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Some Fractional Special Functions and Fractional Moments

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Abstract

In this paper, we introduced a generalized moment generating function (GMGF) and showed that fractional moments, that is moments of order $nq-th$ of a certain distribution, was expressed in Caputa fractional derivation of GMGF in zero, n bieng a positive integer and $0 < q \leq 1$. The case $q = 1$ is reduced to the integer moments.

Keywords: *Fractional calculus (FC), Mittag-leffler function, Beta distribution, confluent hypergeometric function.*

1 Introduction

Classical moment generating function of random variable X, $M_X(t)$, is expectation of exponential function and multiplication $t^n \setminus n!$ at its Mac-laurin extention one, are the integer moments of that distribution. In this work paper we introduced a generalized moment generating ffunction GMGF.

In first, we defined the Mittag-leffler-type function $E_q(x)^q$, which we will call the generalized exponential function. this function is the produce of a Mittag-leffler function and a power function. This function allows us to directly obtain GMGF of a random variable, by using the fractional Taylor series (discussed in [3]). our mains of FC, for this generalization, were Caputa and Riemann-Liouville operatprs. Taking into consideration the presented method, the fractional moments of a certain distribution was expressed in Caputa fractional derivation of GMGF in zero. The interesting point is the relationship between fractional moments and fractional special functions.The special functions used here were generalized fractional confluent hypergeometric function (discussed in [3,4]) and generalized fractional Bell polynomials (discussed in [5]). For example, the fractional moments of Beta distribution was expressed in Caputa fractional derivation of generalized confluent hypergeometric function in zero. Moreover, the fractional moments of poisson distribution was expressed in Caputa fractional derivation of generalized Bell polynomials in zero. Therefore, this generating, in several aspects, corresponds with results from FC, namely (i) generalized Taylor series by Caputa fractional derivation, (ii) generalized fractional special functions.

The fractional Taylor series of an infinitely fractionally differentiable function is based on the Fundamental theorem of FC. By using of Fundamental theorem of FC, one can say that the right Caputa derivative operastion and the right Riemann-Liouville integral operation are inverse to each other. The reasons for using of Caputa derivative are (i) the Fundamental theorem of FC and (ii) using of the generalized exponential function expectation for defining of the GMGF of original probability distribution which produces arbitrary moments. It has also been shown that the GMGF equals with expectation of the generalized exponential function and therefor fractional moments concides with Caputa fractional derivation of GMGF in zero. The case $q = 1$ is reduced to the classical moment generating function of random variable X.

2 Problem Formulations

Definition 2.1 *Let $f(x)$ is a function defined on the interval $[a, b]$. Let q be a positive real number. The right Riemann-Liouville fractional integral is defined by:*

$$ {}_aI_x^q f(x) = \frac{1}{\Gamma(q)} \int_a^x (x-t)^{q-1} f(t)dt, \qquad -\infty \leq a < x < \infty \tag{1}$$

And also the right Riemann Liouville fractional derivative" is defined by:

$$ {}_aD_x^q f(x) = (\frac{d}{dx})^n({}_aI_x^{n-q}) = \frac{1}{\Gamma(n-q)}(\frac{d}{dx})^n \int_a^x \frac{f(t)}{(x-t)^{q-n+1}}dt, \qquad (2)$$

for $n = [q] + 1, a < x$ *[1,2].*

Definition 2.2 *Let* $n = [q] + 1$ *, the right Caputa fractional derivative is defined by*

$$ {}_a^cD_x^q f(x) = {}_a I_x^{n-q} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-q)} \int_a^x (x-t)^{n-q-1} \frac{d^n}{dt^n} f(t)dt, \qquad (3)$$

and the sequential fractional derivatives is given by:

$$ {}_a^cD_x^{kq} = \underbrace{{}_a^cD_x^q \ {}_a^cD_x^q \ \dots \ {}_a^cD_x^q}_{k-times} \qquad (4)$$

[1,2].

Definition 2.3 *Let* $f(x)$ *be a function defined on the right neighborhood of* a*, and be an infinitely fractionally- differentiable function at* a *, that is to say, all* $({}_a^cD_x^q)^k f(x), (k = 0, 1, 2, ...)$ *exist. The formal fractional right Riemann-Liouville Taylor series of a function is*

$$ f(x) = \sum_{k=0}^{\infty} ({}_a^cD_x^q)^k f(x)|_{x=a} \times [({}_aI_x^q)^k(1)] \qquad (5)$$

expilicity

$$ ({}_aI_x^q)^k(1) = \tfrac{1}{\Gamma(kq+1)}(x-a)^{kq}$$

where ${}_a^cD_x^q$ *is the right Caputa fractional derivative and* ${}_aI_x^q$ *is the right Riemann- Liouville fractional integral [3].*

Definition 2.4 *We define the generalized exponential function,* $E_q(x^q)$*, by the series*

$$ \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)}. \qquad (6)$$

The explicit solutions to the equation

$$({}_0^C D_x^q y)(x) - \lambda y(x) = 0, \qquad (x > 0,\ \ n-1 < q \leq n,\ \ n \in N,\quad \lambda \in R) \quad (7)$$

is in terms of this function, that is

$$y(x) = E_q(\lambda x^q). \quad (8)$$

Sequential fractional derivative of the function gives

$$({}_0^C D_x^{kq} y) = \lambda^k y. \quad (9)$$

and in general case

$${}_a^C D_x^q\ E_q((x-a)^q) = E_q((x-a)^q) \quad (10)$$

In addition, The generalized exponential function satisfied

$$E_q(0) = 1 \quad (11)$$

and

$$E_q(\lambda x^q) E_q(\lambda y^q) = E_q(\lambda (x+y)^q), \quad (12)$$

That is,$E_q(x^q)$ is the fractional analogue of $exp(x)$ The fractional Taylor series of this function is as following:

$$E_q((x-a)^q) = \sum_{k=0}^{\infty} [({}_a I_x^q)^k (1)] = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)}, \quad (13)$$

because ${}_a^C D_x^q\ E_q((x-a)^q)|_{x=a} = 1$. It can be seen that,

$$L\{E_q((x)^q)\} = \frac{s^{q-1}}{s^q - 1}, \quad (14)$$

where L is Laplace transform. With substitutions $q = 1,\ \ a = 0$, the results (7) t0 (14) valid for the elementary exponential function.

Definition 2.5 *The generalized moment generating function* $\tilde{M}_X(t)$*of any random variable* X*, is defined by*

$$\tilde{M_X}(t) = E[E_q((Xt)^q)] \quad (15)$$

where $E_q((xt)^q)$ *is the generalized exponential function and in the special case* $q = 1$ *we obtain the ordinary moment generating function*

$$M_X(t) = E[exp(Xt)]. \quad (16)$$

Definition 2.6 *Let $q \in [n-1, n)$ and $n = 1, 2, ...$ The generalized Bell polynomials of order q and $(-q)$ are*

$$B_q(x) = e^{-x}\ D_x^q e^{-x} \tag{17}$$

and

$$B_{-q}(x) = e^{-x}\ I_x^q e^{-x} \tag{18}$$

where D_x^q is the right Riemann–Liouville fractional derivative and I_x^q is the right Riemann–Liouville fractional integral [5].

3 Conclusion

Theorem 3.1

(i) *Suppose that the GMGF of a random variable X is finite in some open interval contaning zero. Then, all the moments exist and*

$$\tilde{M}_X(t) = \sum_{k=0}^{\infty} E(X^{kq}) \frac{t^{kq}}{\Gamma(kq+1)} \tag{19}$$

and the GMGF is infinitely fractionally differentiable in the open interval, and for $0 < q \leq 1$ and $k = 1, 2, ...$

$$E(X^{kq}) = ({}_0^C D_x^q)^k \tilde{M}_X(t)\mid_{t=0} \tag{20}$$

in the special case $q = 1$, we have

$$E(X^k) = ({}_0^C D_x)^k M_X(t)\mid_{t=0} = M_X^{(k)}(t)\mid_{t=0} \tag{21}$$

(ii) *If X and Y are independent random variables, then $X+Y$ has the GMGF*

$$\tilde{M}_{X+Y}(t) = \tilde{M}_X(t)\tilde{M}_Y(t). \tag{22}$$

Proof: (i) Since the fractional Mac-Lourin series of $E_q((x)^q)$ is

$\mathrm{E}_q((x)^q) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)}$

we have

$\tilde{M}_X(t) = E\{E_q((Xt)^q)\} = E\{\sum_{k=0}^{\infty}(Xt^{kq})\frac{t^{kq}}{\Gamma(kq+1)}\}$

$=\sum_{k=0}^{\infty} E(X^{kq})\frac{t^{kq}}{\Gamma(kq+1)}$.

In the other hand, by using (9), we have:

$({}_0^cD_x^q)^k\tilde{M_X}(t) = ({}_0^cD_x^q)^k(E\{E_q((Xt)^q)\})$

$=\text{E}\{({}_0^cD_x^q)^k(E_q((Xt)^q)\} = E\{X^{kq}E_q((Xt)^q)\}.$

(ii) By using (12), we have:

$\tilde{M_{X+Y}}(t) = E\{(E_{kq}((t(X+Y))^{kq})\} = E\{(E_{kq}((tX)^{kq}E_{kq}((tY)^{kq})\}.$

Theorem 3.2 *If $X \sim Beta(\alpha,\beta)$, then*

(i) *the GMGF equals*

$$\tilde{M_X}(t) =_1 \tilde{F}_1(\alpha;\alpha+\beta;t) = \sum_{k=0}^{\infty}\frac{(\alpha)_k^q}{(\alpha+\beta)_k^q}\frac{t^{kq}}{\Gamma(kq+1)} \tag{23}$$

where ${}_1\tilde{F}_1(\alpha;\alpha+\beta;t)$ denotes the generalized confluent hypergeometric function and

$$(a)_k^q = \begin{cases} 1 & ,k=0 \\ a & ,k=1 \\ (a)_1^q(a+\frac{\Gamma(q+1)}{\Gamma(1)})...(a+\frac{\Gamma(kq-q+1)}{\Gamma(kq-2q+1)}) & ,k\geq 2 \end{cases}$$

(ii) *the $(qk)^{th}$ moment equals*

$$E[X^{qk}] = \frac{(\alpha)_k^q}{(\alpha+\beta)_k^q} \tag{24}$$

in the special case $q=1$, we have

$$M_X(t) =_1 F_1(\alpha;\alpha+\beta;t) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\alpha+\beta)_k}\frac{t^k}{\Gamma(k+1)} \tag{25}$$

where

$$E[X^k] = \frac{(\alpha)_k}{(\alpha+\beta)_k}. \tag{26}$$

Proof: The proof (i) is considering simply with definition 1. For (ii), we have

$\text{E}_q((Xt)^q) = \sum_{k=0}^{\infty}(Xt^{kq})\frac{t^{kq}}{\Gamma(kq+1)}$

and so

$\tilde{M_X}(t) = E\{E_q((Xt)^q)\} = \sum_{k=0}^{\infty} E(X^{kq}) \frac{t^{kq}}{\Gamma(kq+1)}$

comparing recent express to (i), we obtain

$\mathrm{E}[\mathrm{X}^{qk}] = \frac{(\alpha)_k^q}{(\alpha+\beta)_k^q}$.

Theorem 3.3 *If $X \sim Poisson(\lambda)$, then*

(i) *the GMGF equals*

$$\tilde{M_X}(t) = \sum_{k=0}^{\infty} B_{qk}(\lambda) \frac{t^{kq}}{\Gamma(kq+1)} \tag{27}$$

where $B_{qk}(\lambda)$ denotes the generalized Bell polynomials and

(ii) *the $(qk)^{th}$ moment equals*

$$E[X^{qk}] = B_{qk}(\lambda) \tag{28}$$

in the special case $q = 1$, we have

$$M_X(t) = e^{\lambda(e^t-1)} = \sum_{k=0}^{\infty} B_k(\lambda) \frac{t^k}{\Gamma(k+1)} \tag{29}$$

that

$$E[X^k] = B_k(\lambda). \tag{30}$$

Proof: The proof is similar to the proof of theorem 3.2, we only out line this point that, for $0 < q < 1$, the right Caputa fractional derivative coincides with the right fractional Riemann- Liouville derivative in the following case:

$$({}_0^C D_x^q y)(x) = ({}_0 D_x^q y)(x), \qquad q \notin N_0. \tag{31}$$

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