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OSCILLATORY PROPERTIES OF SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH SEVERAL RETARDED ARGUMENTS

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Abstract. The impulsive differential equation

$$\begin{aligned} &x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) = 0, \quad t \neq \xi_k, \\ &\Delta x(\xi_k) = b_k x(\xi_k) \end{aligned}$$

with several retarded arguments is considered, where $p_i(t) \geq 0$, $1 + b_k > 0$ for $i = 1, \dots, m$, $t \geq 0$, $k \in \mathbb{N}$. Sufficient conditions for the oscillation of all solutions of this equation are found.

§ 1. Introduction

In the past two decades the number of investigations of the oscillatory and nonoscillatory behavior of solutions of functional differential equations has been growing constantly. The greater part of works on this subject published up to 1977 are given in [1]. In the monographs [2] and [3], published in 1987 and 1991 respectively, the oscillatory and asymptotic properties of solutions of various classes of functional differential equations are systematically studied.

The first work in which the oscillatory properties of impulsive differential equations with retarded argument of the form

$$\begin{aligned} &x'(t) + p(t)x(t-\tau) = 0, \quad t \neq t_k, \\ &\Delta x(t_k) = b_k x(t_k) \end{aligned} \tag{$*$}$$

are investigated is the paper of Gopalsamy and Zhang [4]. In it the authors give sufficient conditions for the oscillation of all solutions under the

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assumption that $p(t)$ is a positive function, and sufficient conditions are found for the existence of a nonoscillatory solution if $p(t) \equiv p = \text{const} > 0$.

In [5] more general conditions for the oscillation of solutions of equation $(*)$ are found, when this equation has a retarded argument ($\tau > 0$). In [6] similar results are obtained when equation $(*)$ has an advanced argument ($\tau < 0$).

In the present work sufficient conditions for the oscillation of solutions of impulsive differential equations with several retarded arguments are found.

§ 2. Preliminary notes

Consider the impulsive differential equation with several retarded arguments

$$\begin{aligned} &x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) = 0, \quad t \neq \xi_k, \\ &\Delta x(\xi_k) = b_k x(\xi_k), \end{aligned} \tag{1}$$

where $\Delta x(\xi_k) = x(\xi_k^+) - x(\xi_k)$, together with the impulsive differential inequalities

$$\begin{aligned} &x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) \leq 0, \quad t \neq \xi_k, \\ &\Delta x(\xi_k) = b_k x(\xi_k) \end{aligned} \tag{2}$$

and

$$\begin{aligned} &x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) \geq 0, \quad t \neq \xi_k, \\ &\Delta x(\xi_k) = b_k x(\xi_k) \end{aligned} \tag{3}$$

provided that the following conditions are met:

A1. $0 < \tau_1 < \cdots < \tau_m$.
A2. The sequence $\{\xi_k\}$ is such that

$$0 < \xi_1 < \xi_2 < \cdots, \qquad \lim_{k\to\infty} \xi_k = \infty.$$

A3. $p_i \in C(\mathbb{R}_+, \mathbb{R}_+), \quad i = 1, 2, \ldots, m.$
A4. $b_k > -1$ for $k \in \mathbb{N}$.

Remark 1. If $1 + b_k \leq 0$ for an infinite number of $k \in \mathbb{N}$, then every nonzero solution $x(t)$ of equation (1) is oscillatory, since $x(\xi_k^+) = (1 + b_k)x(\xi_k)$. Therefore the case $b_k < -1$ is not interesting for consideration and condition A4 is quite natural.

Consider the sequences $\{\xi_n\}_{n=1}^{\infty}$ (the jump points) and $\{\xi_n+\tau_i\}_{n=1}^{\infty}$, $i=1,2,\dots,m$, and let

$$\{\xi_n\}_{n=1}^{\infty}\cap\{\xi_n+\tau_1\}_{n=1}^{\infty}\cap\cdots\cap\{\xi_n+\tau_m\}_{n=1}^{\infty}=\varnothing.$$

Let $\{t_k\}_1^{\infty}$ be the sequence with the following properties:

(i) $t_1<t_2<\cdots$.

(ii) $\{t_k\}_1^{\infty}=\{\xi_n\}_1^{\infty}\cup\{\xi_n+\tau_1\}_1^{\infty}\cup\cdots\cup\{\xi_n+\tau_m\}_1^{\infty}$.

Clearly $\lim\limits_{k\to\infty} t_k=\infty$.

Definition 1. The function $x(t)$ is said to be a *solution* of equation (1) if:

1. $x(t)$ is continuous in $[-\tau,\infty)\setminus\{\xi_n\}_{n=1}^{\infty}$, and is continuous from the left for $t=\xi_n, \quad n=1,2,\dots$.
2. $x(t)$ is differentiable in $(0,\infty)\setminus\{t_k\}_1^{\infty}$.
3. $x(t)$ satisfies equation (1) for $t\in(0,t_1)$ and $t\in\bigcup\limits_{i=1}^{\infty}(t_i,t_{i+1})$.
4. $x(\xi_k^+)=(1+b_k)x(\xi_k)$.

Analogously solutions of the inequalities (2) and (3) are defined. Details about the general theory of differential equations with impulses can be found in [7].

Definition 2. The solution $x(t)$ of the inequality (2) is said to be *eventually positive* if there exists $t_0>0$ such that $x(t)>0$ for $t\ge t_0$.

Definition 3. The solution $x(t)$ of the inequality (3) is said to be *eventually negative* if there exists $t_0>0$ such that $x(t)<0$ for $t\ge t_0$.

Definition 4. The solution $x(t)$ of equation (1) is said to be *oscillatory* if the set of its zeros is unbounded above, otherwise it is said to be *nonoscillatory.*

Let the function $u(t)$ satisfy conditions 1, 2, and 4 from Definition 1 and let $u(t)$ be nonincreasing in $(t_k,t_{k+1}), k=0,1,2,\dots (t_0=0)$. Define the function $\psi(t)$ by

$$\psi(t)=\prod_{0\le\xi_k<t}(1+b_k). \tag{4}$$

(If $t\le\xi_1, \quad \psi(t)=1$.) Let

$$v(t)=\frac{u(t)}{\psi(t)}, \quad t\ge 0. \tag{5}$$

We shall establish some properties of $v(t)$ which we shall use in the proof of the main results. To this end we need the following result.

Lemma 1 ([8], p. 330). *Let* f *have the properties:*

(i) $f \in C([a,b],\mathbb{R})$.

(ii) *There exists a finite derivative* $f'(x)$ *in* (a,b) *with the possible exception of countably many points.*

(iii) $f' \in L_1(a,b)$.

Then f *is absolutely continuous and*

$$\int_a^x f'(t)\,dt = f(x) - f(a), \quad x \in [a,b].$$

Lemma 2. *The function* $v(t)$ *defined by (5) has the following properties:*

(i) $v \in C([0,\infty),\mathbb{R})$.

(ii) $v(t)$ *is nonincreasing in* $[0,\infty)$.

(iii) $v(t)$ *is differentiable for* $t \in \bigcup\limits_{i=0}^{\infty}(t_i,t_{i+1})$ *and if* $0 \le a < b$ *then*

$$\int_a^b v'(t)\,dt = v(b) - v(a).$$

Proof. **(i)** We have to prove only that $v(t)$ is continuous at the points $\xi_1,\xi_2,\dots$. Since $u(t)$ and $\psi(t)$ are continuous from the left at these points it suffices to prove that $\lim\limits_{t\to\xi_k+} v(t) = v(\xi_k)$ but this follows easily from the following relations:

$$\lim_{t\to\xi_k+} v(t) = \lim_{t\to\xi_k+}\frac{u(t)}{\psi(t)} = \frac{\lim\limits_{t\to\xi_k+} u(t)}{\lim\limits_{t\to\xi_k+}\psi(t)} = \frac{u(\xi_k^+)}{\prod\limits_{i=1}^{k}(1+b_i)} =$$

$$= \frac{(1+b_k)u(\xi_k)}{\prod\limits_{i=1}^{k}(1+b_i)} = \frac{u(\xi_k)}{\prod\limits_{i=1}^{k-1}(1+b_i)} = \frac{u(\xi_k)}{\psi(\xi_k)} = v(\xi_k).$$

(ii) Since $u(t)$ is nonincreasing in (t_i,t_{i+1}), $i = 0,1,\dots$, and $\psi(t)$ is constant in each of these intervals, we conclude that $v(t)$ is nonincreasing in (t_i,t_{i+1}), $i = 0,1,\dots$. The fact that $v(t)$ is continuous implies that $v(t)$ is nonincreasing in $[0,\infty)$.

(iii) The fact that the derivative $v'(t)$ exists at each point $t \in \bigcup\limits_{i=0}^{\infty}(t_i,t_{i+1})$ is obvious. The already proved properties (i) and (ii) imply that $v'(t)$ is summable in any finite interval. Then (iii) follows immediately from Lemma 1. $\square$

§ 3. Main Results

Theorem 1. *Suppose that:*

1. *Conditions* A1—A4 *hold.*
2. *There exists an unbounded increasing sequence* $\{\gamma_n\}$ *such that for each* $n \in \mathbb{N}$,

$$\int_{\gamma_n-\tau_1}^{\gamma_n} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1} ds \ge 1. \tag{6}$$

Then:

1. *Inequality* (2) *has no eventually positive solution.*
2. *Inequality* (3) *has no eventually negative solution.*
3. *Each solution of equation* (1) *is oscillatory.*

Proof. First we shall prove that inequality (2) has no eventually positive solutions. Suppose that this is not true and let $x(t)$ be an eventually positive solution of (2). Without loss of generality we may assume that $x(t) > 0$ for $t \ge -\tau_m$. We define the function $y(t)$ by $y(t) = x(t)\psi(t)^{-1}$, where $\psi(t)$ was defined by (4). Then the inequality (2) takes the form

$$y'(t) + \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1} y(t-\tau_i) \le 0, \quad t \ne t_k,\ t \ge 0. \tag{7}$$

Since $x(t) > 0$, (2) implies that $x(t)$ is nonincreasing in each interval (t_i, t_{i+1}), $i = 0, 1, \ldots$. Thus $x(t)$ satisfies the same conditions as the function $u(t)$. Then (5) and Lemma 2 imply that $y(t)$ is a continuous nonincreasing function. Let N be a large enough integer such that $\gamma_n \ge \tau_1$ for $n \ge N$. We integrate (7) from $\gamma_n - \tau_1$ to γ_n and by making use of assertion (iii) of Lemma 2 obtain

$$y(\gamma_n) - y(\gamma_n - \tau_1) + \int_{\gamma_n-\tau_1}^{\gamma_n} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1} y(s-\tau_i) ds \le 0.$$

Since $y(t)$ is nonincreasing,

$$y(s-\tau_i) \ge y(\gamma_n - \tau_i) \ge y(\gamma_n - \tau_1), \quad i = 1, 2, \ldots, m,$$

for $s \in [\gamma_n - \tau_1, \gamma_n]$. Then

$$y(\gamma_n) + y(\gamma_n - \tau_1) \left\{ \int_{\gamma_n-\tau_1}^{\gamma_n} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1} ds - 1 \right\} \le 0. \tag{8}$$

The last inequality and the fact that $y(t)$ is positive imply that for $n \geq N$ the following inequality holds:

$$\int_{\gamma_n-\tau_1}^{\gamma_n} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i\leq\xi_k<s} (1+b_k)^{-1}\, ds < 1,$$

which contradicts (6).

In order to prove that (3) has no eventually negative solution it suffices to note that if $x(t)$ is a solution of (3), then $-x(t)$ is a solution of (2). From assertions 1 and 2 it follows that equation (1) has neither eventually positive nor eventually negative solutions. Thus each solution of (1) is oscillatory. □

Theorem 2. *Suppose that:*

1. *Conditions* A1—A4 *hold.*
2. *There exists an unbounded increasing sequence* $\{\gamma_n\}$ *such that for each* $n \in \mathbb{N}$ *and for some* $i = i(n) \in \{1, \ldots, m\}$ *the inequalities*

$$\int_{\gamma_n-\tau_i}^{\gamma_n} p_i(s) \prod_{s-\tau_i\leq\xi_k<s} (1+b_k)^{-1} ds \geq 1 \tag{9}$$

hold.

Then:

1. *Inequality* (2) *has no eventually positive solution.*
2. *Inequality* (3) *has no eventually negative solution.*
3. *Each solution of equation* (1) *is oscillatory.*

The proof of Theorem 2 is analogous to the proof of Theorem 1.

Theorem 3. *Let conditions* A1—A4 *hold and a sequence of disjoint intervals* $\{(\alpha_n, \beta_n)\}_{n=1}^{\infty}$ $(\alpha_1 > 0)$ *exist such that* $\beta_n - \alpha_n \geq 2\tau_m$, $\lim_{n\to\infty}(\beta_n - \alpha_n) = \infty$ *and*

$$\liminf_{t\to\infty} \int_{t-\tau_1}^{t} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i\leq\xi_k<s} (1+b_k)^{-1} ds > \frac{1}{e} \tag{10}$$

for $t \in \bigcup_{n=1}^{\infty}(\alpha_n + \tau_m, \beta_n)$. *Then:*

1. *Inequality* (2) *has no eventually positive solution.*
2. *Inequality* (3) *has no eventually negative solution.*
3. *Each solution of equation* (1) *is oscillatory.*

Proof. Suppose that $x(t)$ is an eventually positive solution of (2). Without loss of generality we may assume that $x(t) > 0$, $t \geq -\tau_m$. We define the function $y(t)$ as in Theorem 1. Then inequality (2) takes form (7), and the function $y(t)$ is continuous, positive, and nonincreasing for $t \geq 0$. Choose K such that

$$\liminf_{t\to\infty} \int\limits_{t-\tau_1}^{t} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \leq \xi_k < s} (1+b_k)^{-1}\, ds > K > \frac{1}{e},$$

$$t \in \bigcup_{n=1}^{\infty} (\alpha_n + \tau_m, \beta_n).$$

Then there exists an integer N_1 such that

$$\int\limits_{t-\tau_1}^{t} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \leq \xi_k < s} (1+b_k)^{-1}\, ds \geq K \tag{11}$$

for $t \in \bigcup\limits_{n=N_1}^{\infty} (\alpha_n + \tau_m, \beta_n)$. We integrate (7) from $\beta_n - \tau_1$ to t and by making use of Lemma $2_{(iii)}$ we obtain

$$y(t) - y(\beta_n - \tau_1) + \int\limits_{\beta_n-\tau_1}^{t} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \leq \xi_k < s} (1+b_k)^{-1} y(s-\tau_i) ds \leq 0,$$

$t \in [\beta_n - \tau_1, \beta_n]$, $n \geq N_1$. Since $\beta_n - \tau_m - \tau_1 > 0$ and $y(t)$ is nonincreasing for $t \geq 0$, we have

$$y(s-\tau_i) \geq y(s-\tau_1) \geq y(t-\tau_1), \quad i = 1, 2, \ldots, m,$$

for $s \in [\beta_n - \tau_1, t]$ and $t \in [\beta_n - \tau_1, \beta_n]$. Then

$$y(t) - y(\beta_n - \tau_1) + y(t-\tau_1) \int\limits_{\beta_n-\tau_1}^{t} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \leq \xi_k < s} (1+b_k)^{-1}\, ds \leq 0.$$

Having in mind that $y(t)$ is a positive function, from the last inequality we obtain for $n \geq N_1$

$$y(t-\tau_1) A_n(t) \leq y(\beta_n - \tau_1)\,, \quad t \in [\beta_n - \tau_1, \beta_n], \tag{12}$$

where

$$A_n(t) = \int\limits_{\beta_n-\tau_1}^{t} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \leq \xi_k < s} (1+b_k)^{-1}\, ds.$$

Analogously, from the inequality

$$y(\beta_n) - y(t) + \int_t^{\beta_n} \sum_{i=1}^m p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1} y(s-\tau_i)\, ds \le 0$$

we obtain

$$y(\beta_n - \tau_1) B_n(t) \le y(t), \quad t \in [\beta_n - \tau_1, \beta_n], \tag{13}$$

where

$$B_n(t) = \int_t^{\beta_n} \sum_{i=1}^m p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1}\, ds.$$

Since $A_n(t)$ and $B_n(t)$ are continuous for $t \in [\beta_n - \tau_1, \beta_n]$ and

$$A_n(t) + B_n(t) = \int_{\beta_n - \tau_1}^{\beta_n} \sum_{i=1}^m p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1}\, ds \ge K,$$

there exists $\gamma_n \in (\beta_n - \tau_1, \beta_n)$ such that $A_n(\gamma_n) \ge \frac{K}{2}$ and $B_n(\gamma_n) \ge \frac{K}{2}$ for $n \ge N_1$. Then from (12) and (13) we obtain

$$\frac{y(\gamma_n - \tau_1)}{y(\gamma_n)} \le \frac{4}{K^2}\,, \quad n \ge N_1\,, \quad \gamma_n \in (\beta_n - \tau_1, \beta_n). \tag{14}$$

On the other hand, it follows from (7) that

$$y'(t) + y(t) \sum_{i=1}^m p_i(t) \prod_{t-\tau_i \le \xi_k < t} (1+b_k)^{-1} \le 0$$

for $t \ge \tau_m$ and in particular for $t \in \bigcup_{n=N_1}^{\infty} (\alpha_n + \tau_m, \beta_n)$, or

$$\frac{y'(t)}{y(t)} + \sum_{i=1}^m p_i(t) \prod_{t-\tau_i \le \xi_k < t} (1+b_k)^{-1} \le 0.$$

From Lemma 1 applied to the function $\ln y(t)$ it follows that

$$\ln \frac{y(t)}{y(t-\tau_1)} + \int_{t-\tau_1}^{t} \sum_{i=1}^m p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1}\, ds \le 0$$

for $t \in \bigcup_{n=N_1}^{\infty} [\alpha_n + \tau_m + \tau_1, \beta_n]$. From (11) we obtain

$$\ln \frac{y(t-\tau_1)}{y(t)} \ge K\,, \quad t \in \bigcup_{n=N_1}^{\infty} [\alpha_n + \tau_m + \tau_1, \beta_n]$$

or equivalently

$$\frac{y(t-\tau_1)}{y(t)} \ge e^K \ge eK\,, \quad t \in \bigcup_{n=N_1}^{\infty} [\alpha_n + \tau_m + \tau_1, \beta_n].$$

Since $y(t-\tau_1) \ge eKy(t)\,, \quad t \in \bigcup_{n=N_1}^{\infty} (\alpha_n + \tau_m + \tau_1, \beta_n)$, it follows from (7) that

$$y'(t) + eKy(t) \sum_{i=1}^{m} p_i(t) \prod_{t-\tau_i \le \xi_k < t} (1+b_k)^{-1} \le 0$$

and, as above,

$$\ln \frac{y(t)}{y(t-\tau_1)} + eK \int_{t-\tau_1}^{t} \sum_{i=1}^{m} p_i(s) \prod_{s-\tau_i \le \xi_k < s} (1+b_k)^{-1}\, ds \le 0$$

for $t \in \bigcup_{n=N_1}^{\infty} [\alpha_n + \tau_m + 2\tau_1, \beta_n]$ or

$$\ln \frac{y(t-\tau_1)}{y(t)} \ge eK^2\,, \quad t \in \bigcup_{n=N_1}^{\infty} [\alpha_n + \tau_m + 2\tau_1, \beta_n].$$

Thus

$$\frac{y(t-\tau_1)}{y(t)} \ge e^{eK^2} \ge e^2K^2\,, \quad t \in \bigcup_{n=N_1}^{\infty} [\alpha_n + \tau_m + 2\tau_1, \beta_n].$$

Repeating the above procedure, we arrive at

$$\frac{y(t-\tau_1)}{y(t)} \ge (eK)^r\,, \quad t \in \bigcup_{n=N_1}^{\infty} [\alpha_n + \tau_m + r\tau_1, \beta_n].$$

If r is sufficiently large, then $(eK)^r > \frac{4}{K^2}$. Since $\lim_{n\to\infty} (\beta_n - \alpha_n) = \infty$ there exists an integer $N_2 \ge N_1$ such that $[\beta_n - \tau_1, \beta] \subseteq [\alpha_n + \tau_m + r\tau_1, \beta_n]$ for $n \ge N_2$. It follows from (14) that

$$\frac{4}{K^2} \ge \frac{y(\gamma_n - \tau_1)}{y(\gamma_n)} \ge (eK)^r > \frac{4}{K^2}\,, \quad n \ge N_2.$$

The contradiction obtained shows that (2) has no eventually positive solutions.

The proof of assertions 2 and 3 is carried out as in Theorem 1. □

Remark 2. The assertion of Theorem 3 is still valid if $p(t) > 0$ only on $\bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$. The proof with insignificant changes is the same.

Remark 3. Theorem 3 generalizes the result of Gopalsamy and Zhang in several directions. The trivial generalization (the procedure is very familiar for delay equations without impulses) is that we have several delays while it is one in [4], and we impose the integral condition (10) on $p(t)$ only on a sequence of disjoint intervals and not on the whole axis. The nontrivial extension is that there are no restrictions on the inter-jump distance (in [4] it is greater than the delay) and also that $b_k \in (-1, \infty)$ while in [4], $b_k \in (0, \infty)$. In the case of one delay and integral condition on the whole axis (10) takes the form

$$\liminf_{t\to\infty} \int\limits_{t-\tau}^{t} p(s) \prod_{s-\tau\le\xi_k<s} (1+b_k)^{-1}\, ds > \frac{1}{e}. \tag{15}$$

If the sequence $\{b_k\}$ is bounded from above and the number of jumps in $[t-\tau, t]$ is also bounded, then

$$l = \liminf_{t\to\infty} \prod_{t-\tau\le\xi_k<t} (1+b_k)^{-1} > 0.$$

It is easy to see that

$$\liminf_{t\to\infty} \int\limits_{t-\tau}^{t} p(s) \prod_{s-\tau\le\xi_k<s} (1+b_k)^{-1}\, ds \ge l \liminf_{t\to\infty} \int\limits_{t-\tau}^{t} p(s)\, ds.$$

Since

$$\liminf_{t\to\infty} \prod_{t-\tau\le\xi_k<t} (1+b_k)^{-1} = \frac{1}{\limsup\limits_{t\to\infty} \prod_{t-\tau\le\xi_k<t}(1+b_k)},$$

the integral condition (15) can be replaced by

$$\liminf_{t\to\infty} \int\limits_{t-\tau}^{t} p(s)\, ds > \frac{\limsup\limits_{t\to\infty} \prod_{t-\tau\le\xi_k<t}(1+b_k)}{e}. \tag{16}$$

For the sake of comparison the corresponding condition in [4] is

$$\liminf_{t\to\infty} \int\limits_{t-\tau}^{t} p(s)\, ds > \frac{1+\sup\limits_{k} b_k}{e} \tag{17}$$

with $b_k > 0$ and $\xi_{k+1} - \xi_k > \tau$. If $\xi_{k+1} - \xi_k > \tau$, then (16) takes the form

$$\liminf_{t\to\infty} \int\limits_{t-\tau}^{t} p(s)\, ds > \frac{1+\limsup\limits_{k} b_k}{e},$$

which is again slightly better than (17).

Analogously to Theorem 3 we can prove the following theorem:

Theorem 4. *Let conditions* A1—A4 *hold and there exist a sequence of disjoint intervals* $\{(\alpha_n, \beta_n)\}_{n=1}^{\infty}$ $(\alpha_1 > 0)$ *such that* $\beta_n - \alpha_n \geq 2\tau_m$, $\lim_{n\to\infty}(\beta_n - \alpha_n) = \infty$, *and there exist* $K > 0$ *and* N_1 *such that the inequalities*

$$\int_{t-\tau_i}^{t} p_i(s) \prod_{s-\tau_i \leq \xi_k < s} (1 + b_k)^{-1}\, ds \geq K > \frac{1}{e} \tag{18}$$

hold for any $n \geq N_1$ *and* $t \in (\alpha_n + \tau_m, \beta_n)$ *and for some* $i = i(n) \in \{1, 2, \ldots, m\}$. *Then:*

1. *Inequality* (2) *has no eventually positive solution.*
2. *Inequality* (3) *has no eventually negative solution.*
3. *Each solution of equation* (1) *is oscillatory.*

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