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POSITIVE SOLUTIONS TO SECOND ORDER SINGULAR DIFFERENTIAL EQUATIONS INVOLVING THE ONE-DIMENSIONAL M-LAPLACE OPERATOR

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Abstract. We consider a class of second order quasilinear differential equations with singular ninlinearities. Our main purpose is to investigate in detail the asymptotic behavior of their solutions defined on a positive half-line. The set of all possible positive solutions is classified into five types according to their asymptotic behavior near infinity, and sharp conditions are established for the existence of solutions belonging to each of the classified types.

1. Introduction

This paper is concerned with positive solutions to second order singular differential equations of the form

$$(p(t)|y'|^{m-2}y')' + q(t)y^{-n} = 0,\ t \geq a, \tag{A}$$

where the following conditions are assumed to hold:=

(a) $m > 1$ and $n > 0$ are constants;

(b) $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty),\ a \geq 0$;

(c) $p(t)$ satisfies

$$\int_a^\infty (p(t))^{-\frac{1}{m-1}} dt < \infty. \tag{1.1}$$

By a (positive) solution of (A) on $J \subset [a, \infty)$ is meant a function $y : J \to (0, \infty)$ which is continuously differentiable together with $p|y'|^{m-2}y'$ and satisfies the equation at every point of J. We will be mainly interested in the case where J is a half-line $[t_0, \infty)$, $t_0 \geq a$. A solution is said to be

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proper if it can be continued to ∞ and *singular* otherwise. It is clear that a singular solution necessarily vanishes at the right end point of its maximal interval of existence which is bounded.

The equation (A) always has singular solutions, and it is= shown that all of its solutions are singular unless $\left[(p(t))^{-1}\int_a^t q(s)ds\right]^{\frac{1}{m-1}}$ is integrable on $[a,\infty)$. Otherwise, (A) posseses proper solutions on which our attention in this paper will be focused. We classify the set of all possible positive proper solutions of (A) into five types according to their asymptotic behavior as $t \to \infty$ (Section 2), and then establish conditions (preferably sharp) for the existence of a proper solution of each of these five types (Section 3). Thus we are able to characterize the situations in which (A) has proper solutions of all but one of the classified types. Since the differential operator $(p(t)|y'|^{m-2}y')'$ in (A), which is a natural generalization of the Sturm–Liouville operator $(p(t)y')'$, can also be regarded as a one-dimensional polar form of the m-Laplace operator $\Delta_m u = \operatorname{div}(|Du|^{m-2}Du)$, Du denoting the gradient of u in $\mathbb{R}^N$, the results for (A) are expected to apply to singular partial differential equations of the type

$$\operatorname{div}\left(|Du|^{m-2}Du\right) + c(|x|)u^{-n} = 0 \tag{1.2}$$

to provide nontrivial information about their positive spherically symmetric solutions defined in exterior domains. Analysis based on this expectation is made in Section 4, in which other examples illustrating our main results for (A) are also presented.

We notice that singular equations of the form (A) with $p(t)$ satisfying the condition

$$\int_a^\infty (p(t))^{-\frac{1}{m-1}}\,dt = \infty \tag{1.3}$$

have been considered by Kurokiba, Kusano and Wang [1]. They have made a fairly complete study of the structure of positive solutions, both singular and proper, of (A), by examining how the trajectories of the solutions emanating from a fixed point $y(t_0) = y_0 > 0$ alter depending on the change of the initial gradient $y'(t_0) = y_1 \in \mathbb{R}$. The transition from (1.2) to (1.1) seems to make it difficult to directly apply their approach to the present situation, and this observation motivated us to look at the problem from a slightly different angle.

Differential equations with singular nonlinearities have received considerable attention in recent years; for the literature more or less related to the present work we refer to the papers [1–11].

2. Classification of Proper Solutions

A) *Singular solutions.* We first observe that there always exist singular solutions of (A) emanating from any given point on the positive part of the line $t = t_0,\ t_0 \geq a$. Let $y(t)$ be a positive solution of (A) determined by the initial conditions

$$y(t_0) = \eta > 0,\ (p(t_0))^{\frac{1}{m-1}} y'(t_0) = -\zeta < 0.$$

From=20(A), $p(t)|y'(t)|^{m-2}y'(t)$ is decreasing, and so $y'(t) < 0$ as long as $y(t)$ exists. Integrating (A) rewritten as

$$(-p(t)(-y'(t))^{m-1})' + q(t)(y(t))^{-n} = 0$$

twice from t_0 to t yields

$$y(t) = \eta - \int_{t_0}^{t} \left[(p(s))^{-1} \left(\zeta^{m-1} + \int_{t_0}^{s} q(r)(y(r))^{-n} dr \right) \right]^{\frac{1}{m-1}} ds. \quad (2.1)$$

Let $\eta > 0$ and $t_1 > t_0$ be fixed and choose $\zeta > 0$ so that

$$\eta < \zeta \int_{t_0}^{t_1} (p(s))^{-\frac{1}{m-1}} ds.$$

Then it follows from (2.1) that $y(t)$ must tend to zero as t approaches some point $t_2 < t_1$ beyond which $y(t)$ cannot be continued to the right as a solution of (A). Thus $y(t)$ is a singular solution.

B) *Proper solutions.* We intend to classify the set of proper positive solutions of (A) according to their asymptotic behavior as $t \to \infty$. Here and in what follows extensive use will be made of the function $\pi(t)$ defined by

$$\pi(t) = \int_{t}^{\infty} (p(s))^{-\frac{1}{m-1}} ds,\ t \geq a. \quad (2.2)$$

Because of (1.1), $\pi(t) \to 0$ as $t \to \infty$. Let $y(t)$ be such a solution on $[t_0, \infty),\ t_0 \geq a$. From (A) we see that $p(t)|y'(t)|^{m-2}y'(t)$ is decreasing for $t \geq t_0$, so that either

$$p(t)|y'(t)|^{m-2}y'(t) > 0 \text{ for } t \geq t_0 \quad (2.3)$$

or there is $t_1 \geq t_0$ such that

$$p(t)|y'(t)|^{m-2}y'(t) < 0 \text{ for } t \geq t_1. \quad (2.4)$$

Suppose that (2.3) holds. Then, $y'(t) > 0,\ t \geq t_0$, and

$$p(t)(y'(t))^{m-1} \leq p(t_0)(y'(t_0))^{m-1},\ t \geq t_0,$$

or equivalently

$$y'(t) \leq p(t_0)^{\frac{1}{m-1}} y'(t_0)(p(t))^{-\frac{1}{m-1}},\ t \geq t_0.$$

Integrating the above shows that

$$y(t) \leq y(t_0) + (p(t_0))^{\frac{1}{m-1}} y'(t_0)\pi(t_0),\ t \geq t_0. \tag{2.5}$$

Thus, $y(t)$ is bounded above on $[t_0, \infty)$ and increases to a finite positive limit as $t \to \infty$. Clearly, $p(t)|y'(t)|^{m-2}y'(t) = p(t)(y'(t))^{m-1}$ tends to a nonnegative constant as $t \to \infty$.

Suppose that (2.4) holds. Then, $y'(t) < 0,\ t \geq t_1$, and

$$-p(t)(-y'(t))^{m-1} \leq -p(t_1)(-y'(t_1))^{m-1},\ t \geq t_1,$$

or

$$-y'(t) \geq (p(t_1))^{\frac{1}{m-1}}(-y'(t_1))(p(t))^{-\frac{1}{m-1}},\ t \geq t_1.$$

An integrating of the above gives

$$y(t) \geq (p(t_1))^{\frac{1}{m-1}}(-y'(t_1))\pi(t),\ t \geq t_1, \tag{2.6}$$

which shows that $y(t)$ may decrease to zero as $t \to \infty$ but cannot decay faster than any constant mulitiple of $\pi(t)$. In this case $p(t)|y'(t)|^{m-2}y'(t) = -p(t)(-y'(t))^{m-1}$ goes to a finite negative limit or $-\infty$ as $t \to \infty$.

The above observations suggest us to distinguish the following five possiblities in the asymptotic behavior of positive proper solutions of the equation (A):

(I) $\lim_{t\to\infty} y(t) = \text{const} > 0,\qquad \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = \text{const} \geq 0;$

(II) $\lim_{t\to\infty} y(t) = \text{const} > 0,\qquad \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = \text{const} < 0;$

(III) $\lim_{t\to\infty} y(t) = \text{const} > 0,\qquad \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = -\infty;$

(IV) $\lim_{t\to\infty} y(t) = 0,\qquad \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = -\infty;$

(V) $\lim_{t\to\infty} y(t) = 0,\qquad \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = \text{const} < 0.$

Positive solutions of types (IV) and (V) are often termed *decaying solutions* of (A). A solution $y(t)$ of type (V) decays as $t \to \infty$ exactly like a constant multiple of $\pi(t)$, since (V) is equivalent to $\lim_{t\to\infty} y(t)/\pi(t) = \text{const} > 0$. The speed of decay of a soluiton $y(t)$ of type (IV) is slower than $\pi(t)$ as $t \to \infty$, since (IV) is equivalent to $\lim_{t\to\infty} y(t)/\pi(t) = \infty$.

C) *Integral equations.* Let us now derive integral equations for proper positive solutions of (A) corresponding to the classification given above. The integral equaitons will prove to be crucial in the study to be made in the next section regarding the existence (and nonexistence) of proper solutions of (A) of the five classified types.

Let $y(t)$ be a solution of type (I) on $[t_0, \infty)$. Rewriting the equation (A) as

$$(p(t)(y'(t))^{m-1})' + q(t)(y(t))^{-n} = 0, \ t \geq t_0,$$

and integrating it twice from t to ∞, we have

$$y(t) = c - \int_t^\infty \left[(p(s))^{-1}\left(w^{m-1} + \int_s^\infty q(r)(y(r))^{-n} dr\right)\right]^{\frac{1}{m-1}} ds, \ t \geq t_0, \quad (2.7)$$

where

$$c = \lim_{t\to\infty} y(t) > 0 \text{ and } \omega = \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) \geq 0.$$

An immediate consequence of (2.7) is that $q(t)(y(t))^{-n}$ is integrable on $[t_0, \infty)$.

Let $y(t)$ be a solution of type (II) on $[t_1, \infty)$. Noting that

$$(-p(t)(-y'(t))^{m-1})' + q(t)(y(t))^{-n} = 0, \ t \geq t_1, \quad (2.8)$$

and integrating (2.8) twice from t to ∞, we obtain

$$y(t) = c + \int_t^\infty \left[(p(s))^{-1}\left(\omega^{m-1} - \int_s^\infty q(r)(y(r))^{-n} dr\right)\right]^{\frac{1}{m-1}} ds, \ t \geq t_1, \quad (2.9)$$

where

$$c = \lim_{t\to\infty} y(t) > 0 \text{ and } \omega = -\lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) > 0.$$

Likewise, $q(t)(y(t))^{-n}$ is integrable on $[t_1, \infty)$ for a solution $y(t)$ of type (II).

To obtain an integral equation for a solution $y(t)$ of type (III) or (IV) we first integrate (2.8) from t_1 to t:

$$p(t)(-y'(t))^{m-1} = p(t_1)(-y'(t_1))^{m-1} + \int_{t_1}^t q(s)(y(s))^{-n} ds, \ t \geq t_1, \quad (2.10)$$

or equivalently

$$-y'(t) = \left[(p(t))^{-1}\left(p(t_1)(-y'(t_1))^{m-1} + \right.\right.$$
$$\left.\left. + \int_{t_1}^t q(s)(y(s))^{-n} ds\right)\right]^{\frac{1}{m-1}}, \ t \geq t_1. \quad (2.11)$$

Since $p(t)(-y'(t))^{m-1} \to \infty$ as $t \to \infty$, (2.10) implies that $q(t)(y(t))^{-n}$ is not integrable on $[t_1, \infty)$. An integration of (2.11) over $[t, \infty)$ then yields

$$y(t) = c + \int_t^\infty \left[(p(s))^{-1} \left(p(t_1)(-y'(t_1))^{m-1} + \int_{t_1}^s q(r)(y(r))^{-n} dr \right) \right]^{\frac{1}{m-1}} ds, \quad t \ge t_1, \tag{2.12}$$

where $c = \lim_{t\to\infty} y(t) \ge 0$; $c > 0$ if $y(t)$ is of type (III) and $c = 0$ if $y(t)$ is of type (IV). It should be noticed in this case that the function $\left[(p(t))^{-1} \int_{t_1}^t q(s)(y(s))^{-n} ds \right]^{\frac{1}{m-1}}$ is integrable on $[t_1, \infty)$.

Finally consider a solution $y(t)$ of type (V). As a result of repeated integration of (2.8) over $[t, \infty)$ it follows that $y(t)$ satisfies the integral equation

$$y(t) = \int_t^\infty \left[(p(s))^{-1} \left(\omega^{m-1} - \int_s^\infty q(r)(y(r))^{-n} dr \right) \right]^{\frac{1}{m-1}} ds, \ t \ge t_1, \tag{2.13}$$

where $\omega = -\lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) > 0$. The integrability of $q(t)(y(t))^{-n}$ on $[t_1, \infty)$ follows readily from (2.13).

3. Existence of Proper Solutions

A) *Increasing proper solutions.* We begin by noting that for the existence of a positive proper solution of type (I) of (A) it is necessary that $q(t)$ is integrable on $[a, \infty)$, that is,

$$\int_a^\infty q(t) dt < \infty. \tag{3.1}$$

In fact, if $y(t)$ is of type (I) on $[t_0, \infty)$, then since $y(t) \le y(\infty) = \lim_{t\to\infty} y(t) < \infty$, $t \ge t_0$, we have, by the integrability of $q(t)(y(t))^{-n}$,

$$(y(\infty))^{-n} \int_{t_0}^\infty q(t) dt \le \int_{t_0}^\infty q(t)(y(t))^{-n} dt < \infty,$$

which implies (3.1). It turns out that (3.1) is also a sufficient condition for (A) to have a type (I)-solution.

Theorem 3.1. *There exists a positive proper solution* $y(t)$ *of* (A) *such that*

$$\lim_{t\to\infty} y(t) = \text{const} > 0, \ \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = \text{const} \geq 0 \qquad (3.2)$$

if and only if (3.1) *holds.*

Proof. Suppose that (3.1) holds. Let $t_0 \geq a$ be fixed. There exist constants $c > 0$ and $\omega \geq 0$ such that

$$\int_{t_0}^{\infty} \left[(p(t))^{-1} \left(\omega^{m-1} + c^{-n} \int_{t_0}^{\infty} q(s) ds \right) \right]^{\frac{1}{m-1}} dt \leq c. \qquad (3.3)$$

It suffices, for example, to take an arbitrary $\omega \geq 0$ and choose $c > 0$ large enough. Define the set $Y \subset C[t_0, \infty)$ and the mapping $\mathcal{F} : Y \to C[t_0, \infty)$ by

$$Y = \{ y \in C[t_0, \infty) : \ c \leq y(t) \leq 2c, \ t \geq t_0 \} \qquad (3.4)$$

and

$$(\mathcal{F}y)(t) = 2c - \int_t^{\infty} \Bigg[(p(s))^{-1} \Bigg(\omega^{m-1} + + \int_s^{\infty} q(r)(y(r))^{-n} dr \Bigg) \Bigg]^{\frac{1}{m-1}} ds, \ t \geq t_0. \qquad (3.5)$$

We will demonstrate the existence of a fixed point of $\mathcal{F}$ in Y, which clearly provides a solution of type (I) of (A) (cf. (2.7)). The Schauder-Tychonoff fixed point theorem will be used for this purpose. It should be shown that (i) $\mathcal{F}$ maps Y into itself, (ii) $\mathcal{F}$ is continuous and (iii) $\mathcal{F}(Y)$ is relatively compact in $C[t_0, \infty)$. That $\mathcal{F}(Y) \subset Y$ is a trivial consequence of (3.3). To prove that $\mathcal{F}$ is continuous, let $\{y_\nu\}$ be a sequence in Y converging to y in the topology of $C[t_0, \infty)$, which means that $\{y_\nu(t)\}$ converges to $y(t)$ uniformly on compact subintervals of $[t_0, \infty)$. Then, using the Lebesgue convergence theorem, we can show that $\{(\mathcal{F}y_\nu)(t)\}$ converges to $(\mathcal{F}y)(t)$ uniformly on compact subintervals of $[t_0, \infty)$. This implies that $\{\mathcal{F}y_\nu\}$ converges to $\mathcal{F}y$ in $C[t_0, \infty)$, establishing the continuity of $\mathcal{F}$. The relative compactness of $\mathcal{F}(Y)$ follows from the uniform boundedness (cf. (3.4)) and the local equicontinuity of this set, the latter being a consequence of the inequality

$$0 \leq (\mathcal{F}y)'(t) \leq \left[(p(t))^{-1} \left(\omega^{m-1} + c^{-n} \int_{t_0}^{\infty} q(s) ds \right) \right]^{\frac{1}{m-1}}, \ t \geq t_0.$$

Thus $\mathcal{F}$ has a fixed point $y \in Y$, which gives rise to a positive proper solution $y(t)$ of (A) on $[t_0, \infty)$ with the property that $\lim_{t\to\infty} y(t) = 2c$ and $\lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = \omega$. $\square$

B) *Decreasing proper solutions.* The remaining solutions of (A) belonging to the types (II)-(V) are all (eventually) decreasing solutions. Let $y(t)$ be a solution of type (II) which is decreasing on $[t_1, \infty)$. Then, $y(t) \le y(t_1)$ for $t \ge t_1$, and combining this with the integrability of $q(t)(y(t))^{-n}$ on $[t_1, \infty)$ we have

$$(y(t_1))^{-n} \int_{t_1}^{\infty} q(s)ds \le \int_{t_1}^{\infty} q(s)(y(s))^{-n} ds < \infty,$$

which implies (3.1). Actually (3.1) is a condition characterizing the existence of a type (II)-solution of (A), as the following theorem asserts.

Theorem 3.2. *There exists a positive proper solution* $y(t)$ *of* (A) *such that*

$$\lim_{t\to\infty} y(t) = \text{const} > 0, \ \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = \text{const} < 0 \tag{3.6}$$

if and only if (3.1) *holds.*

Proof. We need only to prove the "if "part of the theorem. Let $t_1 \ge a$ be fixed. For an arbitrary fixed constant $\omega > 0$ take a constant $c > 0$ such that

$$\omega \int_{a}^{\infty} (p(t))^{-\frac{1}{m-1}} dt \le c \text{ and } \omega^{1-m} \int_{t_1}^{\infty} q(t)dt \le (2c)^n, \tag{3.7}$$

which is possible by (1.1) and (3.1). Consider the set $Y \subset C[t_1, \infty)$ and the integral operator $\mathcal{F} : Y \to C[t_1, \infty)$ defined by

$$Y = \{y \in C[t_1, \infty) : \ c \le y(t) \le 2c, \ t \ge t_1\}$$

and

$$(\mathcal{F}y)(t) = c + \int_{t}^{\infty} \left[(p(s))^{-1} \left(\omega^{m-1} - \int_{s}^{\infty} q(r)(y(r))^{-n} dr \right) \right]^{\frac{1}{m-1}} ds, \ t \ge t_1.$$

The condition (3.7) ensures that $\mathcal{F}$ is well defined on Y and maps Y into itself. The continuity of $\mathcal{F}$ and the relative compactness of $\mathcal{F}(Y)$ can be proved exactly as in the proof of Theorem 3.1, and so there exists an element $y \in Y$ such that $y = \mathcal{F}y$ by the Schauder-Tychonoff fixed point theorem. Since the fixed element $y(t)$ obtained satisfies the integral equation (2.9), it provides a positive solution of (A) defined on $[t_1, \infty)$ and satisfying (3.6): $\lim_{t\to\infty} y(t) = c$ and $\lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = -\omega$. This finishes the proof. $\square$

We now turn to positive solutions of types (III) and (IV). We observed in Section 2 that $q(t)(y(t))^{-n}$ is nonintegrable but $\left[(p(t))^{-1}\int_{t_1}^{t} q(s)(y(s))^{-n}ds\right]^{\frac{1}{m-1}}$ is integrable on $[t_1,\infty)$ for any solution of (A) of type (III) or type (IV). If $y(t)$ is of type (III), then, since $y(t_1) \geq y(t) \geq y(\infty) > 0$ for $t \geq t_1$, we see that

$$(y(\infty))^{-n}\int_{t_1}^{\infty} q(t)dt \geq \int_{t_1}^{\infty} q(t)(y(t))^{-n}dt = \infty \tag{3.8}$$

and

$$\begin{aligned}(y(t_1))^{-\frac{n}{m-1}}&\int_{t_1}^{\infty}\left[(p(t))^{-1}\int_{t_1}^{t} q(s)ds\right]^{\frac{1}{m-1}}dt\\ &\leq \int_{t_1}^{\infty}\left[(p(t))^{-1}\int_{t_1}^{t} q(s)(y(s))^{-n}ds\right]^{\frac{1}{m-1}}dt < \infty,\end{aligned} \tag{3.9}$$

whereas if $y(t)$ is of type (IV), then since $y(t_1) \geq y(t) \geq k\pi(t)$ for $t \geq t_1$, we find that (3.9) remains to hold but, instead of (3.8),

$$k^{-n}\int_{t_1}^{\infty} q(t)(\pi(t))^{-n}dt \geq \int_{t_1}^{\infty} q(t)(y(t))^{-n}dt = \infty.$$

Summarizing the above, in order for (A) to have a positive proper solution of type (III) it is necessary that

$$\int_{a}^{\infty} q(t)dt = \infty \text{ and } \int_{a}^{\infty}\left[(p(t))^{-1}\int_{a}^{t} q(s)ds\right]^{\frac{1}{m-1}}dt < \infty, \tag{3.10}$$

and in order for (A) to have a proper solution of type (IV) it is necessary that

$$\int_{a}^{\infty} q(t)(\pi(t))^{-n}dt = \infty \text{ and } \int_{a}^{\infty}\left[(p(t))^{-1}\int_{a}^{t} q(s)ds\right]^{\frac{1}{m-1}}dt < \infty. \tag{3.11}$$

We will show that (3.10) is also sufficient for the existence of a type (III)-solution of (A).

Theorem 3.3. *There exists a positive proper solution $y(t)$ of* (A) *such that*

$$\lim_{t\to\infty} y(t) = \text{const} > 0, \ \lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = -\infty \tag{3.12}$$

if and only if (3.10) *is satisfied.*

Proof. Suppose that (3.10) holds. For fixed $t_1 \geq a$ choose $c > 0$ so large that

$$\int_{t_1}^{\infty} \left[(p(t))^{-1} \int_{t_1}^{t} q(s)ds \right]^{\frac{1}{m-1}} dt \leq c^{\frac{m+n-1}{m-1}}.$$

It is easy to verify on the basis of the Schauder-Tychonoff theorem that the mapping $\mathcal{F}$ defined by

$$(\mathcal{F}y)(t) = c + \int_{t}^{\infty} \left[(p(s))^{-1} \int_{t_1}^{s} q(r)(y(r))^{-n} dr \right]^{\frac{1}{m-1}} ds, \ t \geq t_1,$$

has a fixed point y in the set $Y = \{y \in C[t_1, \infty) : \ c \leq y(t) \leq 2c, \ t \geq t_1\}$. The fixed element $y = y(t)$ satisfies the integral equation (2.12) (with $y'(t_1) = 0$), and hence it gives a positive solution of (A) on $[t_1, \infty)$. From the integral equation it follows that $\lim_{t\to\infty} y(t) = c$ and that

$$(p(t))^{\frac{1}{m-1}} y'(t) = -\left[\int_{t_1}^{t} q(s)(y(s))^{-n} ds \right]^{\frac{1}{m-1}} \leq$$

$$\leq -(2c)^{-\frac{n}{m-1}} \left[\int_{t_1}^{t} q(s)ds \right]^{\frac{1}{m-1}}, \ t \geq t_1,$$

which implies that $\lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = -\infty$ (see the first condition in (3.10)). Thus we have been able to construct a positive proper solution with the required asymptotic property (3.12), completing the proof. □

So far we have been unable to prove or disprove that (3.11) is a necessary and sufficient condition for (A) to possess a proper solution of type (IV). Only a sufficient condition will be given in the following theorem.

Theorem 3.4. *There exists a positive proper solution $y(t)$ of* (A) *such that*

$$\lim_{t\to\infty} y(t) = 0 \ \textit{and} \ \lim_{t\to\infty} \frac{y(t)}{\pi(t)} = \infty \tag{3.13}$$

if

$$\int_a^\infty q(t)dt = \infty \ \text{and} \ \int_a^\infty \left[(p(t))^{-1} \int_a^t q(s)(\pi(s))^{-n} ds\right]^{\frac{1}{m-1}} dt < \infty. \quad (3.14)$$

Proof. Let $t_1 \geq a$ be fixed and $k > 0$ be any constant. Put

$$K = \int_{t_1}^\infty \left[(p(t))^{-1}\left(k^{m-1} + k^{-n}\int_{t_1}^t q(s)(\pi(s))^{-n} ds\right)\right]^{\frac{1}{m-1}} dt$$

and define

$$Y = \{y \in C[t_1, \infty) : \ k\pi(t) \leq y(t) \leq K, \ t \geq t_1\}.$$

Using the Schauder–Tychonoff theorem one can show that the integral operator $\mathcal{F}$ given by

$$(\mathcal{F}y)(t) = \int_t^\infty \left[(p(s))^{-1}\left(k^{m-1} + \int_{t_1}^s q(r)(y(r))^{-n} dr\right)\right]^{\frac{1}{m-1}} ds, \ t \geq t_1,$$

has at least one fixed point $y = y(t)$ in Y, which satisfies the integral equation (2.12) with $c = 0$ and $(p(t_1))^{\frac{1}{m-1}} y'(t_1) = -k$. It is trivial to see that $\lim_{t\to\infty} y(t) = 0$. That $\lim_{t\to\infty} y(t)/\pi(t) = \infty$, which is equivalent to $\lim_{t\to\infty} (p(t))^{\frac{1}{m-1}} y'(t) = -\infty$, is concluded by using the first condition in (3.14) in the inequality

$$(p(t))^{\frac{1}{m-1}} y'(t) = -\left[k^{m-1} + \int_{t_1}^t q(s)(y(s))^{-n} ds\right]^{\frac{1}{m-1}} \leq$$

$$\leq -\left[k^{m-1} + K^{-n}\int_{t_1}^t q(s) ds\right]^{\frac{1}{m-1}}, \ t \geq t_1.$$

Our final task is to study the problem of existence of a type (V)-solution of (A). If $y(t)$ is such a solution on $[t_1, \infty)$, then $\lim_{t\to\infty} y(t)/\pi(t) = \text{const} > 0$ and there exists a constant $k > 0$ such that $y(t) \leq k\pi(t)$ for $t \geq t_1$. The last inequality combined with the integrability of $q(t)(y(t))^{-n}$ (cf. (2.13)) shows that

$$k^{-n}\int_{t_1}^\infty q(t)(\pi(t))^{-n} dt \leq \int_{t_1}^\infty q(t)(y(t))^{-n} dt < \infty,$$

where $\pi(t)$ is defined in (2.2). Thus a necessary condition for (A) to have a solution of type (V) is

$$\int_a^\infty q(t)(\pi(t))^{-n}dt < \infty. \qquad \square \tag{3.15}$$

Theorem 3.5. *There exists a positive proper solution $y(t)$ of* (A) *such that*

$$\lim_{t\to\infty} \frac{y(t)}{\pi(t)} = \text{const} > 0 \tag{3.16}$$

if and only if (3.15) *holds.*

Proof. It suffices to demonstrate the sufficiency of (3.15). Let $t_1 \geq a$ be fixed and let $\omega > 0$ be a constant such that

$$\int_{t_1}^\infty q(t)(\pi(t))^{-n}dt \leq (2^{m-1}-1)\omega^{m+n-1}.$$

It is verified in a routine manner that the integral operator $\mathcal{F}$ defined by

$$(\mathcal{F}y)(t) = \int_t^\infty \left[(p(s))^{-1}\left((2\omega)^{m-1} - \int_s^\infty q(r)(y(r))^{-n}dr\right)\right]^{\frac{1}{m-1}} ds,\ t \geq t_1,$$

is well defined on the set

$$Y = \{y \in C[t_1,\infty) :\ \omega\pi(t) \leq y(t) \leq 2\omega\pi(t),\ t \geq t_1\}$$

and maps Y into a relatively compact subset of Y. Therefore there exists a fixed element $y = y(t)$ of $\mathcal{F}$ in Y which solves the integral equation (2.13) with ω replace by 2ω and hence gives a positive decaying solution satisfying (3.16). $\square$

A close look at the hypotheses and conclusions of the theorems proven above enable us to find a criterion for the nonexistence of positive proper solutions for the equation (A).

Theorem 3.6. *All positive solutions of* (A) *are singular if and only if*

$$\int_a^\infty \left[(p(t))^{-1}\int_a^t q(s)ds\right]^{\frac{1}{m-1}} dt = \infty. \tag{3.17}$$

4. Examples and Application to PDE

We will give examples illustrating the theorems obtained in Section 3 and demonstrate the applicability of the results for (A) to the qualitative study of certain singular partial differential equations involving the m-Laplace operator.

A) Let us consider the equation

$$(p(t)|y'|^{m-2}y')' + \lambda(p(t))^{-\frac{1}{m-1}}(\pi(t))^l y^{-n} = 0,\ t \geq a, \tag{4.1}$$

where $l,\ m > 1,\ n > 0$ and $\lambda > 0$ are constants, $p(t)$ is a positive continuous function on $[a, \infty)$ satisfying (1.1) and $\pi(t)$ is defined by (2.2). This is a special case of (A) with $q(t) = \lambda(p(t))^{-\frac{1}{m-1}}(\pi(t))^l$, for which the main integral conditions required in Theorems 3.1$-$3.6 read as follows:

$$\int_a^\infty q(t)dt < \infty \Longleftrightarrow l > -1; \tag{4.2}$$

$$\int_a^\infty \left[(p(t))^{-1}\int_a^t q(s)ds\right]^{\frac{1}{m-1}} dt < \infty \Longleftrightarrow l > -m; \tag{4.3}$$

$$\int_a^\infty q(t)(\pi(t))^{-n}dt < \infty \Longleftrightarrow l > n-1; \tag{4.4}$$

$$\int_a^\infty \left[(p(t))^{-1}\int_a^t q(s)(\pi(s))^{-n}ds\right]^{\frac{1}{m-1}} dt < \infty \Longleftrightarrow n-1 \geq l > n-m. \tag{4.5}$$

Taking this fact into account, we have the following statments regarding the structure of the solution set of (4.1).

(i) If $-\infty < l \leq -m$, then all positive soluion of (4.1) are singular, and vice versa.

(ii) If $-m < l \leq -1$, then (4.1) possesses a positive proper solution of type (III), and vice versa.

(iii) If $-1 < l < \infty$, then (4.1) possesses positive proper solutions1 of types (I) and (II), and vice versa.

(iv) If $n-1 < l < \infty$, then (4.1) possesses a positive proper solution of type (V), and vice versa.

(v) If $n-m < l \leq -1$, then (4.1) possesses a positive proper solution of type (IV).

Remark 4.1. We now compare a necessary condition (3.11) with a sufficient condition (3.14) for the existence of a type (IV)-solution of (A). These conditions applied to (4.1) reduce, respectively, to

$$-m < l \leq n-1 \text{ and } n-m \leq l < -1$$

(cf. the last statement (v)), illustrating the gap between (3.11) and (3.14). An elementary calculation shows that the equation (4.1) has a decaying solution $(\pi(t))^{\frac{m+l}{m+n-1}}$ if $-m < l < n-1$ and $\lambda = (m-1)(n-l-1)(m+l)^{m-1}/(m+n-1)^m$. Since $0 < (m+l)/(m+n-1) < 1$, this solution decays slower than $\pi(t)$ as $t \to \infty$, so that it is a solution of type (IV). This example might suggest the possibility of considerably improving the criterion, given in Theorem 3.4, for the existence of a type (IV)-solution for (A).

A more concrete example of equations of the type (4.1) is

$$(e^{(m-1)t}|y'|^{m-2}y')' + \lambda e^{\mu t} y^{-n} = 0, \ t \geq 0, \tag{4.6}$$

where $m > 1$, $n > 0$ and μ are constants. Restricting our attention to decaying solutions of (4.6), we conclude that (4.6) has a proper solution $y(t)$ such that $\lim_{t\to\infty} e^t y(t) = \text{const} > 0$ if and only if $-\infty < \mu < -n$, and that (4.6) has a solution $z(t)$ such that

$$\lim_{t\to\infty} z(t) = 0 \text{ and } \lim_{t\to\infty} e^t z(t) = -\infty \tag{4.7}$$

if $0 \leq \mu < m-n-1$. For a larger μ -interval, that is, for any $\mu \in (-n, m-1)$, there exists a positive value of λ for which (4.6) possesses a decaying solution $z(t) = e^{-\frac{m-\mu-1}{m+n-1}t}$ satisfying (4.7).

B) It is a simple exercise to verify that the results for (A) can be directly applied to the qualitative study of spherically symmetric positive solutions to singular elliptic partial differential equations of the form

$$\operatorname{div}(|Du|^{m-2}Du) + c(|x|)u^{-n} = 0, \ x \in E_a, \tag{4.8}$$

where $m > 1$ and $n > 0$ are constants, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, $N \geq 2$, $Du = (\partial u/\partial x_1, \cdots, \partial^n/\partial x_N)$, $|\cdot|$ denotes the Euclidean length of an N-vector, $E_a = \{x \in \mathbb{R}^N : |x| > a\}$, $a > 0$ and $c(t)$ is a positive continuous function on $[a, \infty)$. As a matter of fact, a spherically symmetric function $u(x) = y(|x|)$ is a solution of (4.8) if and only if $y(t)$ satisfies the ordinary differential equation

$$(t^{N-1}|y'|^{m-2}y')' + t^{N-1}c(t)y^{-n} = 0, \ t \geq a, \tag{4.9}$$

which is clearly a special case of (A). The condition (1.1) written for (4.9) is equivalent to requiring that $N > m$, in which case the function $\pi(t)$ given by (2.2) becomes

$$\pi(t) = \frac{m-1}{N-m} t^{-\frac{N-m}{m-1}}. \tag{4.10}$$

Assuming that $N > m$, we consider the following special case of (4.8)

$$\operatorname{div}(|Du|^{m-2}Du) + |x|^k u^{-n} = 0,\ x \in E_a, \tag{4.11}$$

where k is a constant. The equation (4.9) then reduces to

$$(t^{N-1}|y'|^{m-2}y')' + t^{N+k-1}y^{-n} = 0,\ t \geq a, \tag{4.12}$$

which, in view of (4.10), is seen to be a special case of (4.1) with $p(t) = t^{N-1}$,

$$l = -\frac{m(N-1)+k(m-1)}{N-m} \text{ and } \lambda = \Big(\frac{N-m}{m-1}\Big)^l.$$

It is easily checked that the conditions (4.2), (4.3), (4.4) and (4.5) applied to (4.12) read $k < -N$, $k < -m$, $k < -N - \frac{n(N-m)}{m-1}$ and $-N - \frac{m(N-m)}{m-1} < k < -m - \frac{n(N-m)}{m-1}$, respectively. Combining this fact with the propositions (i)$-$(v) stated in the preceding subsection, we can say something about the existence and asymptotic behavior of positive symmetric solutions to the equation (4.11).

(i) If $k \geq -m$, then (4.11) has no positive spherically symmetric solutions defined throughout E_a.

(ii) If $-N \leq k < -m$, then (4.11) has a positive symmetric solution $u(x)$ on E_a such that

$$\lim_{|x|\to\infty} u(x) = \text{const} > 0,\ \lim_{|x|\to\infty} |x|^{\frac{N-1}{m-1}}|Du| = \infty.$$

(iii) If $k < -N$, them (4.11) has positive symmetric solutions $u(x)$ and $v(x)$ in E_a such that

$$\lim_{|x|\to\infty} u(x) = \text{const} > 0, \qquad \lim_{|x|\to\infty} |x|^{\frac{N-1}{m-1}}|Du| \geq 0$$

and

$$\lim_{|x|\to\infty} v(x) = \text{const} > 0, \qquad \lim_{|x|\to\infty} |x|^{\frac{N-1}{m-1}}|Du| > 0.$$

(iv) If $k < -N - \dfrac{n(N-m)}{m-1}$, then (4.11) has a positive symmetric decaying solution $u(x)$ with the property that

$$\lim_{|x|\to\infty} |x|^{\frac{N-m}{m-1}} u(x) = \text{const} > 0.$$

(v) If $-N \leq k < -m - \dfrac{n(N-m)}{m-1}$, then (4.11) has a positive symmetric decaying solution $v(x)$ with the property that

$$\lim_{|x|\to\infty} |x|^{\frac{N-m}{m-1}} v(x) = \infty.$$

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