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TEST FOR DISCONJUGACY OF A DIFFERENTIAL INCLUSION OF NEUTRAL TYPE

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ABSTRACT. Generalizations of the disconjugacy for a differential inclusion of neutral type are given. The relation between these generalizations and the existence of solutions of multipoint boundary value problems is discussed and the tests for disconjugacy are established. These tests are applicable to the linear differential equations of neutral type, too.

In [1], the author has introduced the notions of disconjugate and strictly disconjugate linear differential equations of neutral type. They are generalizations of similar notions for ordinary differential equations without delay and are closely related to the existence and uniqueness of the solution of multipoint boundary value problems for linear differential equations of neutral type. The purpose of this paper is to generalize these notions for differential inclusions, to prove a relation between them and the existence of the solution of multipoint boundary value problems for differential inclusions of neutral type, and to give the tests for disconjugacy and strict disconjugacy of differential inclusions of neutral type. These tests will be applicable to the linear differential equations of neutral type also.

Consider the following nth order differential inclusion with delays of neutral type

$$x^{(n)}(t)\in A(t)x^{(n)}(t-\Delta_0(t))+\sum_{i=1}^{n}\sum_{j=1}^{m}B_{ij}(t)x^{(n-i)}(t-\Delta_{ij}(t)),\ n\geq 1, \quad \text{(DI)}$$

where

(i) $A(t)$, $B_{ij}(t)$ $(i=1,\dots,n,\ j=1,\dots,m)$ are lower semicontinuous multivalued maps (see [2, p. 12]) of the interval $I=[T_0,T)$, $T\leq\infty$, the values of which are convex closed subsets of $\mathbb{R}$,

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(ii) the delays $\Delta_0(t)$, $\Delta_{ij}(t)$ $(i = 1, \dots, n,\ j = 1, \dots, m)$ are continuous on I,

(iii) $A + \lambda B = \{a + \lambda b :\ a \in A,\ b \in B\}$ for $A, B \subset \mathbb{R}$ and $\lambda \in \mathbb{R}$.

The initial value problem (IVP) for (DI) is defined as follows. Let $t_0 \in [T_0, T)$ and let a continuous initial vector function $\Phi(t) = (\phi_0(t), \dots, \phi_n(t))$ be given on the initial set

$$E_{t_0} := \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} E_{t_0}^{ij} \cup E_{t_0}^{0},$$

where

$$E_{t_0}^{ij} := \bigl\{t - \Delta_{ij}(t) :\ t - \Delta_{ij}(t) < t_0,\ t \in [t_0, T)\bigr\} \cup \{t_0\}$$

for $i = 1, \dots, n$, $j = 1, \dots, m$ and

$$E_{t_0}^{0} := \bigl\{t - \Delta_0(t) :\ t - \Delta_0(t) < t_0,\ t \in [t_0, T)\bigr\} \cup \{t_0\}.$$

We have to find a solution $x \in C^{(n)}([t_0, T))$ of (DI) satisfying the initial value conditions

$$\begin{gathered} x^{(k)}(t_0) = \phi_k(t_0), \\ x^{(k)}(t - \Delta_{ij}(t)) = \phi_k(t - \Delta_{ij}(t)) \quad \text{if} \quad t - \Delta_{ij}(t) < t_0, \end{gathered} \tag{IV$_1$}$$

$$k = 0, 1, \dots, n-1, \quad i = n - k, \quad j = 1, \dots, m,$$

$$\begin{gathered} x^{(n)}(t_0) = \phi_n(t_0), \\ x^{(n)}(t - \Delta_0(t)) = \phi_n(t - \Delta_0(t)) \quad \text{if} \quad t - \Delta_0(t) < t_0. \end{gathered} \tag{IV$_2$}$$

Remark 1. The set of all solutions of (DI) is not a linear space in general.

Now we shall define some notions and give preliminary results which will be needed in the sequel:

Let $A \subset \mathbb{R}$. Then $|A| := \sup\{|a| :\ a \in A\}$. Let $2^{\mathbb{R}}$ denote the family of all nonempty subsets of $\mathbb{R}$ and, let $cf(\mathbb{R})$ be the set of all nonempty convex closed subsets of $\mathbb{R}$. For $F : I \to 2^{\mathbb{R}}$, we let $\operatorname{graph}(F) := \{(t, x) \in I \times \mathbb{R} :\ t \in I,\ x \in F(t)\}$. A function $f : I \to \mathbb{R}$ such that $f(t) \in F(t)$ for each $t \in I$ will be called a *selection* of F.

Lemma 1 (K. Deimling [2]). *Let $F : I \to cf(\mathbb{R})$ be lower semicontinuous. Then, given $(t_0, x_0) \in \operatorname{graph}(F)$, F has a continuous selection f such that $f(t_0) = x_0$.*

In the sequel we shall assume that for $A(t)$ from (DI) the inequality $|A(t)| \le A < 1$ holds.

Theorem 1. *Let the initial vector function $\Phi(t)$ be continuous and bounded on E_{t_0}, $t_0 \in [T_0, T)$. Then the initial value problem* (DI), (IV$_1$), (IV$_2$) *has at least one solution on $[t_0, T)$.*

Proof. By Lemma 1, we can associate to the initial value problem (DI), (IV_1), (IV_2) the problem of finding a solution $x(t)$ of the differential equation

$$x^{(n)}(t)+a(t)x^{(n)}(t-\Delta_0(t))+\sum_{i=1}^{n}\sum_{j=1}^{m}b_{ij}(t)x^{(n-i)}(t-\Delta_{ij}(t))=0,\quad n\geq 1 \tag{E_n}$$

(where $a(t)$ ($b_{ij}(t)$) is a continuous selection of $A(t)$ ($B_{ij}(t)$)) which satisfies the initial value conditions (IV_1), (IV_2). But this problem, by Theorem 1 from [1], has exactly one solution on $[t_0,T)$ which is also a solution of the initial value problem (DI), (IV_1) and (IV_2).

Now for $t_0\in I$ let us denote by $B'(\mathrm{DI},t_0)$ the set of all solutions of (DI) with constant initial vector function

$$\Phi(t)=\big(\phi_0(t),\dots,\phi_n(t)\big)=(c_0,\dots,c_n)\neq(0,\dots,0),\ \ t\in E_{t_0},$$

and by $B(\mathrm{DI},t_0)$ the set of all $x\in B'(\mathrm{DI},t_0)$ with $\phi_0(t)=0$, $t\in E_{t_0}$. □

Definition 1. A point $\xi\in[t_0,T)$ is called a *zero-point of order p* of a solution $x\in B'(\mathrm{DI},t_0)$ iff $x(\xi)=\cdots=x^{(p-1)}(\xi)=0$ and $x^{(p)}(\xi)\neq 0$. If $x(\xi)=0$ and also $x^{(p)}(\xi)=0$ for each $p\in N$, then we shall say that ξ *is a zero-point of order infinity.*

The first zero-point of $x\in B'(\mathrm{DI},t_0)$ to the right of t_0 which is at least $(n+1)$th consecutive zero (counting also eventual zero $\xi=t_0$ and including multiplicity of zeros), will be denoted by $\eta(x,t_0)$. If such a point does not exist, we put $\eta(x,t_0)=T$.

Definition 2. By the *first adjoint point* to the point $a\in[T_0,T)$ with respect to (DI) we mean the point $\alpha_1(a):=\inf\{\eta(x,a):\ x\in B'(\mathrm{DI},a)\}$.

Definition 3. The differential inclusion (DI) is said to be *strictly disconjugate on an interval $J\subset I$*, iff $a\in J\Rightarrow\alpha_1(a)\notin J$.

Notation 1. Let a continuous vector function $\Phi(t)=(\phi_0(t),\dots,\phi_n(t))$ be defined on E_{t_0} and let $r_0\in\{0,1,\dots,n\}$. Then

$$H(t_0,\Phi,r_0):=\Big\{\big(\phi_0(t),\dots,\phi_{r_0-1}(t),c_0+\phi_{r_0}(t),\dots,c_{n-r_0}+\phi_n(t)\big):\ c_i\in\mathbb{R},\ i=0,1,\dots,n-r_0\Big\}.$$

Notation 2. Let $\Phi(t)=(\phi_0(t),\dots,\phi_n(t))$ be a continuous and bounded vector function defined on E_{t_0}. Let $x(t)$ be a solution of (DI). Then we shall write $x\in\widetilde{H}(t_0,\Phi,r_0)$, iff $x(t)$ is determined by an initial vector function from $H(t_0,\Phi,r_0)$.

In the sequel, we shall consider the following boundary value problem (BVP) for (DI) (see [1]): Let

$$\tau_0, \tau_1, \dots, \tau_p \in J \subset I, \quad \tau_0 < \tau_1 \leq \dots \leq \tau_p, \quad p \leq n,$$
$$r_0 \in \{0, 1, \dots, n\}; \quad r_1, \dots, r_p \in N, \quad r_0 + r_1 + \dots + r_p = n + 1,$$
$$\beta_0^{(1)}, \dots, \beta_0^{(r_0)}, \beta_1^{(1)}, \dots, \beta_p^{(r_p)} \in \mathbb{R},$$

and $\Phi(t) = (\phi_0(t), \dots, \phi_n(t))$ be a continuous and bounded vector function defined on E_{τ_0} such that $\phi_{i-1}(\tau_0) = \beta_0^{(i)}$, $i = 1, \dots, r_0$. The problem is to find a solution $x(t)$ of (DI) which satisfies the conditions

$$x^{(\nu_k - 1)}(\tau_k) = \beta_k^{(\nu_k)}, \;\; \nu_k = 1, \dots, r_k; \;\; k = 1, \dots, p, \;\; x \in \widetilde{H}(\tau_0, \Phi, r_0).$$

Theorem 2. *Let* (DI) *be strictly disconjugate on an interval J. Then each* (BVP) *for* (DI) *has at least one solution.*

Proof. If (DI) is strictly disconjugate on J, then every associate differential equation (E_n) is also strictly disconjugate on J (see [1]) and thus (BVP) for (E_n) has exactly one solution (see Theorem 6 of [1]), which is also a solution of (BVP) for (DI). □

To prove a test for strict disconjugacy of (DI), we shall need the following lemmas.

Lemma 2 (Levin [3]). *Suppose $y \in C^{(n)}([a, b])$ and*

$$|y^{(n)}(t)| \leq \mu, \;\; t \in [a, b],$$
$$y(a_0) = y'(a_1) = \dots = y^{(n-1)}(a_{n-1}) = 0,$$

where $a \leq a_0 \leq a_1 \leq \dots \leq a_{n-1} \leq b$ or $a \leq a_{n-1} \leq \dots \leq a_1 \leq a_0 \leq b$. Then

$$|y^{(n-k)}(t)| \leq \frac{\mu(b-a)^k}{k[\frac{k-1}{2}]![\frac{k}{2}]!}$$

for $a \leq t \leq b$, $k = 1, \dots, n$.

As a consequence of this lemma, we have

Lemma 3. *Suppose $x \in B'(\mathrm{DI}, t_0)$, $t_0 \in [a, b)$ and*

$$|x^{(n)}(t)| \leq \mu \;\; \text{for} \;\; t_0 \leq t \leq b,$$
$$x(a_0) = x'(a_1) = \dots = x^{(n-1)}(a_{n-1}) = 0 \tag{1}$$

where

$$t_0 \leq a_0 \leq a_1 \leq \dots \leq a_{n-1} \leq b \tag{2}$$

or

$$t_0 \le a_{n-1} \le \cdots \le a_1 \le a_0 \le b. \tag{3}$$

Then

$$|x^{(n-k)}(t)| \le \frac{\mu(b-a)^k}{k[\frac{k-1}{2}]![\frac{k}{2}]!} \quad \text{for} \quad t_0 \le t \le b, \ \ k = 1, \dots, n,$$

and

$$\left|x^{(n-i)}(t - \Delta_{ij}(t))\right| \le \frac{\mu(b-a)^i}{i[\frac{i-1}{2}]![\frac{i}{2}]!}$$

for $t_0 \le t \le b$, $i = 1, \dots, n$, $j = 1, \dots, m$.

Theorem 3. *Suppose*

$$|A(t)| \le A < 1, \quad |B_{ij}(t)| \le B_{ij} \tag{4}$$

for all t *in a compact interval* $J = [a,b]$ $(i = 1, \dots, n, \ j = 1, \dots, n)$. *Then the differential inclusion* (DI) *is strictly disconjugate on* J *if*

$$\chi(b-a) < 1 - A, \tag{5}$$

where

$$\chi(h) := \sum_{i=1}^{n} \frac{\sum_{j=1}^{m} B_{ij}}{i[\frac{i-1}{2}]![\frac{i}{2}]!}\, h^i.$$

Proof. Suppose on the contrary that (DI) is not strictly disconjugate on J. Then there is a point $t_0 \in [a,b)$ such that (DI) has a solution $x \in B'(\text{DI},t_0)$ with at least $n+1$ zeros (including multiplicity) in $[t_0, b]$. From this fact and Rolle's theorem, there are points $a_0, a_1, \dots, a_{n-1}$ such that inequalities (2) or (3) and the equalities (1) are fulfilled. The interval $[t_0, a_{n-1}]$ is nondegenerate, since $x(t)$ cannot have a zero of multiplicity $n+1$ at t_0 (since $x \in B'(\text{DI},t_0)$).

Let us denote $a_{n-1} = c$. Applying Lemma 3 to the interval $[t_0, c]$, we obtain

$$\max_{t_0 \le t \le c} \left|x^{(n-i)}(t - \Delta_{ij}(t))\right| \le \frac{\mu(b-a)^i}{i[\frac{i-1}{2}]![\frac{i}{2}]!}\,, \quad i = 1, \dots, n, \ \ j = 1, \dots, m, \tag{6}$$

where $\mu := \max_{t_0 \le t \le c} |x^{(n)}(t)|$. However, for some $\tau \in [t_0, c]$ we have

$$\max_{t_0 \le t \le c} |x^{(n)}(t)| = |x^{(n)}(\tau)| = \mu. \tag{7}$$

From (7), (DI) and (6), we get

$$\mu = |x^{(n)}(\tau)| \le A\mu + \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ij}\big|x^{(n-i)}(\tau - \Delta_{ij}(\tau))\big| \le$$
$$\le A\mu + \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ij}\frac{\mu(b-a)^i}{i[\frac{i-1}{2}]![\frac{i}{2}]!} = \mu\Big(A + \sum_{i=1}^{n}\frac{\sum_{j=1}^{m} B_{ij}}{i[\frac{i-1}{2}]![\frac{i}{2}]!}(b-a)^i\Big).$$

Evidently $\mu > 0$ since otherwise $x(t)$ would coincide on $[t_0, c]$ with a polynomial of degree $m < n$ and $x^{(m)}(t)$ would not vanish on $[t_0, c]$ (but $x^{(m)}(a_m) = 0$). Hence $\chi(b-a) \ge 1 - A$ which is a contradiction with (5). □

Definition 4. Let $a \in I$. By the *adjoint point to the point a* with respect to (DI) we shall mean the point $\alpha(a) := \inf\{\eta(x, a) : x \in B(\mathrm{DI}, a)\}$.

Definition 5. The differential inclusion (DI) is said to be *disconjugate on an interval $J \subset I$* iff the implication $a \in J \Rightarrow \alpha(a) \notin J$ holds for each $a \in J$.

Corollary 1. *If the differential inclusion* (DI) *is strictly disconjugate on an interval J, then it is disconjugate on J.*

Remark 2. As we shall see, the evaluated length of the interval on which (DI) is strictly disconjugate is less then the evaluated length of the interval on which (DI) is disconjugate.

Theorem 4. *Let the differential inclusion* (DI) *be disconjugate on an interval J. Then each boudary value problem for* (DI) *with $r_0 \in \{1, \dots, n\}$ has at least one solution.*

Proof. If (DI) is disconjugate on J, then on this interval the associate differential equation (E_n) is also disconjugate (see [1]). Thus the boundary value problem for (E_n) with $r_0 \in \{1, \dots, n\}$ has exactly one solution which is also a solution of BVP for (DI). □

Lemma 4. *Suppose $x \in B(\mathrm{DI}, t_0)$, $t_0 \in [a, b)$ has at least $n+1$ zeros on $[t_0, b]$. Then*

(i) *there are points $a_0, a_1, \dots, a_{2n}$ such that*

$$t_0 \le a_0 \le a_1 \le \dots \le a_{2n} \le b, \tag{8}$$
$$0 = x(a_0) = x'(a_1) = \dots = x^{(n-1)}(a_{n-1}) = x^{(n)}(a_n) =$$
$$= x^{(n-1)}(a_{n+1}) = \dots = x'(a_{2n-1}) = x(a_{2n}) \tag{9}$$

and

(ii) *the subintervals $[t_0, a_n]$, $[a_n, b]$ are nondegenerate.*

Proof. Let $a_1^{(0)}, \dots, a_{n+1}^{(0)}$ be $n+1$ zeros of $x(t)$ such that

$$t_0 \le a_1^{(0)} \le \cdots \le a_{n+1}^{(0)} \le b.$$

Since $x \in B(\mathrm{DI}, t_0)$, we have that t_0 is a zero of $x(t)$ of multiplicity less than $n+1$. Hence $x(t)$ has at least two distinct zeros in $[t_0, b]$, i.e.,

$$t_0 = a_1^{(0)} < a_{n+1}^{(0)} \le b.$$

By Rolle's theorem there are n zeros $a_1^{(1)}, \dots, a_n^{(1)}$ of $x'(t)$ such that

$$a_i^{(0)} \le a_i^{(1)} \le a_{i+1}^{(0)}, \quad i = 1, \dots, n, \quad \text{and} \quad t_0 \le a_1^{(1)} < a_n^{(1)} \le b.$$

Repeating this process we eventually obtain a zero $a_1^{(n)}$ of $x^{(n)}(t)$ between two zeros $a_1^{(n-1)}$, $a_2^{(n-1)}$ of $x^{(n-1)}(t)$. The points

$$a_0^{(0)}, a_1^{(1)}, \dots, a_1^{(n-1)}, a_1^{(n)}, a_2^{(n-1)}, \dots, a_n^{(1)}, a_{n+1}^{(0)}$$

satisfy (8), (9) and since $t_0 \le a_1^{(n-1)} < a_2^{(n-1)} \le b$ we have $t_0 \ne a_1^{(n)} \ne b$. □

Remark 3. Statement (i) of Lemma 4 is in fact a consequence of Lemma 2 of [3].

Theorem 5. *Suppose that the estimates* (4) *are valid. Then the inclusion* (DI) *is disconjugate on* $[a, b]$ *if*

$$\chi\Big(\frac{b-a}{2}\Big) < 1 - A. \tag{10}$$

Proof. Suppose on the contrary that (DI) is not disconjugate on $[a, b]$. Then there is a point $t_0 \in [a, b)$ such that (DI) has a solution $x \in B(\mathrm{DI}, t_0)$ with at least $n+1$ zeros (including multiplicity) in $[t_0, b]$. By Lemma 4, there are points $a_0, a_1, \dots, a_{2n}$ such that (8), (9) hold and the subintervals $[t_0, a_n]$, $[a_n, b]$ are not degenerate. Let us denote $a_n = c$. One of this subintervals, say $[t_0, c]$, has length at most $\frac{1}{2}(b-a)$. Applying Lemma 3 to this interval, we obtain

$$\max_{t_0 \le t \le c} \big|x^{(n-i)}(t - \Delta_{ij}(t))\big| \le \frac{\mu (b-a)^i}{2^i i [\frac{i-1}{2}]! [\frac{i}{2}]!}, \quad i = 1, \dots, n, \;\; j = 1, \dots, m,$$

where

$$\mu = \max_{t_0 \le t \le c} |x^{(n)}(t)|.$$

However, for some $\tau \in [t_0, c]$ we have

$$\mu = |x^{(n)}(\tau)| \le A\mu + \sum_{i=1}^{n} \sum_{j=1}^{m} B_{ij} \big|x^{(n-i)}(\tau - \Delta_{ij}(\tau))\big| \le$$

$$\leq A\mu + \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ij}\frac{\mu(b-a)^i}{2^i i[\frac{i-1}{2}]![\frac{i}{2}]!} = \mu\Big(A + \sum_{i=1}^{n}\frac{\sum_{j=1}^{m} B_{ij}}{2^i i[\frac{i-1}{2}]![\frac{i}{2}]!}(b-a)^i\Big).$$

Evidently $\mu > 0$ since otherwise $x(t)$ would coincide on $[t_0, c]$ with a polynomial of degree $m < n$ and $x^{(m)}(t)$ would not vanish on $[t_0, c]$ (but $x^{(m)}(a_m) = 0$).

Hence $\chi\Big(\frac{b-a}{2}\Big) \geq 1 - A$ which is a contradiction with (10). □

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