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STRONG INNOVATION AND ITS APPLICATIONS TO INFORMATION DIFFUSION MODELLING IN FINANCE

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Abstract. We consider the mean-variance hedging and utility maximization problems under partial information for diffusion models of the stock price process. The special feature of this paper is that we construct a strong innovation process for the stock price process which allows us to reduce the partial information case to the full information one.

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1. Introduction

We consider two important issues of financial modelling: the mean-variance hedging and utility maximization problems under partial information assuming that only the prices of risky assets (stocks) are observable.

We focus on the diffusion models. The mean-variance hedging problem under partial information was studied in several recent papers: Di Masi, Platen and Runggaldier [4] studied the case, where stock prices are observed at discrete time moments, Schweizer [25], Lasry and Lions [16], and Frey and Runggaldier [6] extended the problem for more general restricted information. In these papers the stock price process is assumed to be martingale under the objective probability, and hence does not include the case, where drift coefficient of the stock price process is not observable. Pham [21] extends the general approach developed by Gourieroux, Laurent and Pham [9] and Rheinländer and Schweizer [24] in full information context to partial information. In particular, Pham's approach includes the case of a nonobservable drift coefficient in the diffusion model. The utility maximization problem under partial information in the complete market framework studied in Detemple [3], Genotte [7], Dothan and Feldman [5] (linear Gaussian models, dynamic programming method), Lakner [14], [15] (diffusion model, martingale approach), Karatzas and Xue [12], and Karatzas and Zhao [13] (the Bayesian case).

Pham and Quenez [22] consider an incomplete market, in particular a stochastic volatility model and combine the stochastic filtering technique and the martingale duality approach to characterize the value function and the optimal portfolio.

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Our approach is based on the following observation: two main tools are used one after the other to solve the mean-variance hedging or portfolio optimization problems in the partial information framework filtering (essentially, the notion of innovation process) and control. But there exist two notions of innovation process: the weak and strong innovations (see Theorem 2.1 below). Weak innovation exists under mild conditions (see, e.g., Meyer [20]). But when this process is used in filtering the solution of the remaining control problems requires considerable effort: e.g., the papers of Pham [21] and Pham and Quenez [22] are essentially devoted to the study of these remaining problems.

The construction of strong innovation is much more difficult. But in situations in which such a process exists and is used in filtering, the remaining control problems for partially observable models become almost trivial, because they are reduced to the corresponding problems whith full information and, hence, with already known solutions. Namely, the mean-variance hedging problem for partial information is reduced to the methodology of Gourieroux, Laurent and Pham [9] and Rheinländer and Schweizer [24] while the utility maximization problem to the methodology of Karatzas, Lehoczky, Shreve and Xu [11] under full information.

Originally, the innovation process was introduced by Shiryaev [26] and Clark [2] and subsequently was studied by many authors in different schemes, see, e.g., the review in [27].

In Section 2 we introduce a strong innovation process for the so called partially observable diffusion scheme and give the conditions under which this process exists.

In Section 3 we consider the mean-variance hadging and the utility maximization problems in different schemes for the stock price process and show how models with partial information can be reduced to models with full information.

In the Appendix we give the proof of Theorem 2.1 and its corollaries.

2. Construction of a strong innovation process for a component of a partially observable diffusion type process

Fix the real number $T > 0$, and integers $d_1 \geq 1$, $d_2 \geq 1$, $d = d_1 + d_2$.

Denote by $(C^l_{[0,T]}, \mathcal{B}^l_{[0,T]})$, $l = d_1$ or d_2, the measurable spaces of continuous l-dimensional functions with the usual uniform metric.

Consider the functionals $A = (A_i(t,x,y))$, $i = \overline{1,d_1}$, $a = (a_i(t,x,y))$, $i = \overline{1,d_2}$, $\overline{B} = \|\overline{B}_{ij}(t,x)\|$, $i = \overline{1,d_1}$, $j = \overline{1,d}$, $\overline{b} = \|\overline{b}_{ij}(t,x,y)\|$, $i = \overline{1,d_2}$, $j = \overline{1,d}$, where $(t,x,y) \in [0,T] \times C^{d_1}_{[0,T]} \times C^{d_2}_{[0,T]}$. Let for each $N = 1,2,\dots$ and for each $x^1, x^2 \in C^{d_1}_{[0,T]}$

$$\tau_N(x^1,x^2) := \inf\Big\{t : t > 0, \sup_{0\leq s\leq t} \max(|x^1(s)|^2, |x^2(s)|^2) > N\Big\},$$

where $\inf\{\varnothing\} = +\infty$ and $|\cdot|$ is the norm in R_{d_1}.

Further, for each $N=1,2,\dots$ introduce the set

$$D_N=\{(t,x^1,x^2)\in[0,T]\times C^{d_1}_{[0,T]}\times C^{d_1}_{[0,T]}:0<t\le\tau_N(x^1,x^2)\}.$$

Denote by $g=(g(t,x,y))$ any of the objects $A_i,a_i,\overline{B}_{ij}$ and $\bar{b}_{ij}$.

Suppose

(1) $g=(g(t,x,y))$ is a non-anticipative nonrandom bounded functional: $|g(t,x,y)|\le\text{const}$, $\forall(t,x,y)\in[0,T]\times C^{d_1}_{[0,T]}\times C^{d_2}_{[0,T]}$;

(2) the functional $g=(g(t,x,y))$ satisfies Lipschitz condition locally in variable x and globally in variable y: for each N, one can find a constant denoted by const_N such that for each $t\in[0,T]$, $x^1,x^2\in C^{d_1}_{[0,T]}$ and $y^1,y^2\in C^{d_2}_{[0,T]}$,

$$|g(t,x^1,y^1)-g(t,x^2,y^2)|^2\le\text{const}_N(|x^1(t)-x^2(t)|^2+|y^1(t)-y^2(t)|^2$$
$$+\int_0^t\Big(|x^1(s)-x^2(s)|^2+|y^1(s)-y^2(s)|^2\Big)ds$$

on the set D_N, where $|\cdot|$ is the norm in the corresponding Euclidean space.

Let a d-dimensional Brownian motion $w=(w,F)=(w(t),\mathcal{F}_t,\,0\le t\le T)$ be given on a filtered complete probability space $(\Omega,\mathcal{F},F=(\mathcal{F}_t),\,0\le t\le T,\,P)$.

Consider the following system of stochastic differential equations (SDEs):

$$\begin{aligned}d\eta(t)&=a(t,\xi,\eta)dt+\bar{b}(t,\xi,\eta)dw(t),\\ d\xi(t)&=A(t,\xi,\eta)dt+\overline{B}(t,\xi)dw(t),\\ \eta(0)&=\eta^0\in R_{d_2},\quad \xi(0)=\xi^0\in R_{d_1}.\end{aligned}\tag{2.1}$$

Remark 2.1. System (2.1) can be rewritten in triangle form (see, Liptser and Shiryaev [18], Lemma 10.4, Theorem 10.3)

$$\begin{aligned}d\eta(t)=&a(t,\xi,\eta)dt+b(t,\xi,\eta)dv(t)+c(t,\xi,\eta)dw(t),\\ d\xi(t)=&A(t,\xi,\eta)dt+B(t,\xi)dw(t),\\ &\eta(0)=\eta^0,\quad \xi(0)=\xi^0,\end{aligned}\tag{2.2}$$

where w and v are independent ($w\perp v$) d_1- and d_2-dimensional Brownian motions, respectively, b, c and B are $d_2\times d_2$-, $d_2\times d_1$- and $d_1\times d_1$-matrices, respectively. All coefficients satisfy conditions (1) and (2).

It is well-known that under these conditions there exists a pathwise unique strong (i.e., F^w-adapted) solution (η,ξ) of system (2.1) (or (2.2)).

Introduce the matrix $B^2(t,x):=\overline{B}(t,x)\overline{B}^*(t,x)$ (the superscript * means transposition) and suppose

(3) there exists a constant $\lambda>0$, such that

$$\sum_{i,j=1}^{d_1}B^2_{ij}u_iu_j\ge\lambda|u|^2$$

for each $(t,x)\in[0,T]\times C^{d_1}_{[0,T]}$ and $u=(u_1,\dots,u_{d_1})\in R_{d_1}$.

Denote by B the symmetric positive square root of the matrix B^2. (Note that this matrix participates in the second equation of (2.2).) Then from condition (3) it follows that there exists the inverse matrix $B^{-1}(t,x)$ of the matrix B, which is a bounded functional of the variables (t,x). Let

$$m = \Big(m(t,x),\ \ 0 \le t \le T,\ \ x \in C^{d_1}_{[0,T]}\Big)$$

be a d_1-dimensional non-anticipative functional with the property: P-a.s., for almost all t, $0 \le t \le T$,

$$m(t,\xi) = E(A(t,\xi,\eta)|\mathcal{F}^\xi_t) \tag{2.3}$$

(such a functional exists, see [18], Lemma 4.9), where $F^\xi = (\mathcal{F}^\xi_t)$, $0 \le t \le T$, is the P-augmentation of the filtration generated by ξ, $E(\cdot|\mathcal{F}^\xi_t) = E(\cdot|\mathcal{F}^\xi_t)(t,\omega)$ is a (t,ω)-measurable modification of conditional expectation.

Define the process $\overline{w}$ by the relation

$$\overline{w}(t) = \int_0^t B^{-1}(s,\xi)(d\xi(s) - m(s,\xi)ds), \quad 0 \le t \le T, \tag{2.4}$$

and denote by $F^{\overline{w}} = (\mathcal{F}^{\overline{w}}_t)$, $0 \le t \le T$, the P-augmentation of the filtration generated by $\overline{w}$.

Theorem 2.1. *In scheme* (2.1)*, under conditions* (1)*,* (2) *and* (3)*, there exists a strong innovation process $\overline{w}$ for the process ξ. That is:*
(a) *the process $\overline{w} = (\overline{w}, F^\xi)$ is a (F^ξ, P)-Brownian motion,*
(b) *$F^{\overline{w}} = F^\xi(\text{mod } P)$.*
The process $\overline{w}$ is given by (2.4)*.*

Let the process $\xi = (\xi(t), 0 \le t \le T)$ satisfy the SDE

$$d\xi(t) = A(t,\xi,\alpha)dt + \overline{B}(t,\xi)dw(t), \xi(0) = \xi^0 \in R_{d_1}, \tag{2.5}$$

where $\alpha = (\alpha, F) = (\alpha(t), \mathcal{F}_t, 0 \le t \le T)$ is a d_2-dimensional F-adapted process independent of w ($\alpha \perp w$) with values in some measurable space $(\mathcal{A}, \mathcal{B}_\mathcal{A})$.

Let the coefficient $\overline{B}$ satisfy conditions (1)–(3), and the coefficient $A = (A(t,x,a), (t,x,a) \in [0,T] \times C^{d_1}_{[0,T]} \times \mathcal{A})$ satisfy condition (1) and the local Lipschitz condition in the variable x, for each fixed $a \in \mathcal{A}$, with const_N which does not depend on the variable a.

Corollary 2.1. *In scheme* (2.5) *there exists a strong innovation process $\overline{w}$ for the process ξ, with $m(t,\xi) = E(A(t,\xi,\alpha)|\mathcal{F}^\xi_t), 0 \le t \le T$.*

Finally, consider the scheme

$$d\xi(t) = \mu(t)dt + \overline{B}(t,\xi)dw(t), \ \ \xi(0) = \xi^0 \in R_{d_1}, \tag{2.6}$$

where $\mu = (\mu(t), 0 \le t \le T)$ is a F-adapted , bounded d_1-dimensional process, independent of w ($\mu \perp w$), and $\overline{B}$ satisfies conditions (1)–(3).

Corollary 2.2. *In scheme* (2.6) *there exists a strong innovation process* $\overline{w}$ *for the process* ξ, *with* $m(t,\xi) = E(\mu(t)|\mathcal{F}_t^\xi)$, $0 \le t \le T$.

Remark 2.2. Suppose in scheme (2.1), η^0 and ξ^0 are random variables independent of the process w, with $E(|\xi^0|_1^2 + |\eta^0|_2^2) < \infty$, where $|\cdot|_i$ is the norm in R_{d_i}, $i = 1, 2$.

Then under the conditions of Theorem 2.1 a strong innovation process $\overline{w}$ exists in the sense that

$$F^{\overline{w},\xi^0} = F^\xi \,(\mathrm{mod}\ P).$$

The same holds true for schemes (2.5) and (2.6).

3. Partial information diffusion modelling in Finance

We consider diffusion models. Usual settings of the mean-variance hedging and utility maximization problems under full and partial information are as follows.

a. Full information. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $w = (w_1, \dots, w_N)^*$ be a N-dimensional Brownian motion defined on it, $N \ge 1$ be an integer. Denote by $F = (\mathcal{F}_t, 0 \le t \le T)$ the P-augmentation of filtration generated by w. Consider a financial market which consists of one risk-free asset, whose price process is assumed, for simplicity, to be equal to 1 at each date, and n risky assets (stocks) with n-dimensional price process $S = (S_1, \dots, S_n)^*$, whose dynamics is governed by the equation

$$dS(t) = \operatorname{diag} S(t)(\mu(t)dt + \sigma(t)dw(t)), \quad S(0) = S^0 \in R_n^+, \tag{3.1}$$

where μ and σ are F-predictable n and $n \times N$-dimensional processes of appreciation rate and volatility, respectively. It is assumed that $n \le N$, the matrix σ has a full rank, and μ and σ satisfy some type of boundedness (integrability) conditions. If $n = N$ the market is complete; if $n < N$, (3.1) models an incomplete market.

Along with model (3.1), consider the popular model of an incomplete market, the so-called stochastic volatility model

$$\begin{aligned} dS(t) &= \operatorname{diag} S(t)(\mu(t)dt + \sigma(t, S(t), Y(t))dw_1(t), \\ dY(t) &= \delta(t)dt + \rho(t, S(t), Y(t))dw_1(t) + \gamma(t, S(t), Y(t))dw_2(t), \\ S(0) &= S^0 \in R_n^+, \quad Y(0) = Y^0 \in R_{N-n}, \end{aligned} \tag{3.2}$$

where $n < N$, the Brownian motion $w = (w, F) = ((w_1, w_2), F)$, with $w_1 = (w_1, \dots, w_n)^*$ and $w_2 = (w_{n+1}, \dots, w_N)^*$, μ and δ are F-predictable n- and $(N-n)$-dimensional vector-valued processes, σ, ρ and γ are $n \times n$-, $n \times (N-n)$- and $(N-n) \times (N-n)$-matrices of functions, defined on $[0,T] \times R_n \times R_{(N-n)}$.

(i) Mean-variance hedging problem. The space of admissible trading strategies $\Theta(F)$ consists of all R_n-valued F-predictable processes θ, which are

S-integrable, such that $\int_0^T \theta(t)dS(t) \in L^2(P, \mathcal{F}_T)$ and the stochastic integral $\int \theta\, dS$ is a Q-martingale under any $Q \in \mathcal{P}(F)$.

Here $\mathcal{P}(F) = \{Q \backsim P$ on $(\Omega, \mathcal{F}_T) : \frac{dQ}{dP}\Big|_{\mathcal{F}_T} \in L^2(P, \mathcal{F}_T)$ and S is a Q-local martingale$\}$.

It is assumed that there is no arbitrage, i.e., $\mathcal{P}(F) \neq \varnothing$. The process $\theta = (\theta(t))$ represents the number of shares of stocks held at time t, based on information $\mathcal{F}_t$. For a given initial investment $x \in R_1^+$ and trading strategy $\theta \in \Theta(F)$, the self-financed wealth process is defined as $V_t^{x,\theta} = x + \int_0^t \theta(u)dS(u)$, $0 \leq t \leq T$. The $\mathcal{F}_T$-measurable random variable $H \in L^2(P, \mathcal{F}_T)$ models the payoff from financial product at maturity time T. If a hedger starts with the initial investment x and uses the trading strategy θ, the mean-variance hedging problem means to find a trading strategy $\theta^{*,F}(x)$ solution of

$$J_F(x) = \min_{\theta \in \Theta(F)} E(H - V_T^{x,\theta})^2. \tag{3.3}$$

Note that for complete markets the solution of (3.3) is almost trivial.

The mean-variance hedging problem has been solved by Gourieroux, Laurent and Pham [8] and Laurent and Pham [17] for the model (3.1), and by the authors of [8] in [9] and Rheinländer and Schweizer [24] in a general semimartingale setting.

For model (3.2), in the case where $\mu(t) = \mu(t, S(t), Y(t))$, $\delta(t) = \delta(t, Y(t))$, $\rho = 0$, $\gamma(t) = \gamma(t, Y(t))$, under some regularity conditions Laurent and Pham [17] get the explicit expressions for the main objects of the approach of [9], that is for the hedging numeriare and the variance-optimal matringale measure. The variance-optimal martingale measure by means of Bellman equation was studied by Mania and Tevzadze in [19].

(ii) Utility maximization problem. Let the stock price process follow (3.1) (or (3.2)). A portfolio process $\theta = (\theta(t), 0 \leq t \leq T)$ is a F-predictable n-dimensional vector valued process with $\int_0^T |\sigma^*(t)\theta(t)|^2 dt < \infty$, P-a.s. $\theta_i(t)$ is interpreted as proportion of wealth invested into stock i at time t, based on information $\mathcal{F}_t$. Given an initial wealth (nonrandom) $x > 0$ the wealth process corresponding to a portfolio θ is defined by $X^{x,\theta}(0) = x$ and

$$dX^{x,\theta}(t) = X^{x,\theta}(t)\theta^*(t)\mu(t)dt + X^{x,\theta}(t)\theta^*(t)\sigma(t)dw(t).$$

A function $U : (0, \infty) \to R_1$ will be called a utility function if it is strictly concave, strictly increasing, of the class C^1, and satisfies $U'(0+) := \lim_{x \downarrow 0} U'(x) = +\infty$, $U'(\infty) := \lim_{x \to \infty} U'(x) = 0$.

The problem is to maximize the expected utility from the terminal wealth $EU(X^{x,\theta}(T))$ over the class $\mathcal{A}(x, F)$ of portfolio processes θ that satisfy $EU^-[(X^{x,\theta}(T))] < \infty$. The value function of this problem is

$$v(x) = \sup_{\theta \in \mathcal{A}(x,F)} EU(X^{x,\theta}(T)), \quad x > 0. \tag{3.4}$$

This problem for both complete and incomplete markets has been solved by Karatzas, Lohoczky, Shreve and Xu [11].

b) Partial information. Consider an investor who can observe neither the Brownian motion nor drift, but only the stock price process S. This situation is referred to as partial information.

Denote $F^S = (\mathcal{F}_t^S, 0 \le t \le T)$-the P-augmentation of filtration generated by the stock price process S. The mean-variance hedging and the utility maximization problems under partial information means that the hedging strategy and the portfolio must be F^S-predictable processes.

(i′) Mean-variance hedging problem under partial information.
Let $H \in L^2(F, P)$. Denote $H_{F^S} := E(H|\mathcal{F}_T^S)$. Then

$$J_{F^S}(x) = E(H - H_{F^S})^2 + \min_{\theta \in \Theta(F^S)} E(H_{F^S} - V_T^{x,\theta})^2, \tag{3.5}$$

and hence the problem is to minimize the expression $E(H_{F^S} - V_T^{x,\theta})^2$ over the admissible strategies $\theta \in \Theta(F^S)$.

Now we consider some models of the stock price process with partial information and by reducing them to models (3.1) (or (3.2)) with full information we solve problem (3.5) under partial information.

1) Let the d_1-dimensional stock price process $S = (S_1, \dots, S_{d_1})^*$ follow the process

$$dS(t) = \operatorname{diag} S(t)(\mu(t, S, \eta)dt + \overline{\sigma}(t, S)dw(t)), \quad S(0) = S^0 \in R_{d_1}^+, \tag{3.6}$$

where the appreciation rate μ of this stock is influenced by d_2 stochastic factors $\eta = (\eta_1, \dots, \eta_{d_2})^*$ whose dynamics is governed by

$$d\eta(t) = \widetilde{a}(t, S, \eta)dt + \widetilde{b}(t, S, \eta)dw(t), \ \eta(0) = \eta^0 \in R_{d_2}. \tag{3.7}$$

Here μ and $\widetilde{a}$ are d_1 and d_2-dimensional vectors, $\overline{\sigma}$ and $\widetilde{b}$ are $d_1 \times d$ and $d_2 \times d$ matrices of non-anticipative functionals defined on $[0, T] \times C_{[0,T]}^{d_1} \times C_{[0,T]}^{d_2}$, respectively, $w = (w_1, \dots, w_d)$ is a d-dimensional (F, P)-Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, F = (\mathcal{F}_t), 0 \le t \le T, P)$ and $d_1 \ge 1$, $d_2 \ge 1$, $d_1 + d_2 = d$ are fixed integers.

Suppose that the coefficients of system (3.6), (3.7) satisfy conditions (1), (2) and (3) of Theorem 2.1.

If we introduce the process $\xi(t) = \ln S(t)$ (which is well defined since under conditions (1)–(3) $\inf_{0 \le t \le T} S(t) > 0$, P-a.s.) and use the Itô formula we easily arrive at scheme (2.1) with

$$\begin{gathered} A(t, x, y) = \mu(t, e^x, y) - \frac{1}{2}\vec{dg}(\overline{\sigma}\,\overline{\sigma}^*)(t, e^x), \quad a(t, x, y) = \widetilde{a}(t, e^x, y), \\ \overline{B}(t, x) = \overline{\sigma}(t, e^x), \quad \overline{b}(t, x, y) = \widetilde{b}(t, e^x, y), \end{gathered} \tag{3.8}$$

where $\vec{dg}\Gamma$ is the vector $(\gamma_{11}, \dots, \gamma_{d_1 d_1})$ of diagonal elements of the matrix $\Gamma = \|\gamma_{ij}\|$, $i, j = \overline{1, d_1}$.

Note now that using the inequality $|e^x - e^y| \le \frac{e^x + e^y}{2}|x - y|$, $x, y \in R_1$, and the Lipschitz condition we easily get that the new coefficients $A, a, \overline{b}$ and $\overline{B}$ satisfy the local Lipschitz condition in variable x and the global one in variable

y. Trivially, these coefficients are bounded and $\overline{B}$ satisfies condition (3) of Theorem 2.1.

Hence by Theorem 2.1 there exists a strong innovation process $\overline{w}$ for the process ξ given by (2.4), with $F^{\overline{w}} = F^{\xi}$. But evidently $F^{\xi} = F^{S}$. If we use now the inverse change of variable $S(t) = e^{\xi(t)}$ and denote $\sigma = (\overline{\sigma}\,\overline{\sigma}^*)^{1/2}$, we easily get

$$\overline{w}(t) = \int_0^t \sigma^{-1}(u,S)(\operatorname{diag} S(u))^{-1}(dS(u) - \operatorname{diag} S(u) m(u,S)du), \qquad (3.9)$$

with $m(t,S) = E(\mu(t,S,\eta)|\mathcal{F}_t^S)$.

Hence

$$dS(t) = \operatorname{diag} S(t)(m(t,S)dt + \sigma(t,S)d\overline{w}(t)), \quad S(0) = S^0 \in R_{d_1}^+, \quad (3.10)$$

with $F^S = F^{\overline{w}}$, i.e., we construct a strong innovation process $\overline{w}$ for the process S given by (3.9).

Thus we reduced the partial information case (scheme (3.6), (3.7) with given information flow F^S) to the usual complete market model (3.10) with full information. Indeed, recall that $\overline{w} = (\overline{w}, F^S) = (\overline{w}, F^{\overline{w}})$, and m and σ are $F^S = F^{\overline{w}}$-adapted (recall scheme (3.1) with $n = N = d_1$). Hence to solve the mean-variance hedging problem (3.5) we can use the well-known results (see the references in Subsection **a**). For example, we immediately get that the whole hedging risk under partial information is given by the expression

$$E(H - H_{F^S})^2 + \frac{(E^{P^S} H_{F^S} - x)^2}{E(\xi_T)^2}, \quad \text{(see (3.5))},$$

where P^S is a unique martingale measure with $\frac{dP^S}{dP}|_{\mathcal{F}_T^S} = \xi_T$,

$$\xi_T = \exp\Big(- \int_0^T \lambda^*(u)d\overline{w}(u) - \frac{1}{2}\int_0^t |\lambda(u)|^2 du\Big), \quad \lambda(t) = \sigma^{-1}(t,S)m(t,S).$$

Consider the particular schemes of model (3.6), (3.7):

$$2)\ \ dS(t) = \operatorname{diag} S(t)(\mu(t,S,\alpha)dt + \overline{\sigma}(t,S)dw(t)), \ \ S(0) = S^0 \in R_{d_1}^+, \ \ (3.11)$$

where $\alpha = (\alpha_1, \dots, \alpha_{d_2})^*$, $\alpha \perp w$ is an arbitrary-valued F-adapted process (stochastic factor influenced on μ), and

$$3)\ \ dS(t) = \operatorname{diag} S(t)(\mu(t)dt + \overline{\sigma}(t,S)dw(t)), \ \ S(0) = S^0 \in R_{d_1}^+, \qquad (3.12)$$

where the process $\mu = (\mu_1, \dots, \mu_{d_1})$ is independent of w ($\mu \perp w$), F-adapted and bounded.

Under the conditions of Corollaries 2.1 and 2.2 in both cases there exists a strong innovation process $\overline{w}$ defined by (3.9). Hence the conclusions concerning the mean-variance hedging problem are the same as in 1).

Remark 3.1. The assumption $\mu \perp w$ in case 3 is the limitation of approach.

4) Consider the following stochastic volatility model:

$$\begin{aligned} dS(t) &= \operatorname{diag} S(t)(\mu(t,S,Z,\eta)dt+\overline{\sigma}_1(t,S(t),Z(t))dw(t)),\ S(0)=S^0\in R_n^+,\\ dZ(t) &= \delta(t,S,Z,\eta)dt+\overline{\sigma}_2(t,S(t),Z(t))dw(t)),\ Z(0)=Z^0\in R_m, \end{aligned} \tag{3.13}$$

where the process $\eta=(\eta_1,\dots,\eta_{d_2})^*$ follows (3.7). Here $\mu=(\mu(t,s,z,y))$ and $\delta=(\delta(t,s,z,y))$ are non-anticipative functionals defined on $[0,T]\times C^n_{[0,T]}\times C^m_{[0,T]}\times C^{d_2}_{[0,T]}$, $n\ge 1$, $m\ge 1$, $n+m=d_1$, $d_1+d_2=d$, $\mu=(\mu_1,\dots,\mu_n)^*$, $\delta=(\delta_1,\dots,\delta_m)^*$. Further $\overline{\sigma}_1=(\overline{\sigma}_1(t,s,z))$ and $\overline{\sigma}_2=(\overline{\sigma}_2(t,s,z))$, $(t,s,z)\in[0,T]\times R_n\times R_m$ are $n\times d$ and $m\times d$ matrices, respectively, and $w=(w_1,\dots,w_d)^*$ is a (F,P)-Brownian motion defined on the filtered complete probability space $(\Omega,\mathcal{F},F=(\mathcal{F}_t),0\le t\le T,P)$.

Suppose the coefficients of equations (3.13) and (3.7) satisfy the conditions of Theorem 2.1. Then there exists a unique strong solution (S,Z,η) of system (3.13), (3.7). Introduce the process $\xi=(\xi^1,\xi^2)^*:=(\ln S,Z)^*$. Then, using the Itô formula, we easily get that the process (ξ,η) satisfies system (2.1) with

$$\begin{aligned} A(t,x,y) &= \begin{pmatrix} \mu(t,e^{x^1},z,y)-\frac{1}{2}\vec{dg}(\overline{\sigma}_1\overline{\sigma}_1^*)(t,e^{x^1},z,y)\\ \delta(t,e^{x^1},z,y)\end{pmatrix},\\ \overline{B}(t,x) &= \begin{pmatrix}\overline{\sigma}_1(t,e^{x^1},z,y)\\ \overline{\sigma}_2(t,e^{x^1},z,y)\end{pmatrix},\\ a(t,x,y) &= \widetilde{a}(t,x^{x^1},z,y),\quad \overline{b}(t,x,y)=\widetilde{b}(t,e^{x^1},z,y), \end{aligned}$$

where $x=(x^1,z)^*$, x^1 is an n-dimensional and z is an m-dimensional vector. Such a type of change of variables has been used by Mania and Tevzadze [19].

Then using Theorem 2.1 as in 1) we easily get that there exists a strong innovation process $\overline{w}$ for the process $(S,Z)^*$ defined as follows.

Let

$$\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} := \left(\begin{pmatrix}\overline{\sigma}_1\\ \overline{\sigma}_2\end{pmatrix}\begin{pmatrix}\overline{\sigma}_1\\ \overline{\sigma}_2\end{pmatrix}^*\right)^{\frac{1}{2}},$$

where σ_1 and σ_2 are $n\times d_1$ and $m\times d_1$ matrices. Let further,

$$\operatorname{diag}\begin{pmatrix}S(t)\\ 1\end{pmatrix} := \begin{pmatrix} S_1(t) & & & & & \\ & \ddots & & & & \\ & & S_n(t) & & & 0\\ & 0 & & 1 & & \\ & & & & \ddots & \\ & & & & & 1\end{pmatrix}$$

be a $d_1\times d_1$ - matrix, and suppose $m:=(m_1,m_2)^*$, where $m_1(t,S,Z):=E(\mu(t,S,Z,\eta)|\mathcal{F}_t^{S,Z})$ and $m_2(t,S,Z):=E(\delta(t,S,Z,\eta)|\mathcal{F}_t^{S,Z})$ are an n- and an m-dimensional vectors, respectively.

Then the $(n+m)$-dimensional innovation process $\overline{w} = (\overline{w}, F^{S,Z})$ for the process $(S,Z)^*$ is given by

$$\overline{w}(t) = \int_0^t \begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}^{-1} (u, S(u), Z(u)) \left(\operatorname{diag}\begin{pmatrix}S(u)\\ 1\end{pmatrix}\right)^{-1} \left(d\begin{pmatrix}S(u)\\ Z(u)\end{pmatrix}\right.$$
$$\left. - \operatorname{diag}\begin{pmatrix}S(u)\\ 1\end{pmatrix}\begin{pmatrix}m_1\\ m_2\end{pmatrix}(u,S,Z)du\right).$$

Hence

$$dS(t) = \operatorname{diag} S(t)(m_1(t,S,Z)dt + \sigma_1(t,S(t),Z(t))d\overline{w}(t)), \quad S(0) = S^0,$$
$$dZ(t) = m_2(t,S,Z)dt + \sigma_2(t,S(t),Z(t))d\overline{w}(t)), \quad Z(0) = Z^0.$$

Rewrite the last system in triangle form. We get

$$\begin{aligned} dS(t) &= \operatorname{diag} S(t)(m_1(t,S,Z)dt + \sigma(t,S(t),Z(t))dN(t)),\ S(0)=S^0,\\ dZ(t) &= m_2(t,S,Z)dt + \rho(t,S(t),Z(t))dN(t)\\ &\quad + \gamma(t,S(t),Z(t)))dM(t),\ Z(0) = Z^0, \end{aligned} \tag{3.14}$$

where N and M are independent n and m-dimensional $(F^{S,Z}, P)$-Brownian motions.

Further, by the property of a strong innovation process

$$F^{S,Z} = F^{\overline{w}} (= F^{N,M}). \tag{3.15}$$

Assume (in addition to the conditions of Theorem 2.1) that $\sigma\sigma^*$ is continuous in (t,s,z) and for all (t,s) the function $\sigma\sigma^*(t,s,\cdot)$ is one-to-one from R^m into a subset Σ of the set of $n \times n$ positive definite matrices, and its inverse function denoted by $\mathcal{L}(t,s,\cdot)$ is continuous in $(t,s,z) \in [0,T] \times R_n \times \Sigma$. Here we follow Renault and Touzi [23] or Pham and Quenez [22].

Under these conditions $Z(t) = \mathcal{L}(t,S,\sigma\sigma^*(t,S(t),Z(t))$ and $< S, S^* >_t = \int_0^t \sigma\sigma^*(u, S(u)Z(u))du$. From this easily it follows that $F^Z \subseteq F^S$, and consequently $F^{S,Z} = F^S$. Hence from (3.15) we get

$$F^S = F^{N,M},$$

and we reduced again the case of partial information to the usual incomplete market model of stochastic volatility with full information (see model (3.2)), and to solve problem (3.5) the well-known arguments of [8], [9], or [24] or [17] can be used.

As above, we can easily consider the particular schemes of (3.13), (3.7) (see (3.11) and (3.12)). We do not stop on this.

(ii′) Utility maximization problem under partial information.

We consider the same models 1)–4) for the stock price process as in previous case.

After reducing them to the usual full information form by means of a strong innovation process, to solve the utility maximization problem under partial information we need only to refer to the paper of Karatzas, Lehoczky, Shreve an Xu [11] for both complete and incomplete market models. □

Collect all above results in the following

Theorem 3.1. *If a stock price process S follows the models described in cases* 1)–4)*, then under the conditions given in these models a strong innovation process $\overline{w}$ exists for process S, and both the mean-variance hedging and the utility maximization problems under partial information are reduced to the full information one with well-known solutions.*

Remark 3.2. The approach summarized in Theorem 3.1 is useful for every problem in finance with partial information, when the stock price process is modelled by 1)–4) and the solution of the considered problem in the full information case is already known (example of such problem see the paper by Cvitanič, Lazrak, Quenez and Zapatero [1]).

Remark 3.3. Consider the situation described in Remark 2.2. In this case models 1)–4) under partial information cannot be reduced by means of a strong innovation to the usual full information models. Hence additional effects may arise (in general) under partial information. This fact is mentioned in [15], [13], and [22] in a Bayesian setting for the utility maximization problem.

4. Appendix

Proof of Theorem 2.1. Assertion (a) is a well-known fact. We prove assertion (b).

For simplicity, the proof will be given for $d_1 = d_2 = 1$. In the multidimensional case the proof is completely analogous, the only difference is the cumbersome expressions.

Rewrite (2.1) in the form of (2.2). Let (η, ξ) be a unique strong solution of (2.2). From (2.4) and the second equation of (2.2)

$$dw(t) = d\overline{w}(t) + (B(t,\xi))^{-1}(m(t,\xi) - A(t,\xi,\eta))dt.$$

Substituting the last expression into the first equation of (2.2) we get

$$d\eta(t) = a(t,\xi,\eta)dt + b(t,\xi,\eta)dv(t) + c(t,\xi,\eta)d\overline{w}(t) + (B(t,\xi))^{-1}c(t,\xi,\eta) \times (m(t,\xi) - A(t,\xi,\eta))dt, \quad \eta(0) = \eta^0. \tag{4.1}$$

Fix the first space variable $\xi = x$, $x \in C_{[0,T]}$, in the coefficients of SDE (4.1). It is easy to see that all coefficients are bounded and Lipschitz in the second variable. Hence the solution of (4.1) is pathwise unique for each $\xi = x$. The solution can be constructed by the standard successive approximation method. Thus for each $\xi = x$, we can construct a non-anticipative functional (see, e.g., [10], Ch. IV, Theorem 1.1) $F(t,x,u,v,p) : [0,T] \times C_{[0,T]} \times C_{[0,T]} \times C_{[0,T]} \times R_1 \to R_1$, such that the process $\eta = (\eta(t))$ with $\eta(t) = F(t,\xi,\overline{w},v,\eta^0)$ would be a unique strong solution of (4.1) with given ξ. But $F^{\overline{w}} \subseteq F^{\xi}$ by the construction, see

(2.4). Hence $\overline{w}(t)=\psi(t,\xi)$, where the functional $\psi(t,x):[0,T]\times C_{[0,T]}\to R_1$ is non-anticipative. Consequently for each t

$$\eta(t)=F(t,\xi,\varphi(t,\xi),v,\eta^0). \tag{4.2}$$

Finally, if we denote $a:=(v,p)\in C_{[0,T]}\times R_1$ and define the functional

$$\varphi(t,x,a):=A(t,x,F(t,x,\psi(t,x),v,p)),$$

we easily see that this functional is non-anticipative (as the superposition of such functionals) with the property: for each t, P-a.s.

$$\varphi(t,\xi,(v,\eta^0))=A(t,\xi,\eta). \tag{4.3}$$

Everywhere below to simplify expressions we omit the fixed number η^0.

From (4.3), (2.4) and (4.1) we have that the process ξ satisfies the following SDEs:

$$d\xi(t)=\varphi(t,\xi,v)dt+B(t,\xi)dw(t),\ (=m(t,\xi)dt+B(t,\xi)d\overline{w}(t)),\ \xi(0)=\xi^0. \tag{4.4}$$

Consider the second equation of (4.4). It is easy to see that this equation has a unique in distribution weak solution: if $\widetilde{\xi}=(\widetilde{\xi}(t))$ is any weak solution of (4.4), then

$$\mu_{\widetilde{\xi}}(A)=\mu_\xi(A),\quad \forall A\in\mathcal{B}_{[0,T]}, \tag{4.5}$$

where $\mu_{\widetilde{\xi}}$ and μ_ξ are the distributions of $\widetilde{\xi}$ and ξ, respectively, on the measurable space $(C_{[0,T]},\mathcal{B}_{[0,T]})$. This fact easily follows from the form of densities of these processes (see [18], Theorem 7.19).

Further, we already know that the equation

$$d\xi(t)=m(t,\xi)dt+B(t,\xi)d\overline{w}(t),\quad \xi(0)=\xi^0, \tag{4.6}$$

has a weak solution (this fact follows from the construction of the process $\overline{w}$, see (2.4) and assertion (a) of Theorem 2.1), and $F^{\overline{w}}\subset F^\xi$. Hence we have to prove that $F^\xi\subset F^{\overline{w}}$. For this it is sufficient to show that (4.6) has a pathwise unique strong solution. By the Yamada–Watanabe theorem (see, e.g., [10]) under the assumption that SDE (4.6) has a weak solution, it is sufficient to prove the pathwise uniqueness of the solution of (4.6). Thus, we have to prove that if ξ^1 and ξ^2 are two solutions of (4.6) defined on the same probability space, with the same initial condition $\xi^1(0)=\xi^2(0)=\xi^0$, then

$$P\Big\{\sup_{0\le t\le T}|\xi^1(t)-\xi^2(t)|=0\Big\}=1. \tag{4.7}$$

Denote by μ_v the distribution of the process v on the measurable space $(C_{[0,T]},\mathcal{B}_{[0,T]})$. It is easy to see that all conditions of Theorem 7.23 of [18] are satisfied (recall that v and w are independent processes, the process ξ satisfies the first equation of (4.4) and has the property $F^{\xi} \subset F^{v,w}$). Hence by the Bayes formula we have: for each t, P-a.s.

$$m(t,\xi) = E(\varphi(t,\xi,v)|\mathcal{F}_t^{\xi}) = \int\limits_{C_{[0,T]}} \varphi(t,\xi,a)\rho(t,\xi,a)\mu_v(da), \tag{4.8}$$

where

$$\rho(t,\xi,v) := \exp(f(t,\xi,v)), \tag{4.9}$$

with

$$\begin{aligned} f(t,\xi,v) :=& \int\limits_0^t (B(s,\xi))^{-1}(\varphi(s,\xi,v) - m(s,\xi))d\overline{w}(s) \\ & - \frac{1}{2}\int\limits_0^t (B(s,\xi))^{-2}(\varphi(s,\xi,v) - m(t,\xi))^2 ds. \end{aligned} \tag{4.10}$$

Fix the variable t and introduce the notation:

$$\begin{gathered} \rho^l(t) = \rho(t,\xi^l,v), \quad m^l(t) = m(t,\xi^l), \quad \varphi^l(t) = \varphi(t,\xi^l,v), \quad f^l(t) = f(t,\xi^l,v), \\ a^l(t) = a(t,\xi^l,\eta^l), \quad b^l(t) = b(t,\xi^l,\eta^l), \quad c^l(t) = c(t,\xi^l,\eta^l), \\ B^l(t) = B(t,\xi^l), \quad A^l(t) = A(l,\xi^l,\eta^l), \\ \eta^l(t) = F(t,\xi^l,\psi(t,\xi^l),v), \quad G(t) = \frac{1}{2}(\rho^1(t) + \rho^2(t)), \end{gathered}$$

where ξ^l, $l = 1,2$ are two solutions of SDE (4.6) defined on the initial probability space (for simplicity) with $\xi^l(0) = \xi^0$, $l = 1,2$. Finally, denote by E^{μ} the operator of integration w.r.t. measure μ_v.

Note first that P-a.s.

$$E^{\mu}G(t) = 1. \tag{4.11}$$

Indeed, let $\widetilde{\xi}$ be some weak solution of SDE (4.6). Then $(1 - E^{\mu}\rho(t,\widetilde{\xi},v))^2 = 0 \Leftrightarrow E(1 - E^{\mu}\rho(t,\widetilde{\xi},v))^2 = 0$. But the last expectation is equal to

$$\int\limits_{C_{[0,T]}} (1 - E^{\mu}\rho(t,x,v))^2 \mu_{\widetilde{\xi}}(dx) = \int\limits_{C_{[0,T]}} (1 - E^{\mu}\rho(t,x,v))^2 \mu_{\xi}(dx) = 0,$$

by (4.5) and (4.8).

Fix an integer N. Denote the F-stopping time τ_N by the relation $\tau_N = \inf\{t : t > 0, \sup_{0\le s\le t} \max(|\xi^1(s)|^2, |\xi^2(s)|^2) > N\}$, with $\inf(\varnothing) = +\infty$, and introduce the process $\chi_N = (\chi_N(t))$ by the formula $\chi_N(t,\omega) = I_{]0,\tau_N]}(t,\omega)$, where $]\cdot]$ is a stochastic interval. It is easy to see that if $s \le t$, then $\chi_N(t) = \chi_N(t)\cdot\chi_N(s)$, and hence

$$\begin{aligned} \{\omega : \chi_N(t) = 1\} \subseteq \{\omega : \chi_N(s) = 1\}, \\ \{\omega : \chi_N(s) = 0\} \subseteq \{\omega : \chi_N(t) = 0\}. \end{aligned} \tag{4.12}$$

Below, to simplify the expressions, we omit the index N in $\chi_N(t)$ and const_N (the Lipschitz constant, see condition (2)). Recall that we have fixed the variable t.

From (4.6) we have

$$E(\xi^1(t)-\xi^2(t))^2\chi(t) \le \mathrm{const}\Big[\int_0^t E\chi(s)(m^1(s)-m^2(s))^2ds + \int_0^t E\chi(s)(\xi^1(s)-\xi^2(s))^2ds\Big], \tag{4.13}$$

where we have used the inequality $(a+b)^2 \le 2(a^2+b^2)$, (4.12) and the Lipschitz condition.

Introduce the function $z=(z(t))$ by the relation

$$z(t)=\int_0^t E\chi(s)(m^1(s)-m^2(s))^2ds. \tag{4.14}$$

This function is non-negative and nondecreasing. Hence if $f=(f(t))$ is some non-negative function with $\int_0^T f(t)dt<\infty$, then

$$\int_0^t f(s)z(s)ds \le \int_0^t f(s)z(t)ds \le \mathrm{const}\cdot z(t), \quad 0\le t\le T. \tag{4.15}$$

Denote $u(t)=E\chi(t)(\xi^1(t)-\xi^2(t))^2$. Then (4.13) gives $u(t)\le \mathrm{const}\cdot\Big(z(t)+\int_0^t u(s)ds\Big)$. Solving this inequality and using (4.15), we easily get $u(t)\le \mathrm{const}\cdot z(t)$. Hence if we prove that

$$z(t)=0, \quad 0\le t\le T, \tag{4.16}$$

then we get $u(t)=E\chi(t)(\xi^1(t)-\xi^2(t))^2=0$, $0\le t\le T$. From this the desirable relation (4.7) easily follows:

$$P\{\xi^1(t)\ne\xi^2(t)\}\le P\{\tau_N\le t\}\to 0,$$

as $N\to\infty$, since $\tau_N\uparrow\infty$, as $N\to\infty$.

Thus we have to prove (4.16).

From (4.8) we write

$$m^1(t)-m^2(t)=E^\mu\Big[\varphi^1(t)\rho^1(t)-\varphi^2(t)\rho^2(t)\Big]=E^\mu\Big[(\varphi^1(t)-\varphi^2(t))G(t)\Big] + E^\mu\Big[(\rho^1(t)-\rho^2(t))\frac{1}{2}(\varphi^1(t)+\varphi^2(t))\Big]. \tag{4.17}$$

From the elementary inequality $|e^x-e^y|\le\frac{1}{2}(e^x+e^y)|x-y|$ it follows that $|\rho^1(t)-\rho^2(t)|\le G(t)|f^1(t)-f^2(t)|$. By condition (1) we have $|\varphi^l(t)|\le\mathrm{const}$,

$l = 1, 2$. Hence from (4.17) we get

$$m^1(t) - m^2(t) \leq \text{const}\Big\{E^\mu\Big[|\varphi^1(t) - \varphi^2(t)|\sqrt{G(t)} \cdot \sqrt{G(t)}\Big] + E^\mu\Big[|f^1(t) - f^2(t)|\sqrt{G(t)} \cdot \sqrt{G(t)}\Big]. \tag{4.18}$$

Multiplying both sides of (4.18) by $\chi(t)$, squaring each parts, applying the inequality $(a+b)^2 \leq 2(a^2+b^2)$, the Schwartz inequality and (4.11), and after that averaging w.r.t. measure $P(d\omega)$, we get

$$E^p(m^1(t) - m^2(t))^2\chi(t) \leq \text{const}\Big\{E^pE^\mu[\chi(t)(\varphi^1(t) - \varphi^2(t))^2G(t)] + E^pE^\mu[\chi(t)(f^1(t) - f^2(t))^2G(t)]\Big\}. \tag{4.19}$$

If we show that each summand of (4.19) is less than or equal to the const $z(t)$, then from (4.19) we get (if we integrate the resulting inequality w.r.t. measure dt) that $z(t) \leq \text{const} \int_0^t z(s)ds$, and (4.16) follows from the Gronwall–Bellman lemma.

Thus the theorem will be proved if we show that

$$\begin{aligned} &E^pE^\mu\Big[\chi(t)(\varphi^1(t) - \varphi^2(t))^2G(t)\Big] \leq \text{const} \cdot z(t),\\ &E^pE^\mu\Big[\chi(t)(f^1(t) - f^2(t))^2G(t)\Big] \leq \text{const} \cdot z(t). \end{aligned} \tag{4.20}$$

Show that the first inequality of (4.20) is satisfied. It is easy to see that (condition (2))

$$\begin{aligned} E^pE^\mu\Big[\chi(t)(\varphi^1(t) - \varphi^2(t))^2G(t)\Big] &= E^pE^\mu\Big[\chi(t)(A(t,\xi^1,\eta^1) - A(t,\xi^2,\eta^2))^2G(t)\Big]\\ &\leq \text{const}\, E^pE^\mu\Bigg[\chi(t)(\xi^1(t) - \xi^2(t))^2G(t) + \chi(t)(\eta^1(t) - \eta^2(t))^2G(t)\\ &+\int_0^t \chi(s)(\xi^1(s)-\xi^2(s))^2ds \cdot G(t) + \int_0^t \chi(s)(\eta^1(s)-\eta^2(s))^2ds \cdot G(t)\Bigg]. \end{aligned} \tag{4.21}$$

Denote

$$p_1(t) = \chi(t)(\xi^1(t) - \xi^2(t))^2G(t), \quad p_2(t) = \chi(t)(\eta^1(t) - \eta^2(t))^2G(t),$$

$$p_3(t) = \int_0^t \chi(s)(\xi^1(s) - \xi^2(s))^2ds \cdot G(t),$$

$$p_4(t) = \int_0^t \chi(s)(\eta^1(s) - \eta^2(s))^2ds \cdot G(t),$$

$$p_5(t) = \int_0^t \chi(s)(m^1(s) - m^2(s))^2ds \cdot G(t),$$

$$p_6(t)=\Big(\int_0^t \chi(s)(b^1(s)-b^2(s))dv(s)\Big)^2\cdot G(t),$$

$$p_7(t)=\Big(\int_0^t \chi(s)(c^1(s)-c^2(s))d\overline{w}(s)\Big)^2\cdot G(t),$$

and rewrite (4.21)

$$E^pE^\mu\Big[\chi(t)(\varphi^1(t)-\varphi^2(t))^2G(t)\Big]\le \text{const}\,E^pE^\mu\Big(\sum_{i=1}^4 p_i(t)\Big). \tag{4.22}$$

Using (4.11) and (4.15) we get

$$\begin{aligned}E^pE^\mu p_1(t)&=E^p\chi(t)(\xi^1(t)-\xi^2(t))^2E^\mu G(t)\\&=E^p\chi(t)(\xi^1(t)-\xi^2(t))^2\le \text{const}\,z(t).\end{aligned} \tag{4.23}$$

Further, from (4.1) we write (using conditions (1)–(3))

$$p_2(t)\le \text{const}\cdot\Big(\sum_{i=3}^7 p_i(t)\Big). \tag{4.24}$$

As in (4.23) we easily have

$$E^pE^\mu p_3(t)\le \text{const}\int_0^t z(s)ds\cdot E^\mu G(t)\le \text{const}\cdot z(t) \tag{4.25}$$

and (see (4.14))

$$E^pE^\mu p_5(t)=z(t). \tag{4.26}$$

Now using the Itô formula for $p_l(t)$, $l=4,6,7$, denoting the martingale parts by symbol "mart" we can write

$$p_4(t)=\text{mart}+\int_0^t p_2(s)ds; \tag{4.27}$$

Recall that

$$\overline{w}(t)=w(t)-\int_0^t (B(s,\xi))^{-1}(m(s,\xi)-A(s,\xi,\eta))ds,$$

and w and v are independent, hence $<\overline{w},v>^{P,F}=0$. Using this fact we get

$$\begin{aligned}p_6(t)&=\text{mart}+\int_0^t \chi(s)(b^1(s)-b^2(s))^2G(s)ds\\&\le \text{mart}+\text{const}\cdot\Big(\sum_{i=1}^4\int_0^t p_i(s)ds\Big).\end{aligned} \tag{4.28}$$

The Itô formula applied to $p_7(t)$ gives

$$p_7(t) = \text{mart} + \int_0^t \chi(s)(c^1(s) - c^2(s))^2 G(s)ds$$
$$+ \int_0^t \int_0^s \chi(u)(c^1(u) - c^2(u))d\overline{w}(u)\chi(s)(c^1(s) - c^2(s))$$
$$\times \Big[(\varphi^1(s) - \varphi^2(s))\rho^1(s) + (\varphi^1(s) - \varphi^2(s))\rho^2(s)\Big]ds. \tag{4.29}$$

It is easy to see that

$$\int_0^t \chi(s)(c^1(s) - c^2(s))^2 G(s)ds \leq \text{const} \cdot \Big(\sum_{i=1}^4 \int_0^t p_i(s)ds\Big).$$

Since φ^l and m^l, $l = 1, 2$, are bounded, we have

$$\sum_{i=1}^2 |\psi^l(t) - m^l(t)|\rho^l(t) \leq \text{const} \cdot G(t).$$

Using the inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$, we write

$$\Big|\int_0^s \chi(s)(c^1(u) - c^2(u))d\overline{w}(u)\Big| \, \Big|\chi(s)(c^1(s) - c^2(s))\Big|$$
$$\leq \frac{1}{2}\Big[\Big(\int_0^s \chi(u)(c^1(u) - c^2(u))d\overline{w}(u)\Big)^2 + \chi(s)(c^1(s) - c^2(s))^2\Big].$$

Collecting the above inequalities and using the Lipschitz condition, from (4.29) we have

$$p_7(t) \leq \text{mart} + \text{const}\Big[\sum_{i=1}^4 \int_0^t p_i(s)ds + \int_0^t p_7(s)ds\Big]. \tag{4.30}$$

From (4.24), substituting all above estimates, we easily get

$$p_2(t) \leq \text{mart} + \text{const}\Big[p_3(t) + \sum_{i=1}^4 \int_0^t p_i(s)ds + p_5(t) + \int_0^t p_7(s)ds\Big]. \tag{4.31}$$

Now we need the following simple fact.

Consider the product space $(\Omega, \mathcal{F}, F, P) \times (C_{[0,T]}, \mathcal{B}_{[0,T]}, \mathbb{B}, \mu)$ and let the process $X = (X(t)) = (X(t, \omega, a), 0 \leq t \leq T, \omega \in \Omega, a \in C_{[0,T]})$ be the Itô process, with

$$X(t, \omega, a) = \int_0^t p(s, \omega, a)ds + \int_0^t y_1(t, \omega, a)d\overline{w}(t) + \int_0^t y_2(t, \omega, a)dv(t),$$

where $\overline{w}(t)=\overline{w}(t,\omega,a)=\overline{w}(t,\omega)$ and $v(t)=v(t,\omega,a)=v(t,a)$ are independent Brownian motions. Suppose that $p(t)=p(t,\omega,a)\geq 0$ and

$$\int_0^T E^p E^\mu y_1^2(t)dt+\int_0^T E^p E^\mu y_2^2(t)dt<\infty,$$

i.e., the martingale part of the process X is square integrable.

Then

$$E^p E^\mu X(t)=\int_0^t E^p E^\mu p(s)ds.$$

Note now that from the boundedness of coefficients it follows that we are in the framework of last statement. Hence averaging w.r.t. E^pE^μ (4.30) and (4.31), and then adding them, we write

$$E^p E^\mu (p_2(t)+p_7(t))\leq \text{const}\left[\int_0^t E^p E^\mu (p_2(s)+p_7(s))ds+z(t)\right]$$

(here we used (4.23), (4.24), (4.25), (4.26) and (4.27)).

Solving this inequality we get $E^pE^\mu(p_2(t)+p_7(t))\leq \text{const}\, z(t)$. But $p_7(t)\geq 0$. Hence

$$E^p E^\mu p_2(t)\leq E^p E^\mu (p_2(t)+p_7(t))\leq \text{const}\, z(t). \tag{4.32}$$

Now the desirable inequality (4.20) (the first one) follows from (4.22), (4.23), (4.32), (4.27) and (4.25).

The verification of the second inequality (4.20) is much simpler. Indeed, using explicit formula (4.10) for f^l, $l=1,2$, the boundedness of φ^l, m^l and $(B^l)^{-1}$, $l=1,2$, we easily reduce the desirable inequality to such inequalities, where the differences of type m^1-m^2 or $\varphi^1-\varphi^2$ participate. If we recall the definition of the function z (see (4.14)) and the just verified inequality (4.20), we easily conclude that the desirable inequality holds true as well. We do not stop on the details. The theorem is proved. □

Proof of Corollary 2.1. If in the scheme (2.2) we take $a=c=0$, $b=$Id and instead of v consider the process α, $\alpha\perp w$, we get $\eta(t)=\eta^0+\alpha(t)$ and the proof follows from the previous one. □

Proof of Corollary 2.2. Put in Corollary 2.1 $\varphi(t,x,\alpha)\equiv\varphi(t,\alpha):=\mu(t)$. □

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