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Regular Elements of Semigroups $B_X(D)$ Defined by the Generalized $X-$Semilattice

Barış Albayrak[1], Neşet Aydin[2] and Didem Yeşil Sungur[3]

[1,2,3]Canakkale Onsekiz Mart University, Art and Science Faculty
Department of Mathematics, Canakkale-Turkey
[1]E-mail: balbayrak77@gmail.com
[2]E-mail: neseta@comu.edu.tr
[3]E-mail: dyesil@comu.edu.tr

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Abstract

In this paper, we take $Q = \{T_1, T_2, \ldots, T_{m-3}, T_{m-2}, T_{m-1}, T_m\}$ *subsemilattice of* $X-$*semilattice of unions* D *where the elements* T_i*'s are satisfying the following properties,* $T_1 \subset T_3 \subset \cdots \subset T_{m-3} \subset T_{m-2} \subset T_m$, $T_1 \subset T_3 \subset \cdots \subset T_{m-3} \subset T_{m-1} \subset T_m$, $T_2 \subset T_3 \subset \cdots \subset T_{m-3} \subset T_{m-2} \subset T_m$, $T_2 \subset T_3 \subset \cdots \subset T_{m-3} \subset T_{m-1} \subset T_m$, $T_1 \backslash T_2 \neq \emptyset$, $T_2 \backslash T_1 \neq \emptyset$, $T_{m-2} \backslash T_{m-1} \neq \emptyset$, $T_{m-1} \backslash T_{m-2} \neq \emptyset$, $T_1 \cup T_2 = T_3$, $T_{m-2} \cup T_{m-1} = T_m$. *We will investigate the properties of regular element* $\alpha \in B_X(D)$ *satisfying* $V(D, \alpha) = Q$. *Moreover, we will calculate the number of regular elements of* $B_X(D)$ *for a finite set* X.

Keywords: *Semigroups, Binary relations, Regular elements.*

1 Introduction

Let X be an arbitrary nonempty set and B_X be semigroup of all binary relations on the set X. If D is a nonempty family of subsets of X which is closed under the union then D is called a *complete* $X-$ *semilattice of unions*. The union of all elements of D is denoted by the symbol $\breve{D}$.

Further, let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X$, $T \in D$, $\emptyset \neq D' \subseteq D$ and $t \in \breve{D}$.

Then we have the following notations,

$$y\alpha = \{x \in X \mid (y,x) \in \alpha\},\ Y\alpha = \bigcup_{y \in Y} y\alpha,\ V(D,\alpha) = \{Y\alpha \mid Y \in D\}$$

$$D_t = \{Z' \in D \mid t \in Z'\},\ D'_T = \{Z' \in D' \mid T \subseteq Z'\},\ \ddot{D}'_T = \{Z' \in D' \mid Z' \subseteq T\}$$
$$N(D,D') = \{Z \in D \mid Z \subseteq Z' \text{ for any } Z' \in D'\},\ \Lambda(D,D') = \cup N(D,D')$$

Let f be an arbitrary mapping from X into D. Then one can construct a binary relation α_f on X by $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$. The set of all such binary relations is denoted by $B_X(D)$ and called a *complete semigroup of binary relations* defined by an $X-$semilattice of unions D. This structure was comprehensively investigated in Diasamidze [1].

Let D be a complete $X-$semilattice of unions. If it satisfies $\Lambda(D,D_t) \in D$ for any $t \in \breve{D}$ and $Z = \bigcup_{t \in Z} \Lambda(D,D_t)$ for any nonempty element Z of D, then D is called $XI-$ *semilattice of unions.* $\alpha \in B_X(D)$ is called *idempotent* if $\alpha \circ \alpha = \alpha$ and $\alpha \in B_X(D)$ said to be *regular* if $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X(D)$. Let D' be an arbitrary nonempty subset of the complete $X-$semilattice of unions D. Set $l(D',T) = \cup(D' \backslash D'_T)$. We say that a nonempty element T is a *nonlimiting element* of D' if $T \backslash l(D',T) \neq \emptyset$. Also, a nonempty element T said to be *limiting element* of D' if $T \backslash l(D',T) = \emptyset$.

The family $C(D)$ of pairwise disjoint subsets of the set $\breve{D} = \cup D$ is the *characteristic family* of sets of D if the following hold

a) $\cap D \in C(D)$

b) $\cup C(D) = \breve{D}$

c) There exists a subset $C_Z(D)$ of the set $C(D)$ such that $Z = \cup C_Z(D)$ for all $Z \in D$.

A mapping $\theta : D \to C(D)$ is called *characteristic mapping* if $Z = (\cap D) \cup \bigcup_{Z' \in \hat{D}} \theta(Z')$ for all $Z \in D$. The existence and the uniqueness of characteristic family and characteristic mapping is given in Diasamidze [3]. Moreover, it is shown that every $Z \in D$ can be written as $Z = \theta(\breve{Q}) \cup \bigcup_{T \in \hat{Q}(Z)} \theta(T)$, where $\hat{Q}(Z) = Q \backslash \{T \in Q \mid Z \subseteq T\}$.

Definitions and properties of $\Phi(D,D')$, $\Omega(D)$, $R(D')$ and $R_\varphi(D,D')$ can be found in [1], [2] and [5].

In [5], they found that the properties of regular element $\alpha \in B_X(D)$ which satisfying $V(D,\alpha) = Q$ for $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$ with seven elements. Therefore, we generalized the results which found in [5] for $Q = \{T_1, T_2, \ldots, T_{m-3}, T_{m-2}, T_{m-1}, T_m\}$ with m elements.

Now we state two theorems which will be used later.

Theorem 1.1. *[4, Theorem 10] Let α and σ be binary relations of the semigroup $B_X(D)$ such that $\alpha\circ\sigma\circ\alpha=\alpha$. If $D(\alpha)$ is some generating set of the semilattice $V(D,\alpha)\setminus\{\emptyset\}$ and $\alpha=\bigcup_{T\in V(D,\alpha)}(Y_T^{\alpha}\times T)$ is a quasinormal representation of the relation α, then $V(D,\alpha)$ is a complete $XI-$ semilattice of unions. Moreover, there exists a complete isomorphism φ between the semilattice $V(D,\alpha)$ and $D'=\{T\sigma \mid T\in V(D,\alpha)\}$, that satisfies the following conditions:*

a) $\varphi(T)=T\sigma$ and $\varphi(T)\alpha=T$ for all $T\in V(D,\alpha)$

b) $\bigcup_{T'\in \ddot{D}(\alpha)_T} Y_{T'}^{\alpha}\supseteq\varphi(T)$ for any $T\in D(\alpha)$,

c) $Y_T^{\alpha}\cap\varphi(T)\neq\emptyset$ for all nonlimiting element T of the set $\ddot{D}(\alpha)_T$,

d) If T is a limiting element of the set $\ddot{D}(\alpha)_T$, then the equality $\cup B(T)=T$ is always holds for the set $B(T)=\left\{Z\in\ddot{D}(\alpha)_T \mid Y_Z^{\alpha}\cap\varphi(T)\neq\emptyset\right\}$.

On the other hand, if $\alpha\in B_X(D)$ such that $V(D,\alpha)$ is a complete $XI-$ semilattice of unions and if some complete $\alpha-$isomorphism φ from $V(D,\alpha)$ to a subsemilattice D' of D satisfies the conditions $b)-d)$ of the theorem, then α is a regular element of $B_X(D)$.

Theorem 1.2. *[2, Theorem 6.3.5] Let X be a finite set. If φ is a fixed element of the set $\Phi(D,D')$ and $|\Omega(D)|=m_0$ and q is a number of all automorphisms of the semilattice D then $|R(D')|=m_0\cdot q\cdot|R_{\varphi}(D,D')|$.*

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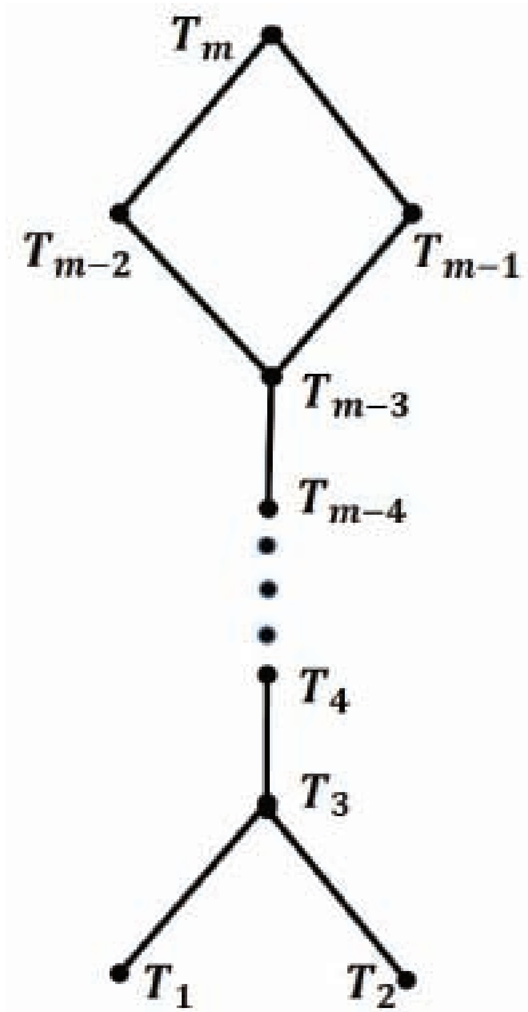
2 Results

Let X be a finite set, D be a complete $X-$semilattice of unions, $m\geq 7$ and $Q=\left\{T_1,T_2,T_3,\ldots,T_{m-3},T_{m-2},T_{m-1},T_m\right\}$ be a $X-$subsemilattice of unions of D satisfies the following conditions.

$$\begin{array}{l}
T_1\subset T_3\subset T_4\subset\cdots\subset T_{m-3}\subset T_{m-2}\subset T_m,\\
T_2\subset T_3\subset T_4\subset\cdots\subset T_{m-3}\subset T_{m-2}\subset T_m,\\
T_1\subset T_3\subset T_4\subset\cdots\subset T_{m-3}\subset T_{m-1}\subset T_m,\\
T_2\subset T_3\subset T_4\subset\cdots\subset T_{m-3}\subset T_{m-1}\subset T_m,\\
T_{m-1}\setminus T_{m-2}\neq\emptyset, T_{m-2}\setminus T_{m-1}\neq\emptyset,\\
T_1\setminus T_2\neq\emptyset, T_2\setminus T_1\neq\emptyset,\\
T_1\cup T_2=T_3, T_{m-2}\cup T_{m-1}=T_m.
\end{array}$$

The diagram of the Q is shown in the following figure.



Let $C(Q) = \{P_i \mid i = 1, 2, \ldots, m\}$ be a characteristic family of sets of Q. The assignment $\varphi(T_i) = P_i$ defines a one to one correspondence between Q and $C(Q)$. Then $T_m = P_m \cup P_{m-1} \cup P_{m-2} \cup \cdots \cup P_1$, $T_{m-1} = P_m \cup P_{m-2} \cup \cdots \cup P_1$, $T_{m-2} = P_m \cup P_{m-1} \cup P_{m-3} \cup \cdots \cup P_1$, $T_{m-3} = P_m \cup P_{m-4} \cup \cdots \cup P_1$, …, $T_4 = P_m \cup P_3 \cup P_2 \cup P_1$, $T_3 = P_m \cup P_2 \cup P_1$, $T_2 = P_m \cup P_1$, $T_1 = P_m \cup P_2$ are obtained.

Now, let us investigate that in which conditions Q is an $XI-$ semilattice of unions. First, we determine the greatest lower bounds of the each semilattice Q_t in Q for $t \in T_m$. Since $T_m = P_m \cup P_{m-1} \cup P_{m-2} \cup \cdots \cup P_1$ and P_i

$(i=1,2,\ldots,m)$ are pairwise disjoint sets, it follows

$$Q_t=\begin{cases} Q & ,t\in P_m \\ \{T_m,T_{m-2}\} & ,t\in P_{m-1} \\ \{T_m,T_{m-1}\} & ,t\in P_{m-2} \\ \{T_m,T_{m-1},T_{m-2}\} & ,t\in P_{m-3} \\ \{T_m,T_{m-1},T_{m-2},T_{m-3}\} & ,t\in P_{m-4} \\ \vdots & \vdots \\ \{T_m,\ldots,T_5\} & ,t\in P_4 \\ \{T_m,\ldots,T_4\} & ,t\in P_3 \\ \{T_m,\ldots,T_3,T_1\} & ,t\in P_2 \\ \{T_m,\ldots,T_3,T_2\} & ,t\in P_1 \end{cases}. \tag{2.1}$$

From the Equation (2.1) the greatest lower bounds for each semilattice Q_t

$$\begin{array}{lll} t\in P_m & \Rightarrow N(Q,Q_t)=\emptyset & \Rightarrow \Lambda(Q,Q_t)=\emptyset \\ t\in P_{m-1} & \Rightarrow N(Q,Q_t)=\{T_{m-2},T_{m-3},\ldots,T_1\} & \Rightarrow \Lambda(Q,Q_t)=T_{m-2} \\ t\in P_{m-2} & \Rightarrow N(Q,Q_t)=\{T_{m-1},T_{m-3},\ldots,T_1\} & \Rightarrow \Lambda(Q,Q_t)=T_{m-1} \\ t\in P_{m-3} & \Rightarrow N(Q,Q_t)=\{T_{m-3},\ldots,T_1\} & \Rightarrow \Lambda(Q,Q_t)=T_{m-3} \\ t\in P_{m-4} & \Rightarrow N(Q,Q_t)=\{T_{m-3},\ldots,T_1\} & \Rightarrow \Lambda(Q,Q_t)=T_{m-3} \\ \vdots & \vdots & \vdots \\ t\in P_4 & \Rightarrow N(Q,Q_t)=\{T_5,\ldots,T_1\} & \Rightarrow \Lambda(Q,Q_t)=T_5 \\ t\in P_3 & \Rightarrow N(Q,Q_t)=\{T_4,\ldots,T_1\} & \Rightarrow \Lambda(Q,Q_t)=T_4 \\ t\in P_2 & \Rightarrow N(Q,Q_t)=\{T_1\} & \Rightarrow \Lambda(Q,Q_t)=T_1 \\ t\in P_1 & \Rightarrow N(Q,Q_t)=\{T_2\} & \Rightarrow \Lambda(Q,Q_t)=T_2 \end{array} \tag{2.2}$$

are obtained. If $t\in P_m$ then $\Lambda(D,D_t)=\emptyset\notin D$. So, $P_m=\emptyset$. Also using the Equation (2.2), we have easily seen that $\bigcup_{t\in T_i}\Lambda(Q,Q_t)\in D$.

Lemma 2.1. *Q is an $XI-$ semilattice of unions if and only if $T_1\cap T_2=\emptyset$*

Proof. $\Rightarrow$: Let Q be an $XI-$ semilattice of unions. Then $P_m=\emptyset$ and $T_1=P_2$, $T_2=P_1$ by Equation (2.1). Therefore $T_1\cap T_2=\emptyset$ since P_1 and P_2 are pairwise disjoint sets. $\Leftarrow$: If $T_1\cap T_2=\emptyset$, then $P_m=\emptyset$. Using the Equation (2.2), we see that $\bigcup_{t\in T_i}\Lambda(Q,Q_t)=T_i$. So, we have Q is an $XI-$ semilattice of unions. □

Lemma 2.2. *Let $G=\{T_1,T_2,\ldots,T_{m-1}\}$ be a generating set of Q. Then the elements $T_1,T_2,T_4,T_5,\ldots,T_{m-1}$ are nonlimiting elements of the set $\ddot{G}_{T_1}$, $\ddot{G}_{T_2}$, $\ddot{G}_{T_4}$, $\ddot{G}_{T_5},\ldots,\ddot{G}_{T_{m-1}}$ respectively and T_3 is limiting element of the set $\ddot{G}_{T_3}$.*

Proof. Definition of $\ddot{D}'_T$ and $l(\ddot{G}_{T_i}, T_i) = \cup(\ddot{G}_{T_i} \setminus \{T_i\})$, $i \in \{1, 2, \ldots, m-1\}$, we find nonlimiting and limiting elements of $\ddot{G}_{T_i}$.

$$\begin{array}{ll}
T_1 \setminus l(\ddot{G}_{T_1}, T_1) = T_1 \setminus \emptyset = T_1 \neq \emptyset, & T_1 \text{ nonlimiting element of } \ddot{G}_{T_1} \\
T_2 \setminus l(\ddot{G}_{T_2}, T_2) = T_2 \setminus \emptyset = T_2 \neq \emptyset, & T_2 \text{ nonlimiting element of } \ddot{G}_{T_2} \\
T_3 \setminus l(\ddot{G}_{T_3}, T_3) = T_3 \setminus T_3 = \emptyset, & T_3 \text{ limiting element of } \ddot{G}_{T3} \\
T_4 \setminus l(\ddot{G}_{T_4}, T_4) = T_4 \setminus T_3 \neq \emptyset, & T_4 \text{ nonlimiting element of } \ddot{G}_{T_4} \\
\vdots & \vdots \\
T_{m-4} \setminus l(\ddot{G}_{T_{m-4}}, T_{m-4}) = T_{m-4} \setminus T_{m-5} \neq \emptyset, & T_{m-4} \text{ nonlimiting element of } \ddot{G}_{T_{m-4}} \\
T_{m-3} \setminus l(\ddot{G}_{T_{m-3}}, T_{m-3}) = T_{m-3} \setminus T_{m-4} \neq \emptyset, & T_{m-3} \text{ nonlimiting element of } \ddot{G}_{T_{m-3}} \\
T_{m-2} \setminus l(\ddot{G}_{T_{m-2}}, T_{m-2}) = T_{m-2} \setminus T_{m-3} \neq \emptyset, & T_{m-2} \text{ nonlimiting element of } \ddot{G}_{T_{m-2}} \\
T_{m-1} \setminus l(\ddot{G}_{T_{m-1}}, T_{m-1}) = T_{m-1} \setminus T_{m-3} \neq \emptyset, & T_{m-1} \text{ nonlimiting element of } \ddot{G}_{T_{m-1}}
\end{array}$$

□

Now, we determine properties of a regular element α of $B_X(Q)$ where $V(D, \alpha) = Q$ and $\alpha = \bigcup_{i=1}^{m} (Y_i^\alpha \times T_i)$.

Theorem 2.3. *Let $\alpha \in B_X(Q)$ with a quasinormal representation of the form $\alpha = \bigcup_{i=1}^{m} (Y_i^\alpha \times T_i)$ such that $V(D, \alpha) = Q$. Then $\alpha \in B_X(D)$ is a regular iff for some complete α-isomorphism $\varphi : Q \to D' \subseteq D$, the following conditions are satisfied:*

$$\begin{array}{l}
Y_1^\alpha \supseteq \varphi(T_1), \\
Y_2^\alpha \supseteq \varphi(T_2), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4), \\
\vdots \\
Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-4}^\alpha \supseteq \varphi(T_{m-4}), \\
Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \supseteq \varphi(T_{m-3}), \\
Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-2}^\alpha \supseteq \varphi(T_{m-2}), \\
Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-1}^\alpha \supseteq \varphi(T_{m-1}), \\
Y_4^\alpha \cap \varphi(T_4) \neq \emptyset, \ldots, \; Y_{m-1}^\alpha \cap \varphi(T_{m-1}) \neq \emptyset
\end{array} \tag{2.3}$$

Proof. Let $G = \{T_1, T_2, \ldots, T_{m-1}\}$ be a generating set of Q. ⇒: Since $\alpha \in B_X(D)$ is regular and $V(D, \alpha) = Q$ is an $XI-$semilattice of unions, by Theorem 1.1, there exits a complete $\alpha-$isomorphism $\varphi : Q \to D'$. By Theorem 1.1

(a), $\varphi(T)\alpha = T$ for all $T \in V(D,\alpha)$. Applying the Theorem 1.1 (b), we have

$$\begin{array}{l} Y_1^\alpha \supseteq \varphi(T_1),\ Y_2^\alpha \supseteq \varphi(T_2), \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3),\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4), \\ \vdots \\ Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-4}^\alpha \supseteq \varphi(T_{m-4}),\ Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \supseteq \varphi(T_{m-3}), \\ Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-2}^\alpha \supseteq \varphi(T_{m-2}),\ Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-1}^\alpha \supseteq \varphi(T_{m-1}) \end{array}$$

Moreover, considering that the elements $T_1, T_2, T_4, T_5, \ldots, T_{m-1}$ are nonlimiting elements of the sets $\ddot{G}_{T_1}$, $\ddot{G}_{T_2}$, $\ddot{G}_{T_4}$, $\ddot{G}_{T_5}, \ldots, \ddot{G}_{T_{m-1}}$ respectively and using the Theorem 1.1 (c), following properties

$$Y_1^\alpha \cap \varphi(T_1) \neq \emptyset,\ Y_2^\alpha \cap \varphi(T_2) \neq \emptyset, Y_4^\alpha \cap \varphi(T_4) \neq \emptyset, \ldots,\ Y_{m-1}^\alpha \cap \varphi(T_{m-1}) \neq \emptyset,$$

are obtained. From $Y_1^\alpha \supseteq \varphi(T_1)$ and $Y_2^\alpha \supseteq \varphi(T_2)$; $Y_1^\alpha \cap \varphi(T_1) \neq \emptyset$, $Y_2^\alpha \cap \varphi(T_2) \neq \emptyset$ always ensured. Also by using $Y_1^\alpha \supseteq \varphi(T_1)$ and $Y_2^\alpha \supseteq \varphi(T_2)$, we get

$$\begin{aligned} Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha &\supseteq \varphi(T_1) \cup \varphi(T_2) \cup Y_3^\alpha = \varphi(T_1 \cup T_2) \cup Y_3^\alpha \\ &= \varphi(T_3) \cup Y_5^\alpha \supseteq \varphi(T_3) \end{aligned}$$

Thus there is no need the condition $Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3)$. Therefore there exists a complete $\alpha-$isomorphism φ which holds given conditions. $\Leftarrow$: Since $V(D,\alpha) = Q$, $V(D,\alpha)$ is an $XI-$semilattice of unions. Hence we have a complete $\alpha-$isomorphism φ satisfying (2.3). Notice that T_3 is a limiting element of the set $\ddot{G}_{T_3}$. By Theorem 1.1 we form $B(T_3) = \left\{ Z \in \ddot{G}_{T_3} \mid Y_Z^\alpha \cap \varphi(T_3) \neq \emptyset \right\}$. It was seen in [5, Theorem 3.4] that $\cup B(T_3) = T_3$. By Theorem 1.1 we conclude that α is the regular element of the $B_X(D)$. □

Now we calculate the number of regular elements α, satisfying the hyphothesis of Theorem 2.3. Let $\alpha \in B_X(D)$ be a regular element which is quasinormal representation of the form $\alpha = \bigcup_{i=1}^{m} (Y_i^\alpha \times T_i)$ and $V(D,\alpha) = Q$. Then there exist a complete $\alpha-$ isomorphism $\varphi : Q \to D' = \{\varphi(T_1), \varphi(T_2), \ldots, \varphi(T_m)\}$ satisfying the hyphothesis of Theorem 2.3. So, $\alpha \in R_\varphi(Q, D')$. We will denote $\varphi(T_i) = \overline{T}_i,\ i = 1, 2, \ldots m$. Diagram of the $D' = \left\{ \overline{T}_1,\ \overline{T}_2, \ldots,\ \overline{T}_m \right\}$ is shown in the following figure.

Then the Equation (2.3) reduced to below equation.

$$\begin{aligned}
&Y_1^\alpha \supseteq \overline{T}_1,\\
&Y_2^\alpha \supseteq \overline{T}_2,\\
&Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \supseteq \overline{T}_4,\\
&\vdots\\
&Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-4}^\alpha \supseteq \overline{T}_{m-4},\\
&Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \supseteq \overline{T}_{m-3},\\
&Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-2}^\alpha \supseteq \overline{T}_{m-2},\\
&Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-1}^\alpha \supseteq \overline{T}_{m-1},\\
&Y_4^\alpha \cap \overline{T}_4 \neq \emptyset, \ldots, \ Y_{m-1}^\alpha \cap \overline{T}_{m-1} \neq \emptyset
\end{aligned} \tag{2.4}$$

On the other hand, $\overline{T}_1$,$\overline{T}_2$,$\overline{T}_4 \backslash \overline{T}_3, \ldots$,$\overline{T}_{m-4} \backslash \overline{T}_{m-5}$, $(\overline{T}_{m-2} \cap \overline{T}_{m-1}) \backslash \overline{T}_{m-4}$, $\overline{T}_{m-1} \backslash \overline{T}_{m-2}$, $\overline{T}_{m-2} \backslash \overline{T}_{m-1}$, $X \backslash \overline{T}_m$ are also pairwise disjoint sets and union of these sets equals X.

Lemma 2.4. *For every $\alpha \in R_\varphi(Q, D')$, there exists an ordered system of disjoint mappings.*

Proof. Let $f_\alpha : X \to D$ be a mapping satisfying the condition $f_\alpha(t) = t\alpha$ for all $t \in X$. We consider the restrictions of the mapping f_α as $f_{1\alpha}$, $f_{2\alpha}, \ldots, f_{(m-1)\alpha}$ on the sets $\overline{T}_1, \overline{T}_2, \overline{T}_4 \backslash \overline{T}_3, \overline{T}_5 \backslash \overline{T}_4, \ldots, \overline{T}_{m-4} \backslash \overline{T}_{m-5}, (\overline{T}_{m-2} \cap \overline{T}_{m-1}) \backslash \overline{T}_{m-4}$, $\overline{T}_{m-1} \backslash \overline{T}_{m-2}$, $\overline{T}_{m-2} \backslash \overline{T}_{m-1}$, $X \backslash \overline{T}_m$, respectively. Now, considering the definition of the sets Y_i^α, $(i = 1, 2, \ldots, m-1)$ together with the Equation (2.4) we have,

$$t \in \overline{T}_1 \Rightarrow t \in Y_1^\alpha \Rightarrow t\alpha = T_1 \Rightarrow f_{1\alpha}(t) = T_1,\ \forall t \in \overline{T}_1$$

$$t \in \overline{T}_2 \Rightarrow t \in Y_2^\alpha \Rightarrow t\alpha = T_2 \Rightarrow f_{2\alpha}(t) = T_2,\ \forall t \in \overline{T}_2$$

$$\begin{aligned} t \in \overline{T}_i \backslash \overline{T}_{i-1}, (i = 4, \ldots, m-4) &\Rightarrow t \in \overline{T}_i \backslash \overline{T}_{i-1} \subseteq \overline{T}_i \subseteq Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_i^\alpha \\ &\Rightarrow t\alpha \in \{T_1, T_2, \ldots, T_i\} \\ &\Rightarrow f_{(i-1)\alpha}(t) \in \{T_1, T_2, \ldots, T_i\},\ \forall t \in \overline{T}_i \backslash \overline{T}_{i-1} \end{aligned}$$

$$\begin{aligned} t \in (\overline{T}_{m-2} \cap \overline{T}_{m-1}) \backslash \overline{T}_{m-4} &\Rightarrow t \in \overline{T}_{m-2} \cap \overline{T}_{m-1} \subseteq Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \\ &\Rightarrow t\alpha \in \{T_1, \ldots, T_{m-3}\} \\ &\Rightarrow f_{(m-4)\alpha}(t) \in \{T_1, \ldots, T_{m-3}\}, \\ &\quad \forall t \in (\overline{T}_{m-2} \cap \overline{T}_{m-1}) \backslash \overline{T}_{m-4} \end{aligned}$$

$$\begin{aligned} t \in \overline{T}_{m-1} \backslash \overline{T}_{m-2} &\Rightarrow t \in \overline{T}_{m-1} \subseteq Y_1^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-1}^\alpha \\ &\Rightarrow t\alpha \in \{T_1, \ldots, T_{m-3}, T_{m-1}\} \\ &\Rightarrow f_{(m-3)\alpha}(t) \in \{T_1, \ldots, T_{m-3}, T_{m-1}\},\ \forall t \in \overline{T}_{m-1} \backslash \overline{T}_{m-2} \end{aligned}$$

$$\begin{aligned} t \in \overline{T}_{m-2} \backslash \overline{T}_{m-1} &\Rightarrow t \in \overline{T}_{m-2} \subseteq Y_1^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-2}^\alpha \\ &\Rightarrow t\alpha \in \{T_1, \ldots, T_{m-3}, T_{m-2}\} \\ &\Rightarrow f_{(m-2)\alpha}(t) \in \{T_1, \ldots, T_{m-3}, T_{m-2}\},\ \forall t \in \overline{T}_{m-2} \backslash \overline{T}_{m-1} \end{aligned}$$

$$t \in X \backslash \overline{T}_m \Rightarrow t \in X \backslash \overline{T}_m \subseteq X = \bigcup_{i=1}^{m} Y_i^\alpha \Rightarrow t\alpha \in Q \Rightarrow f_{(m-1)\alpha}(t) \in Q, \forall t \in X \backslash \overline{T}_m$$

Besides, $Y_i^\alpha \cap \overline{T}_i \neq \emptyset$ so there is an element $t_i \in Y_i^\alpha \cap \overline{T}_{i-1}$. Then $t_i\alpha = T_i$ and $t_i \in \overline{T}_i$. If $t_i \in \overline{T}_{i-1}$, then $t_i \in \overline{T}_{i-1} \subseteq Y_1^\alpha \cup \cdots \cup Y_{i-1}^\alpha$. Thus $t_i\alpha \in \{T_1, \ldots, T_{i-1}\}$ which is a contradiction with the equality $t_i\alpha = T_i$. So, there is an element $t_i \in \overline{T}_i \backslash \overline{T}_{i-1}$ such that $f_{(i-1)\alpha}(t_i) = T_i$. Similarly, $f_{(m-4)\alpha}(t_{m-3}) = T_{m-3}$ for some $t_{m-3} \in \overline{T}_{m-3} \backslash \overline{T}_{m-4}$, $f_{(m-3)\alpha}(t_{m-1}) = T_{m-1}$ for some $t_{m-1} \in \overline{T}_{m-1} \backslash \overline{T}_{m-2}$, $f_{(m-2)\alpha}(t_{m-2}) = T_{m-2}$ for some $t_{m-2} \in \overline{T}_{m-2} \backslash \overline{T}_{m-1}$ since $Y_{m-5}^\alpha \cap \overline{T}_{m-5} \neq \emptyset$, $Y_{m-2}^\alpha \cap \overline{T}_{m-2} \neq \emptyset$, $Y_{m-1}^\alpha \cap \overline{T}_{m-1} \neq \emptyset$. Therefore, for every $\alpha \in R_\varphi(Q, D')$ there exists an ordered system $(f_{1\alpha}, f_{2\alpha}, \ldots, f_{(m-1)\alpha})$. On the other hand, suppose that for $\alpha, \beta \in R_\varphi(Q, D')$ which $\alpha \neq \beta$, be obtained $f_\alpha = (f_{1\alpha}, f_{2\alpha}, \ldots, f_{(m-1)\alpha})$ and $f_\beta = (f_{1\beta}, f_{2\beta}, \ldots, f_{(m-1)\beta})$. If $f_\alpha = f_\beta$, we get

$$f_\alpha = f_\beta \Rightarrow f_\alpha(t) = f_\beta(t),\ \forall t \in X \Rightarrow t\alpha = t\beta,\ \forall t \in X \Rightarrow \alpha = \beta$$

which contradicts to $\alpha \neq \beta$. Therefore different binary relations's ordered systems are different. $\square$

Lemma 2.5. *Let Q be an $XI-$semilattice of unions and $f = (f_1, f_2, \ldots, f_{(m-1)})$ be ordered system from X in the semilattice D such that*

f_1:$\overline{T}_1 \to \{T_1\}$,$f_1(t) = T_1$,
f_2:$\overline{T}_2 \to \{T_1\}$,$f_2(t) = T_2$,
f_{i-1}: $\overline{T}_i \backslash \overline{T}_{i-1} \to \{T_1, \ldots, T_i\}$, $(i = 4, \ldots, m-4)$, $f_{i-1}(t) \in \{T_1, \ldots, T_i\}$
and $f_{i-1}(t_i) = T_i$, $\exists\ t_i \in \overline{T}_i \backslash \overline{T}_{i-1}$,
f_{m-4}: $(\overline{T}_{m-2} \cap \overline{T}_{m-1}) \backslash \overline{T}_{m-4} \to \{T_1, \ldots, T_{m-3}\}$, $f_{m-4}(t) \in \{T_1, \ldots, T_{m-3}\}$
and $f_{m-4}(t_{m-3}) = T_{m-3}$, $\exists\ t_{m-3} \in \overline{T}_{m-3} \backslash \overline{T}_{m-4}$,
f_{m-3}: $\overline{T}_{m-1} \backslash \overline{T}_{m-2} \to \{T_1, \ldots, T_{m-3}, T_{m-1}\}$, $f_{m-3}(t) \in \{T_1, \ldots, T_{m-3}, T_{m-1}\}$
and $f_{m-3}(t_{m-1}) = T_{m-1}$, $\exists\ t_{m-1} \in \overline{T}_{m-1} \backslash \overline{T}_{m-2}$,
f_{m-2}: $\overline{T}_{m-2} \backslash \overline{T}_{m-1} \to \{T_1, \ldots, T_{m-3}, T_{m-2}\}$, $f_{m-2}(t) \in \{T_1, \ldots, T_{m-3}, T_{m-2}\}$
and $f_{m-2}(t_{m-2}) = T_{m-2}$, $\exists\ t_{m-2} \in \overline{T}_{m-2} \backslash \overline{T}_{m-1}$,
f_{m-1}: $X \backslash \overline{T}_m \to Q$, $f_{m-1}(t) \in Q$.

Then $\beta = \bigcup_{x \in X} (\{x\} \times f(x)) \in B_X(D)$ *is regular and* φ *is complete* $\beta-$*isomorphism. So* $\beta \in R_\varphi(Q, D')$.

Proof. First we see that $V(D, \beta) = Q$. Considering $V(D, \beta) = \{Y\beta \mid Y \in D\}$, the properties of f mapping, $\overline{T}_i \beta = \bigcup_{x \in \overline{T}_i} x\beta$ and $D' \subseteq D$, we get $V(D, \beta) = Q$.

Also, $\beta = \bigcup_{T \in V(X^*, \beta)} \left(Y_T^\beta \times T\right)$ is quasinormal representation of β since $\emptyset \notin Q$. From the definition of β, $f(x) = x\beta$ for all $x \in X$. It is easily seen that $V(X^*, \beta) = V(D, \beta) = Q$. We get $\beta = \bigcup_{i=1}^{m} \left(Y_i^\beta \times T_i\right)$. On the other hand

$$t \in \overline{T}_1 \Rightarrow t\beta = f(t) = T_1 \Rightarrow t \in Y_1^\beta \Rightarrow \overline{T}_1 \subseteq Y_1^\beta,$$
$$t \in \overline{T}_2 \Rightarrow t\beta = f(t) = T_2 \Rightarrow t \in Y_2^\beta \Rightarrow \overline{T}_2 \subseteq Y_2^\beta,$$
$$t \in \overline{T}_i, (i = 4, \ldots, m-4) \Rightarrow t\beta \in \{T_1, T_2, \ldots, T_i\} \Rightarrow t \in Y_1^\beta \cup Y_2^\beta \cup \cdots \cup Y_1^\beta \Rightarrow Y_1^\beta \cup Y_2^\beta \cup \cdots \cup Y_1^\beta \supseteq \overline{T_i}$$
$$t \in \overline{T}_{m-3} \Rightarrow t\beta \in \{T_1, \ldots, T_{m-3}\} \Rightarrow t \in Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \Rightarrow Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \supseteq \overline{T}_{m-3}$$
$$t \in \overline{T}_{m-2} \Rightarrow t\beta \in \{T_1, \ldots, T_{m-3}, T_{m-2}\} \Rightarrow t \in Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-2}^\alpha \Rightarrow Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-2}^\alpha \supseteq \overline{T}_{m-2}$$
$$t \in \overline{T}_{m-1} \Rightarrow t\beta \in \{T_1, \ldots, T_{m-3}, T_{m-1}\} \Rightarrow t \in Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-1}^\alpha \Rightarrow Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-1}^\alpha \supseteq \overline{T}_{m-1}$$

Also, for $i = 4, \ldots, m-4$ by using $f_{i-1}(t_i) = T_i$, $\exists\ t_4 \in \overline{T}_i \backslash \overline{T}_{i-1}$, we obtain $Y_i^{\beta} \cap \overline{T}_i \neq \emptyset$. Similarly, $Y_{m-3}^{\beta} \cap \overline{T}_{m-3} \neq \emptyset$, $Y_{m-2}^{\beta} \cap \overline{T}_{m-2} \neq \emptyset$ and $Y_{m-1}^{\beta} \cap \overline{T}_{m-1} \neq \emptyset$. Therefore the mapping $\varphi : Q \to D' = \left\{\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_m\right\}$ to be defined $\varphi(T_i) = \overline{T}_i$ satisfies the conditions in the Equation (2.4) for β. Hence φ is complete $\beta-$isomorphism because of $\varphi(T)\beta = \overline{T}\beta = T$, for all $T \in V(D, \beta)$. By Theorem 2.3, $\beta \in R_{\varphi}(Q, D')$. □

Therefore, there is one to one correspondence between the elements of $R_{\varphi}(Q, D')$ and the set of ordered systems of disjoint mappings.

Theorem 2.6. *Let X be a finite set and Q be an $XI-$ semilattice and $m \geq 7$. If $D' = \{\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_m\}$ is $\alpha-$ isomorphic to Q and $\Omega(Q) = m_0$, then*

$$|R(D')| = 4m_0\left(4^{|\overline{T}_4 \backslash \overline{T}_3|} - 3^{|\overline{T}_4 \backslash \overline{T}_3|}\right)\left(5^{|\overline{T}_5 \backslash \overline{T}_4|} - 4^{|\overline{T}_5 \backslash \overline{T}_4|}\right) \cdots \left((m-4)^{|\overline{T}_{m-4} \backslash \overline{T}_{m-5}|} - (m-5)^{|\overline{T}_{m-4} \backslash \overline{T}_{m-5}|}\right)(m-3)^{|(\overline{T}_{m-2} \cap \overline{T}_{m-1}) \backslash \overline{T}_{m-3}|} \left((m-3)^{|\overline{T}_{m-3} \backslash \overline{T}_{m-4}|} - (m-4)^{|\overline{T}_{m-3} \backslash \overline{T}_{m-4}|}\right)\left((m-2)^{|\overline{T}_{m-1} \backslash \overline{T}_{m-2}|} - (m-3)^{|\overline{T}_{m-1} \backslash \overline{T}_{m-2}|}\right)\left((m-2)^{|\overline{T}_{m-2} \backslash \overline{T}_{m-1}|} - (m-3)^{|\overline{T}_{m-2} \backslash \overline{T}_{m-1}|}\right) m^{|X \backslash \overline{T}_m|}$$

Proof. Lemma 2.4 and Lemma 2.5 show us that the number of the ordered system of disjoint mappings $(f_{1\alpha}, f_{2\alpha}, \ldots, f_{(m-1)\alpha})$ is equal to $|R_{\varphi}(Q, D')|$, which $\alpha \in B_X(D)$ regular element $V(D, \alpha) = Q$ and $\varphi : Q \to D'$ is a complete $\alpha-$isomorphism. The number of the mappings $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}, f_{4\alpha}, \ldots, f_{(m-5)\alpha}$, $f_{(m-4)\alpha}$, $f_{(m-3)\alpha}$, $f_{(m-2)\alpha}$ and $f_{(m-1)\alpha}$ are respectively

$$1, 1, \left(4^{|\overline{T}_4 \backslash \overline{T}_3|} - 3^{|\overline{T}_4 \backslash \overline{T}_3|}\right), \left(5^{|\overline{T}_5 \backslash \overline{T}_4|} - 4^{|\overline{T}_5 \backslash \overline{T}_4|}\right) \cdots \left((m-4)^{|\overline{T}_{m-4} \backslash \overline{T}_{m-5}|} - (m-5)^{|\overline{T}_{m-4} \backslash \overline{T}_{m-5}|}\right), (m-3)^{|(\overline{T}_{m-2} \cap \overline{T}_{m-1}) \backslash \overline{T}_{m-3}|} \left((m-3)^{|\overline{T}_{m-3} \backslash \overline{T}_{m-4}|} - (m-4)^{|\overline{T}_{m-3} \backslash \overline{T}_{m-4}|}\right), \left((m-2)^{|\overline{T}_{m-1} \backslash \overline{T}_{m-2}|} - (m-3)^{|\overline{T}_{m-1} \backslash \overline{T}_{m-2}|}\right), \left((m-2)^{|\overline{T}_{m-2} \backslash \overline{T}_{m-1}|} - (m-3)^{|\overline{T}_{m-2} \backslash \overline{T}_{m-1}|}\right), m^{|X \backslash \overline{T}_m|}$$

The number q of all automorphisms of the semilattice Q is 4. These are

$$I_Q = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots T_{m-2} & T_{m-1} & T_m \\ T_1 & T_2 & T_3 & \cdots T_{m-2} & T_{m-1} & T_m \end{pmatrix} \quad \tau_1 = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots T_{m-2} & T_{m-1} & T_m \\ T_2 & T_1 & T_3 & \cdots T_{m-2} & T_{m-1} & T_m \end{pmatrix}$$
$$\tau_2 = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots T_{m-2} & T_{m-1} & T_m \\ T_1 & T_2 & T_3 & \cdots T_{m-1} & T_{m-2} & T_m \end{pmatrix} \quad \tau_3 = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots T_{m-2} & T_{m-1} & T_m \\ T_2 & T_1 & T_3 & \cdots T_{m-1} & T_{m-2} & T_m \end{pmatrix}.$$

Therefore by using, one to one correspondence between the elements of $R_{\varphi}(Q, D')$ and the set of ordered systems of disjoint mappings and Theorem 1.2,

$$|R(D')| = 4m_0\left(4^{|\overline{T}_4 \backslash \overline{T}_3|} - 3^{|\overline{T}_4 \backslash \overline{T}_3|}\right)\left(5^{|\overline{T}_5 \backslash \overline{T}_4|} - 4^{|\overline{T}_5 \backslash \overline{T}_4|}\right) \cdots \left((m-4)^{|\overline{T}_{m-4} \backslash \overline{T}_{m-5}|} - (m-5)^{|\overline{T}_{m-4} \backslash \overline{T}_{m-5}|}\right)(m-3)^{|(\overline{T}_{m-2} \cap \overline{T}_{m-1}) \backslash \overline{T}_{m-3}|} \left((m-3)^{|\overline{T}_{m-3} \backslash \overline{T}_{m-4}|} - (m-4)^{|\overline{T}_{m-3} \backslash \overline{T}_{m-4}|}\right)\left((m-2)^{|\overline{T}_{m-1} \backslash \overline{T}_{m-2}|} - (m-3)^{|\overline{T}_{m-1} \backslash \overline{T}_{m-2}|}\right)\left((m-2)^{|\overline{T}_{m-2} \backslash \overline{T}_{m-1}|} - (m-3)^{|\overline{T}_{m-2} \backslash \overline{T}_{m-1}|}\right) m^{|X \backslash \overline{T}_m|}$$

are obtained. □

By taking $m = 7$ in Theorem 2.6 one gets the following corollary which is given in [5, Theorem 3.7].

Corollary 2.7. *[5, Theorem 3.7] Let X be a finite set and Q be an $XI-$ semilattice. If $D' = \{\overline{T}_1, \overline{T}_2, \overline{T}_3, \overline{T}_4, \overline{T}_5, \overline{T}_6, \overline{T}_7\}$ is $\alpha-$ isomorphic to Q and $\Omega(Q) = m_0$, then*

$$|R(D')| = 4 \cdot m_0 \cdot 4^{\left|(\overline{T}_5 \cap \overline{T}_6) \setminus \overline{T}_4\right|} \cdot \left(4^{\left|\overline{T}_4 \setminus \overline{T}_3\right|} - 3^{\left|\overline{T}_4 \setminus \overline{T}_3\right|}\right) \cdot \left(5^{\left|\overline{T}_6 \setminus \overline{T}_5\right|} - 4^{\left|\overline{T}_6 \setminus \overline{T}_5\right|}\right) \cdot \left(5^{\left|\overline{T}_5 \setminus \overline{T}_6\right|} - 4^{\left|\overline{T}_5 \setminus \overline{T}_6\right|}\right) \cdot 7^{\left|X \setminus \overline{T}_7\right|}$$

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