J_t(u, X^A), u \in \widehat{U}_d^0)$ and hence the set $(J_t(u, X^A) - J_t(v, X^A), u, v \in \widehat{U}_d^0)$, is stochastically bounded for each $t \in R_+$. Therefore it is sufficient to show that for each $H \in \mathcal{H}$ and $u \in \widehat{U}_d^0$ there exists a pair $u^1, u^2 \in \widehat{U}_d^0$ such that

$$\int_0^t H_s d(J_s(u, X^A) - X_s^b) = J_t(u^1, X^A) - J_t(u^2, X^A). \tag{14}$$

Without loss of generality we can assume that the function $H \in \mathcal{H}$ has the same intervals of constancy as the function u.

For $H = \sum_{i \leq n-1} H_i I_{]t_i, t_{i+1}]}$ and $u(\omega, t, da) = \sum_{i \leq n-1} u_i(\omega, da) I_{]t_i, t_{i+1}]}$ let us define

$$u^1(\omega, t, da) = \sum_{i \leq n-1} (H_i^+(\omega) u_i(\omega, da) + (1 - H_i^+(\omega)) \varepsilon_{\omega,t}^b(da)) I_{]t_i, t_{i+1}]},$$
$$u^2(\omega, t, da) = \sum_{i \leq n-1} (H_i^-(\omega) u_i(\omega, da) + (1 - H_i^-(\omega)) \varepsilon_{\omega,t}^b(da)) I_{]t_i, t_{i+1}]},$$

where $H^+ = \max(H, 0), H^- = -\min(H, 0)$ and ε^b is the measure concentrated at the point $b \in A$ for each ω, t.

It is easy to check that for u^1, u^2 defined above equality (14) is true and hence the family of semimartingales $(J(u, X^A) - X^b, u \in \widehat{U}_d^0)$ satisfies the condition U.T.

Denote by Y^a (respectively, by $J(u, Y^A)$) the difference $X^a - X^b$ (respectively, $J(u, X^A) - X^b$). Since b is fixed, we omit (for convenience) the index

b (e.g., we write Y^a instead of $Y^{a,b}$ and $M^{a,\lambda}$ instead of $M^{a,b,\lambda}$ below). Let

$$\tilde{Y}_t^{a,\lambda} = \sum_{s\le t} \Delta(Y_s^a) I_{[|\Delta(Y_s^a)|>\lambda]},$$
$$\tilde{X}_t^{u,\lambda} = \sum_{s\le t} \Delta(J_s(u,Y^A)) I_{[|\Delta(J_s(u,Y^A))|>\lambda]},$$

and let

$$Y_t^a - \tilde{Y}_t^{a,\lambda} = M^\lambda(t,a) + B^\lambda(t,a), M^\lambda(\cdot,a) \in \mathcal{M}_{\text{loc}}, B^\lambda(\cdot,a) \in \mathcal{A}_{\text{loc}},$$
$$J_t(u,Y^A) - \tilde{Y}_t^{u,\lambda} = M^{u,\lambda}(t) + B^{u,\lambda}(t), M^{u,\lambda} \in \mathcal{M}_{\text{loc}}, B^{u,\lambda} \in \mathcal{A}_{\text{loc}},$$

be the canonical decomposition of the special semimartingales $Y^a - \tilde{Y}^{a,\lambda}$ and $J(u,Y^A) - \tilde{Y}^{u,\lambda}$, respectively.

Since $Y^a \in S$ (for any $a \in A$), one can write the process $J(u,Y^a)$ in the form

$$J_t(u,Y^A) = \sum_{a\in A^0} \int_0^t I_{(u_s=a)} d(Y_s^a)$$

for each $u \in U_d^0$ and it is easy to see that for each $u \in U^0$ (the more so for each $u \in U_d^0$, but not for each $\widehat{U}_d^0$)

$$\tilde{Y}_t^{u,\lambda} = \sum_{a\in A_0} nt_0^t I_{(u_s=a)} d\tilde{Y}_s^{a,\lambda}.$$

Therefore the uniqueness of the canonical decomposition of the special semimartingales implies that for every $u \in U_d^0$

$$M^{u,\lambda} = J(u,M^A),\ B^{u,\lambda} = J(u,B^A), \tag{15}$$

where

$$J_t(u,M^A) = \sum_i (M^\lambda(t_{i+1}\wedge t, u_{t_i}) - M^\lambda(t_i \wedge t, u_{t_i})) =$$
$$= \sum_{a\in A_0} \int_0^t I_{[u_s=a]} dM_s^{\lambda,a}, \in \mathcal{M}_{\text{loc}}^2,$$
$$J_t(u,B^A) = \sum_i (B^\lambda(t_{i+1}\wedge t, u_{t_i}) - B^\lambda(t_i \wedge t, u_{t_i})) =$$
$$\sum_{a\in A_0} \int_0^t I_{[u_s=a]} dB_s^{\lambda,a}, \in \mathcal{A}_{\text{loc}},$$

and an easy calculation shows that the square characteristic of the martingale $J(u, M^A)$ is equal to

$$\langle J(u, M^A)\rangle_t = \sum_{a\in A_0} \int_0^t I_{[u_s=a]} d\langle M^{\lambda,a}\rangle_s.$$

Since the family of semimartingales $(J(u, Y^A), u \in \widehat{U}_d^0)$ satisfies the condition U.T., it follows from Proposition 1 that the family of random variables $(\langle M^{u,\lambda}\rangle_t, \mathrm{Var}(B^{u,\lambda})_t, \mathrm{Var}(\tilde{Y}^{u,\lambda})_t, u \in \widehat{U}_d^0)$ is stochastically bounded for each $t \geq 0$. Therefore from equality (15) we have

$$\lim_N \sup_{u\in U_d^0} P(\langle J(u, M^A)\rangle_t \geq N) = 0, \tag{16}$$

$$\lim_N \sup_{u\in U_d^0} P(\mathrm{Var}(J(u, B^A))_t \geq N) = 0, \tag{17}$$

and

$$\lim_N \sup_{u\in U_d^0} P(\mathrm{Var}(\tilde{Y}^{u,\lambda})_t \geq N) = 0. \tag{18}$$

Let for each $t \in R_+$

$$K_t^M = \operatorname{ess\,sup}_{u\in U_d^0} \langle J(u, M^A)\rangle_t,$$
$$K_t^B = \operatorname{ess\,sup}_{u\in U_d^0} |J_t(u, B^A)|.$$

Evidently, $K_s^M \leq K_t^M$, $K_s^B \leq K_t^B$ a.s. for each $s \leq t$. Let us prove that $P(K_t^M < \infty) = P(K_t^B < \infty) = 1$ for every $t \in R_+$.

It follows from (16) and Lemma 1 that there exists a measure Q equivalent to P, with bounded density, such that (for each $t \in R_+$)

$$\sup_{u\in U_d^0} E^Q\langle J(u, M^A)\rangle_t < \infty. \tag{19}$$

Suppose that

$$Q(\operatorname{ess\,sup}_{u\in U_d^0} \langle J(u, M^A)\rangle_t = \infty) > \alpha > 0. \tag{20}$$

Then there exists some $0 \leq s \leq t$ for which (a.s.)

$$E^Q(\operatorname{ess\,sup}_{u\in U_d^0}(\langle J(u, M^A)\rangle_t - \langle J(u, M^A)\rangle_s)/\mathcal{F}_s) = \infty.$$

For each fixed $s, t \in R_+, s \leq t$, consider the family of random variables

$$G_{s,t}^Q(u) = E^Q(\langle J(u, M^A)\rangle_t - \langle J(u, M^A)\rangle_s/\mathcal{F}_s), u \in U^0.$$

This family satisfies the ε-lattice property (with $\varepsilon = 0$). Indeed, if $u^1, u^2 \in U_d^0$ then we define $u^3 \in U_d^0$ by

$$u^3 = I_{\Omega\times[0,s]}v + I_{D\times]s,t]}u^1 + I_{D^c\times]s,t]}u^2,$$

where $D = \{\omega : G_{s,t}(u^1) \ge G_{s,t}(u^2)\}$ and v is an arbitrary element of U_d^0, for which we have

$$G_{s,t}(u^3) \ge \max(G_{s,t}(u^1), G_{s,t}(u^2)).$$

Therefore according to Lemma 16.11 of [9]

$$E^Q(\operatorname{ess\,sup}_{u\in U_d^0}(\langle J(u,M^A)\rangle_t - \langle J(u,M^A)\rangle_s)/\mathcal{F}_s) = \\ = \operatorname{ess\,sup}_{u\in U_d^0} E^Q(\langle J(u,M^A)\rangle_t - \langle J(u,M^A)\rangle_s/\mathcal{F}_s). \tag{21}$$

Thus from (20) and (21) we have

$$\operatorname{ess\,sup}_{u\in U_d^0} E^Q(\langle J(u,M^A)\rangle_t - \langle J(u,M^A)\rangle_s/\mathcal{F}_s) = \infty \ \text{ a.s.}$$

which contradicts (19), since, according to the definition of ess sup (keeping the lattice property for G^Q in mind) there exists a sequence $(u^n, n \ge 1) \in U_d^0$ such that

$$E^Q(\langle J(u^n,M^A)\rangle_t - \langle J(u^n,M^A)\rangle_s/\mathcal{F}_s) \to \infty \ \text{ a.s.}$$

The equality $P(K_t^B < \infty) = 1$ is proved in a similar way.

Let us prove now that

$$\langle M^a, M^a\rangle_t \ll K_t^M, t \in R_+, \\ (\operatorname{Var} B^a)_t \ll K_t^B, t \in R_+,$$

for every $a \in A$.

Without loss of generality we can assume that the random variables K_t^M and K_t^B are integrable for each $t \in R_+$ (otherwise, one can use a localizing sequence of stopping times $(\tau_n, n \ge 1)$ with $EK_{t\wedge\tau_n}^M < \infty$ for each $n \ge 1$. Such a sequence exists, since $P(K_t^M + B_t^M < \infty) = 1$ for each $t \in R_+$, and the processes K^M, K^B are predictable).

Similarly to the above, we can show that for each fixed $s, t \in R_+, s \le t$, the family of random variables

$$G_{s,t}(u) = E(\langle J(u,M^A)\rangle_t - \langle J(u,M^A)\rangle_s/\mathcal{F}_s), u \in U_d^0$$

also satisfies the ε-lattice property (with $\varepsilon = 0$) and equality (21) is also valid, if we replace E^Q by the mathematical expectation E with respect to the basic measure P.

Therefore using (21) we obtain

$$\begin{aligned} &E(K_t^M - K_s^M/\mathcal{F}_s) = \\ &= E(\operatorname{ess\,sup}_{u\in U_d^0}\langle J(u,M^A)\rangle_t - \operatorname{ess\,sup}_{u\in U_d^0}\langle J(u,M^A)\rangle_s/\mathcal{F}_s) \ge \\ &\ge \operatorname{ess\,sup}_{u\in U_d^0} E(\langle J(u,M^A)\rangle_t/\mathcal{F}_s) - \operatorname{ess\,sup}_{u\in U_d^0}\langle J(u,M^A)\rangle_s = \\ &= \operatorname{ess\,sup}_{u\in U_d^0} E(\langle J(u,M^A)\rangle_t - \operatorname{ess\,sup}_{u\in U_d^0}\langle J(u,M^A)\rangle_s/\mathcal{F}_s) \ge \\ &\ge \operatorname{ess\,sup}_{u\in U_d^0} E(\langle J(u,M^A)\rangle_t - \langle J(u,M^A)\rangle_s/\mathcal{F}_s) = \\ &= E(\operatorname{ess\,sup}_{u\in U_d^0}(\langle J(u,M^A)\rangle_t - \langle J(u,M^A)\rangle_s)/\mathcal{F}_s) \ge \\ &\ge E(\langle M^a\rangle_t - \langle M^a\rangle_s/\mathcal{F}_s), \end{aligned}$$

and hence the process $(K_t^M - \langle M^a\rangle_t, t \ge 0)$ is a submartingale. Since every predictable local martingale of finite variation is constant (see, e.g., [10]), we conclude that the process $(K_t^M - \langle M^a\rangle_t, t \ge 0)$ is increasing for each $a \in A$. Evidently, this implies that the measure $d\langle M^a\rangle_t(\omega)$ is absolutely continuous with respect to the measure $dK_t^M(\omega)$ for almost all $\omega \in \Omega$, i.e., $\langle M^a\rangle \ll K^M$ for each $a \in A$.

Indeed, suppose H is some bounded positive predictable process such that $E\int_0^t H_s dK_s^M = 0$. Then

$$E\int_0^t H_s d\langle M^a\rangle_s + E\int_0^t H_s d(K_s^M - \langle M^a\rangle_s) = 0$$

implies that $E\int_0^t H_s d\langle M^a\rangle_s = 0$, since $K^M - \langle M^a\rangle$ is an increasing process.

In a similar way one can prove that

$$(\operatorname{Var} B^a)_t \ll K_t^B, t \in R_+,$$

for each $a \in A$. Thus the increasing process $K_t = K_t^M + K_t^B$ dominates the characteristics of semimartingales $(Y^a, a \in A)$. $\square$

Let $X^A = (X^a, a \in A)$ be a family of semimartingales and let for each $u \in \widehat{U}^0$

$$\bar{J}_t(u, X^A) = \sum_{a\in A^0}\int_0^t u_s(a)dX_s^a.$$

Evidently,

$$\bar{J}_t(u, X^A) = \sum_{a\in A^0}\int_0^t I_{(u_s=a)}dX_s^a$$

if $u \in U^0$, and

$$\bar{J}(u, X^A) = J(u, X^A)$$

for every $u \in \widehat{U}_d^0$.

Theorem 3. *X^A is a regular family of semimartingales if and only if for each sequence $(u^n, n \geq 1) \in \widehat{U}^0$ uniformly converging to some $u \in \widehat{U}$, the corresponding integral sums $\widehat{J}_t(u^n, X^A)$ are fundamental in probability uniformly on every compact.*

Proof. The necessity part of the theorem is proved in [3].

Sufficiency. Let for each sequence $(u^n, n \geq 1) \in \widehat{U}^0$ converging uniformly to some $u \in \widehat{U}$

$$\sup_{s \leq t} |\widehat{J}_t(u^n, X^A) - \widehat{J}(u^m, X^A)| \to 0 \tag{22}$$

as $n \to \infty, m \to \infty$, for every $t \geq 0$. We must show that the family of semimartingales X^A is regular. Since $\bar{J}(u, X^A) = J(u, X^A)$ for every $u \in \widehat{U}_d^0$, it follows from Theorem 2 that X^A is a dominated family of semimartingales, i.e., condition (B) of Definition 1 is satisfied.

Let us show that the family X^A satisfies conditions (C), (D).

First we prove that there exists a μ^K-a.e. continuous modification of the Radon–Nicodým derivative

$$\varphi(a, a') = d\langle M^a - M^{a'} \rangle / dK.$$

Let $A_0^m = (a_1^m, a_2^m, \dots, a_{i(m)}^m)$ be a $\frac{1}{m}$-net of the compact A. Suppose that the function $\varphi(a, a')$ is separable (or we can choose such a version).

The function

$$\varphi^*(a, a') = \lim_m \sup_{i: r(a_i^m, a) \leq \frac{1}{m}} \sup_{j: r(a_j^m, a') \leq \frac{1}{m}} \varphi(a_i^m, a_j^m)$$

is upper continuous a.e. with respect to the measure μ^K and it is easy to see that the μ^K-a.e. continuous modification of the derivative of $\varphi(a, a')$ exists if and only if

$$\varphi_t^*(u_t, u_t) = 0, \ 0 \leq t < \infty,$$

μ^K-a.e. for all $u \in U$. Let there exist $u \in U$ such that

$$\mu^K[(\omega, t) : \varphi^*(u, u) > 0] > 0.$$

We can construct two sequences $(u^m, m \geq 1), (v^m, m \geq 1) \in U^0$ (i.e., u^m and v^m are taking a finite number of values for each m), such that $u^m \to u, v^m \to u$ uniformly and

$$\varphi(u^m, v^m) \to \varphi^*(u, u), \ \mu^K - a.e, \tag{23}$$

as $m \to \infty$.

Put $u_t^m(\omega) = a^m_{i_m(\omega,t)}, v_t^m(\omega) = a^m_{j_m(\omega,t)}$, where the indices $a^m_{i_m}$ and $a^m_{j_m}$ are selected so that

$$r(a^m_{i_m(\omega,t)}, u_t(\omega)) \le \frac{1}{m},\ r(a^m_{j_m(\omega,t)}, u_t(\omega)) \le \frac{1}{m}$$

and

$$\varphi(a^m_{i_m(\omega,t)}, a^m_{j_m(\omega,t)}) \ge \sup_{i:r(a_i^m,u_t(\omega))\le\frac{1}{m}} \ \sup_{j:r(a_j^m,u_t(\omega))\le\frac{1}{m}} \varphi(a_i^m, a_j^m) - \frac{1}{m}.$$

Since for each $u, v \in U^0$

$$\langle \bar{J}(u, M^A) - \bar{J}(v, M^A)\rangle_t = \int_0^t \varphi_s(u_s, v_s) dK_s$$

we have

$$\liminf_m E\langle \bar{J}(u^m, M^A) - \bar{J}(v^m, M^A)\rangle_t = \liminf_m E \int_0^t \varphi_s(u_s^m, v_s^m) dK_s \ge$$

$$\ge E\int_0^t \varphi_s^*(u_s, u_s) dK_s > 0 \tag{24}$$

for some $t \ge 0$. Since the family of semimartingales $\bar{J}(u, X^A), u\widehat{U}^0)$ satisfies the condition U.T. (the proof is similar to the corresponding assertion of Theorem 2) and

$$\sup_{s\le t} |\bar{J}_s(u^n, X^A) - \bar{J}_s(u^m, X^A)| \to 0, m, n \to \infty, \tag{25}$$

it follows from Proposition 2 that

$$\sup_{s\le t} |\bar{J}_s(u^n, M^A) - \bar{J}_s(u^m, M^A)| \to 0, \quad m, n \to \infty, \tag{26}$$

$$\sup_{s\le t} |\bar{J}_s(u^n, B^A) - \bar{J}_s(u^m, B^A)| \to 0, \quad m, n \to \infty, \tag{27}$$

where

$$\bar{J}(u, M^A) = \sum_{a\in A^0} \int_0^t I_{(u_s=a)} dM_s^{a,\lambda}, \bar{J}(u, B^A) = \sum_{a\in A^0} \int_0^t I_{(u_s=a)} dB_s^{a,\lambda}$$

coincide with canonical decomposition terms of the special semimartingale $\bar{J}(u, X^A) - \sum \Delta \bar{J}(u, X^A) I_{(|\bar{J}(u,X^A)|>\lambda)}$ for each $u \in U^0$.

Obviously, (24) implies that the continuity condition (26) for $J(u, M^A)$ with respect to u does not hold and therefore $\varphi(a, a')$ is continuous in (a, a') with respect to the measure μ^K.

The existence of a μ^K-a.e. continuous modification of the derivative $g_t(a) = dB(t, a)/dK_t$ is proved analogously.

Let us prove now that the family X^A satisfies condition (D). Evidently, for each $u \in U^0$

$$\langle \bar{J}(u, M^A) \rangle_t = \int_0^t \psi_s(u_s, u_s) dK_s$$

and it is easy to see that the μ^K-a.e. continuity of $\varphi(a, a')$ implies the μ^K-a.e. continuity of the function $\psi(a, b)$ along the diagonal (i.e., there exists a μ^K-a.e. continuous modification of the function $\psi(a, a)$). Therefore

$$\int_0^t \sup_{a \in A} \psi_s(a, a) dK_s = \operatorname{ess\,sup}_{u \in U^0} \langle J(u, M^A) \rangle_t < \infty \quad \text{a.s.}$$

for each $t \in R_+$. The inequality

$$\int_0^t \sup_{a \in A} g_s(a) dK_s < \infty$$

is proved in a similar way.

Since

$$\Delta J_t(u, X^A) = \sum_{a \in A^0} I_{(u_s = a)}(X(t, a) - X(t-, a)) = X(t, u(t)) - X(t-, u(t)),$$

relation (18) can be rewritten as

$$\lim_{n \to \infty} \sup_{u \in U^0} P\{\sum_{s \le t} |X(t, u(t)) - X(t-, u(t))| I_{(|X(t,u(t)) - X(t-,u(t))| > \lambda)} > n\} = 0,$$

which implies that

$$\operatorname{ess\,sup}_{u \in U^0} \sum_{s \le t} |X(t, u(t)) - X(t-, u(t))| I_{(|X(t,u(t)) - X(t-,u(t))| > \lambda)} < \infty$$

a.s. for each $t \in R_+$.

Since the continuity property (22) of the functional $\bar{J}$ implies that a function ΔX_t^a is continuous with respect to a in probability uniformly on each $[0,t]$ and A is a compact subset of some metric space we obtain that

$$\sum_{s\le t}\sup_{a\in A}\Delta|X(s,a)|I_{(\sup_{a\in A}\Delta|X(s,a)|>\lambda)}=$$
$$=\operatorname{ess\,sup}_{u\in U}\sum_{s\le t}|X(t,u(t))-X(t-,u(t))|I_{(|X(t,u(t))-X(t-,u(t))|>\lambda)}=$$
$$=\operatorname{ess\,sup}_{u\in U^0}\sum_{s\le t}|X(t,u(t))-X(t-,u(t))|I_{(|X(t,u(t))-X(t-,u(t))|>\lambda)}<\infty$$

a.s. for each $t\ge 0$, and hence condition (D) is fulfilled. $\square$

Remark. A usual stochastic integral, evidently, corresponds to the case $X(t,a)=aX(t), a\in R$, where X is a semimartingale. It is easy to see that if u is a locally bounded predictable process then the usual stochastic integral $\int_0^t u_s dX_s$ satisfies relations (3), (4), and hence the stochastic integral $u\cdot M$ coincides (up to an evanescent set) with the stochastic line integral with respect to the family of martingales $(aM, a\in R)$ along the curve u.

Conversely, let $X=(X_t,t\ge 0)$ be an adapted continuous process and consider the family $X^A=(aX,a\in R)$. For an elementary predictable process $u\in\mathcal{H}$

$$J_t(u,X^A)=\sum_i u_i[X(t_{i+1}\wedge t)-X(t_i\wedge t)]$$

and it follows from Theorem 1 that the continuity of the functional $J(u)$ with respect to the class $\{u\in U_d^0:\ |u|\le 1\}$ implies that the process $X=X^1-X^0$ is a semimartingale (since the process $X^0=0$ is a semimartingale).

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