Gen. Math. Notes, Vol. 1, No. 2, December 2010, pp. 130-137
ISSN 2219-7184; Copyright ©ICSRS Publication, 2010
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The $Geom^X/G/1$ Queue with Bernoulli Feedback and Server Setup/Closedown Times

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(Received 13.11.2010, Accepted 25.11.2010)

Abstract

In this paper ,we discuss the $Geom^X/G/1$ queue with Bernoulli feedback and server setup/closedown times.We derive the probability generating function (PGF) of the steady-state queue length immediately after a service completion by the embedded Markov chain and the PGF of the queue length immediately after an arbitrary slot boundary .We also get the PGF of the stationary waiting times under first come first service (FCFS) service discipline .From the results ,we obtain the conclusion the steady-state queue length and waiting times have the property of stochastic decomposition .Finally,we get the generating function of busy period and give several special cases to verify the effect of our model.

Keywords: *Bernoulli feedback;bulk arrive ;set-up times;close-down times; stochastic decomposition.*

2000 MSC No: primary 60K25;Secondary 90B22.

1 Introduction

In recent years,the discrete time vacation queue is used so widely in the telecommunications systems and computer communication networks that the discrete-time queuing system model and analysis is very important and very meaningful.Therefore,since Meisling first proposed and studied this type of queuing system in 1958[1], the discrete time queueing model[2−5] with various kinds of vacation rules has been extensively studied,while the discrete-time queuing system with Bernoulli feedback is few studied,recently,Yingyuan Wei etc[6] analyzed the $Geom/G/1$ queueing system with multiple adaptive vacations and Bernoulli feedback by full probability decomposition method.Based on this,in this paper, we apply the embedded Markov chain method to discuss the $Geom^X/G/1$ queue with Bernoulli feedback and server setup/closedown times,and obtain the probability generating function of the steady-state queue length, the probability generating function of the stationary waiting times and the property of stochastic decomposition results of them.And we also analyze the busy period of the system model and get the generating function of busy period .Finally ,we give several special cases of our model. Such model often arise in real life.For example, a remote multi-channel communication system will transmit again when the data transmission error occurs.

2 Model description

The $Geom^X/G/1$ queue with Bernoulli feedback and server setup/closedown times studied here is as follows:

(1)We consider the epoch n to clarify the state of the system ,and suppose that the arrivals of the customers occurs only at $t = n^-$ time (that moment on the eve of $t = n$) $n = 0, 1, 2, \cdots$; the start and end of service occur at $t = n^+$ time (that moment on the end of $t = n$) $n = 0, 1, 2, \cdots$.This model is known as the late model system.

(2)Customers arrive in batches.There is only a server in the system, service rule is first come first serve.When a customer has served ,one will be immediately discharged to wait for the next service at the end of the queue with the probability $1 - \alpha$,or leave the system forever with probability $\alpha(0 < \alpha < 1)$. That means,the total service number of a customer ξ obeys the geometric distribution of the parameter α

$$P(\xi = k) = \alpha\overline{\alpha}^{k-1}, k = 0, 1, 2, \cdots .$$

(3)One cycle begins as soon as the system become empty .At the moment of the last customer leaving the system , the server starts to take a closedown time.If there are customers arrival during the closedown period,service will begin after the closedown period until the queuing system is empty .Otherwise ,the server starts the consecutive vacations ,and the maximum number of vacations is denoted by$H, H = 1, 2, \cdots$.After each vacation,the server checks the system to see if there are customers waiting and decides the action to take according to the state of the system .There are three cases:Case 1:if there is any customer waiting and the server will no longer take another vacation and begin setup.

Case 2: if there is no customer waiting and the total number of vacations taken is still less than H,the server will repeat the same distribution on a independent vacation, and will enter a setup period on condition that there is any customer arriving after the kth vacation time ,$k = 1, 2, \cdots, H$. Case 3: if there is no customer waiting and the number of vacations taken is equal to H,the sever will stay idle and wait for the next arrival.

There are J customers arriving during the vacation time,the setup time and the closedown time, The probability distributions of whose are respectively listed as

$$C_j^{(V)} = \sum_{l=1}^{\infty} v_l \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} p^k \overline{p}^{l-k} P(\sum_{i=1}^{k} \Lambda_i = j),$$
$$C_j^{(U)} = \sum_{l=1}^{\infty} u_l \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} p^k \overline{p}^{l-k} P(\sum_{i=1}^{k} \Lambda_i = j),$$
$$C_j^{(C)} = \sum_{l=1}^{\infty} c_l \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} p^k \overline{p}^{l-k} P(\sum_{i=1}^{k} \Lambda_i = j),$$
$$C_0^{(V)} = \sum_{l=0}^{\infty} v_l \overline{p}^l = V(\overline{p}).$$

Let J represent the actual number of vacations taken by the server ,then $J = \min\{H, k : V^{k-1} < T < V^k\}$. Let A_I and A_V denote the event that the first customer arriveto an empty system occurs in an idle state and a server's vacation state,respectively,$h_{(j)}$ and $H(z)$ stand for the probability and PGF of H.We have

$$P(A_I) = \sum_{i=1}^{\infty} h_{(i)} \sum_{k=i}^{\infty} P(V_{(i)} = k)\overline{p}^k = H(V(\overline{p})), \qquad P(A_v) = 1 - H(V(\overline{p})).$$

(4)Above each random variables are independent.

Let $\widetilde{S}$ represent each customer's total service time, we have by the complete probability formula

$$\widetilde{g}_j = p(\widetilde{S} = j) = \sum_{k=0}^{j} p(\xi = k)p(\widetilde{S} = j \mid \xi = k) = \sum_{k=0}^{j} \alpha\overline{\alpha}^{k-1} p(\sum_{i=1}^{k} S_i = j), j = 1, 2, \cdots$$

And the PGFs and the average of the total service time are respectively listed as

$$\widetilde{G}(z) = \sum_{j=1}^{\infty} \widetilde{g}^j z^j = \frac{\alpha G(z)}{1 - (1 - \alpha)G(z)}, |z| < 1, \quad E(\widetilde{S}) = [\frac{d\widetilde{G}(z)}{dz}]|_{z=1} = \frac{g}{\alpha}.$$

During busy period,define L_n be the number of customers when the nth customer leaves the system and $\{L_n, n \geq 1\}$ be the Markov chain of the discrete-time queue length process.Then we have

$$L_{n+1} = \begin{cases} L_n - 1 + A, & Ł_n \geq 1, \\ Q_b - 1 + A, & Ł_n = 0. \end{cases}$$

where A_{n+1} represents the number of customers who arrive during the internal of the $(n+1)$th customer's total service times ,Q_b stands for the number of customers when a busy period begins. Then the probability distributions and PGFs of them are listed as

$k_l = p(A = l) = \sum\limits_{j=1}^{\infty} p(\widetilde{S} = j) \sum\limits_{k=0}^{j} \begin{pmatrix} j \\ k \end{pmatrix} p^k \overline{p}^{j-k} P(\sum\limits_{i=1}^{k} \Lambda_i = l), l = 0, 1, 2, \cdots.$

$b_i = p(Q_b = i) = (1 - C(\overline{p}))C_i^{(C)} + C(\overline{p})[H(V(p))C_{i-1}^{(U)} + \frac{1-H(V(\overline{p}))}{1-V(\overline{p})} \sum\limits_{k=1}^{i} C_k^{(V)} C_{i-k}^{(U)},$
$j = 0, 1, 2, \cdots.$

$Q_b(z) = \sum\limits_{i=1}^{\infty} k_l z^i = (1 - C(\overline{p}))C(\overline{p} + p\Lambda(z)) + C(\overline{p})U(\overline{p} + p\Lambda(z))[H(V(p)) + \frac{1-H(V(\overline{p}))}{1-V(\overline{p})}(V(\overline{p} + p\Lambda(z)) - V(\overline{p}))].$

which yields

$$E(Q_b) = (1 - C(\overline{p}))p\lambda E(C) + C(\overline{p})p\lambda E(U) + C(\overline{p})\frac{1 - H(V(\overline{p}))}{1 - V(\overline{p})}p\lambda E(V).$$

Whether a customer arrive to in the closedown period we can get for

$$h_j = P(Q_b + A - 1 = j) = \sum_{i=1}^{j+1} b_i k_{k+1-i}$$

Based on Foster rule,we can confirm that the Markov chain $\{L_n, n \geq 1\}$ is positive recurrent if and only if $\rho = \frac{p\lambda}{\alpha} < 1$(see[7]).

We use the following notations throughout the paper:

$\overline{x} = 1 - x$:any real number $x \in [0, 1]$. T : arrival time interval.

$\Lambda; \lambda_k, k = 0, 1, 2, \cdots, \Lambda(z), \lambda$:random batch size and its probability,PGF and mean.

$S; g_k, k = 0, 1, 2, \cdots; G(z), g, S_{(i)}$:the service time and its probability ,PGF,mean and of the ith customer.

$V; v_k, k = 0, 1, 2, \cdots; V(z); V^{(k)}$:the vacation time and its probability,PGF and sum of k vacations.

$U; u_k, k = 0, 1, 2, \cdots; U(z)$:the setup time and its probability and PGF.

$C; c_k, k = 0, 1, 2, \cdots; C(z)$:the closedown time and its probability and PGF.

$\theta, \theta(z); \Theta, \Theta(z)$:busy period caused by a customer and a batch and their PGFs.

3 Stochastic decomposition results

The system is positive recurrent If $\rho < 1$, then we are listed as

$$\beta = \frac{E(Q_b)}{\lambda} = (1 - C(\overline{p}))pE(C) + C(\overline{p})pE(U) + C(\overline{p})\frac{1 - H(V(\overline{p}))}{1 - V(\overline{p})}pE(V).$$

Theorem 1. When $\rho < 1$,the steady-state queue length immediately after the service completions ,denoted by Π,can be decomposed into the sum of two independent random variables:$\Pi = \Pi_0 + \Pi_d$,whereΠ_0 is queue length of the $Geom^X/G/1$ queue with Bernoulli feedback and no vacation, Π_d is the additional queue length,the PDFs of them is listed as

$$\Pi_0(z) = \frac{(1-\rho)(1-z)\widetilde{G}(\overline{p}+p\Lambda(z))}{\widetilde{G}(\overline{p}+p\Lambda(z))-z}, \qquad \Pi_d(z) = \frac{1-Q_b(z)}{\beta(1-\Lambda(z))}. \tag{1}$$

where
$\widetilde{G}(\overline{p}+p\Lambda(z)) = \frac{\alpha G(\overline{p}+p\Lambda(z))}{1-(1-\alpha)G(\overline{p}+p\Lambda(z))}$, $Q_b(z) = (1-C(\overline{p}))C(\overline{p}+p\Lambda(z))+C(\overline{p})U(\overline{p}+p\Lambda(z))[H(V(p)) + \frac{1-H(V(\overline{p}))}{1-V(\overline{p})}(V(\overline{p}+p\Lambda(z))-V(\overline{p}))]$.

Proof Denote $\pi_k^+, k \geq 0$be the steady-state distribution of the queue length ,which satisfies $\Pi\widetilde{P} = \Pi$,that means $\pi_j^+ = \pi_0^+ h_j + \sum_{i=1}^{j+1}\pi_j^+ k_{j+1-i}, j = 0,1,2,\cdots$.Multiplying z^j on both sides of the distribution function and summing over $j = 0,1,2,\cdots$,we obtain

$$\Pi^+(z) = \pi_0^+\sum_{j=0}^{\infty} h_j z^j + \sum_{j=0}^{\infty}\sum_{i=1}^{j+1} z^j = \pi_0^+ R(z) + \frac{1}{z}A(z)[\Pi^+(z) - \pi_0^+] \tag{2}$$

where
$A(z) = \sum_{l=0}^{\infty} k_l z^l = \widetilde{G}(\overline{p}+p\Lambda(z)), R(z) = \sum_{j=0}^{\infty} b_j z^j = \frac{1}{z}A(z)Q_b(z)$.
Substituting $R(Z)$ and $A(Z)$ into (2),which yields

$$\Pi^+(z) = \pi_0^+\frac{\widetilde{G}(\overline{p}+p\Lambda(z))(1-Q_b(z))}{\widetilde{G}(\overline{p}+p\Lambda(z))-z} \tag{3}$$

Based on the normalization condition $\Pi^+(1) = 1$ and L'Hospital rule ,we get

$$\pi_0^+ = (1-\rho)(\lambda\beta)^{-1} \tag{4}$$

Substituting(4) into (3) ,which yields

$$\Pi^+(z) = \frac{(1-\rho)(1-\Lambda(z))\widetilde{G}(\overline{p}+p\Lambda(z))}{\widetilde{G}(\overline{p}+p\Lambda(z))-z}\frac{1-Q_b(z)}{\lambda\beta(1-\Lambda(z))} \tag{5}$$

This is the probability generating function of the queue length when the customer will leave the sever.It should be pointed out that the queue length which remain when the customer leave the server is not usually of stationary distribution $\{\pi_k, k \geq 0\}$, that means $\pi_k = \lim_{t\to\infty} p(N(t) = k)$, but is the distribution of the queue of the regeneration time queue .However,the PGF of the general Stationary distribution is as

$$\Pi(z) = \frac{\lambda(1-z)}{(1-\Lambda(z))}\Pi^+(z) = \frac{(1-\rho)(1-z)\widetilde{G}(\overline{p}+p\Lambda(z))}{\widetilde{G}(\overline{p}+p\Lambda(z))-z}\frac{1-Q_b(z)}{\beta(1-\Lambda(z))} \tag{6}$$

As for (6),we can have the stochastic decomposition (1) and mean of the additional queue length
$E(\Pi_d) = \frac{p^2\lambda}{2\beta}\{(1-C(\overline{p}))E[C(C-1)]+C(\overline{p})E[U(U-1)]+C(\overline{p})\frac{1-H(V(\overline{p}))}{1-V(\overline{p})}[2E(U)E(V)+E[V(V-1)]]\}$.

Theorem 2. When $\rho < 1$,the stationary batch waiting times immediately after the service completions ,denoted by W, can be decomposed into the sum of two independent random variables: $W = W_0+W_d$,where W_0 is the stationary batch waiting time of the $Geom^X/G/1$ queue with Bernoulli feedback and no vacation, W_d is the additional delay,the PDFs of them is listed as

$$W_0(s) = \frac{(1-\rho)(1-s)}{\Lambda[\widetilde{G}(s)] - s - \overline{p}} \frac{1-\Lambda[\widetilde{G}(s)]}{\lambda(1-\widetilde{G}(s))} \tag{7}$$

$$W_d(s) = \frac{\left\{1-(1-C(\overline{p}))C(s) + C(\overline{p})U(s)[H(V(p)) + \frac{1-H(V(\overline{p}))}{1-V(\overline{p})}(V(s)-V(\overline{p}))]\right\}}{\beta(1-s)} \tag{8}$$

Where

$$\Lambda[\widetilde{G}(s)] = \Lambda[\frac{\alpha G(s)}{1-(1-\alpha)G(s)}].$$

Proof. Because W is composed by two parts W_g and W_f,where W_g is the waiting times of the batch, W_f is the waiting times of the customer in the batch.Based the FCFS rule,the number of customers batch to reach when service is completed in the existing system is equivalent to the one within the interval time.So,$W_g(s)$ has the relationship

$$\begin{aligned}\Pi_g(z) &= \frac{(1-\rho)(1-\Lambda_g(z))\widetilde{G}_g(\overline{p}+p\Lambda_g(z))}{\widetilde{G}_g(\overline{p}+p\Lambda_g(z))-z} \frac{[1-Q_b(z)]_g}{\beta(1-\Lambda_g(z))} \\ &= W_g(\overline{p}+p\Lambda_g(z))\widetilde{G}_g(\overline{p}+p\Lambda_g(z))\end{aligned} \tag{9}$$

where
$[Q_b(z)]_g = (1-C(\overline{p}))C(\overline{p}+p\Lambda_g(z))+C(\overline{p})U(\overline{p}+p\Lambda_g(z))[H(V(p))+\frac{1-H(V(\overline{p}))}{1-V(\overline{p})}(V(\overline{p}+p\Lambda_g(z))-V(\overline{p}))]$.

Also, because the service time of the $Geom^X/G/1$ queue with Bernoulli feedback and server setup/closedown times can be regarded as $\sum_{i=1}^{\Lambda}\widetilde{G}_i$, which is the service time of the Bernoulli feedback $Geom/G/1$ queue model with multiple adaptive vacations and server setup/closedown times .Thus

$$\widetilde{G}_g(s) = \Lambda(\widetilde{G}(s)) \tag{10}$$

Making $s = \overline{p} + p\Lambda(z)$,substituting (10) into (9),which yields

$$W(s) = \frac{(1-\rho)(1-s)}{\Lambda[\widetilde{G}(s)]-s-\overline{p}} \frac{p\left\{1-(1-C(\overline{p}))C(s)-C(\overline{p})U(s)[H(V(p))+\frac{1-H(V(\overline{p}))}{1-V(\overline{p})}(V(s)-V(\overline{p}))]\right\}}{\beta(1-s)} \tag{11}$$

we use a result of "discrete-time form" renewal theory to solve $W_f(s)$. In the batch queue for the length Λ,Let η stand for the customer number before the customer,which is nonnegative integer random variable,then the waiting times of the customer in the batch is $\sum_{i=1}^{\eta}\widetilde{G}_i$, the corresponding PGF is

$$W_f(s) = \frac{1-\Lambda[\widetilde{G}(s)]}{\lambda(1-\widetilde{G}(s))} \tag{12}$$

As for (11) and (12) ,we have

$$W(s) = \frac{(1-\rho)(1-s)}{\Lambda[\widetilde{G}(s)]-s-\overline{p}}\frac{1-\Lambda[\widetilde{G}(s)]}{\lambda(1-\widetilde{G}(s))} \frac{p\left\{1-(1-C(\overline{p}))C(s)-C(\overline{p})U(s)[H(V(p))+\frac{1-H(V(\overline{p}))}{1-V(\overline{p})}(V(s)-V(\overline{p}))]\right\}}{\beta(1-s)} \tag{13}$$

As for (13),we can have the stochastic decomposition (3.7),(3.8),and mean of the additional queue length
$E(W_d) = \frac{p}{2\beta}\{(1-C(\overline{p}))E[C(C-1)]+C(\overline{p})E[U(U-1)]+C(\overline{p})\frac{1-H(V(\overline{p}))}{1-V(\overline{p})}[2E(U)E(V)+E[V(V-1)]]\}$.

4 Analysis of busy period

As for the batch arrival $Geom/G/1$ system,we have known that there are two kinds of busy period caused by a single customer and customers of a batch,respectively.Thus the busy period caused by the number of customers who arrive at the setup and vacations is listed as $\theta_v = \sum_{j=1}^{Q_b}\theta_i$,the corresponding PGF $\theta_v(z) = Q_b(\theta(z))$.The busy period caused by a batch customers is listed as $\Theta = \sum_{i=1}\Lambda\theta_i$, the corresponding PGF $\Theta(z) = \Lambda(\theta(z))$.

Because $\theta = A+\Theta_1+\cdots+\Theta_{N(A)}$, $\Theta = U_\xi+\theta_1+\cdots+\theta_{U_\xi}$,where$U_\xi = \sum_{i=1}\xi A_i$,so the PGF of busy period caused by a customer is listed as

$\theta(z) = E(z^\theta) = \sum_{n=1}^{\infty} z^n \sum_{j=1}^{n}(\Theta(z))^j \begin{pmatrix} n \\ j \end{pmatrix} p^j\overline{p}^{n-j}p(A=n) = \widetilde{G}(z-zp+zp\Theta(z))$

Thus we know

$$\theta_v(z) = Q_b(\widetilde{G}(z-zp(1-\Lambda(Theta(z)))), \qquad E(\theta_v) = E(Q_b)E(\theta) = \frac{g}{\alpha}\frac{E(Q_b)}{1-\rho}.$$

5 Conclusion

Comparison theorem 1 and 2,we have $E(\Pi_d) = p\lambda E(W_d)$,which He testes Little formula established.

Case 1. If we consider the case $\alpha = 1$,the model changes into the $Geom^X/G/1$ queue model with multiple adaptive vacations and server setup/closedown times, the results have been given Wei Sun,Naishuo Tian et al [5].

Case 2. If we consider the case $\Lambda = 1, H = \infty$ and $U = C = 0$,the model turns into the Bernoulli feedback on $Geom/G/1$ queue model with multiple vacations, the steady-state queue length have been given Yingyuan Wei et al [6],and the stationary waiting time is

$$W(s) = \frac{(1-\rho)(1-s)}{p[\widetilde{G}(s)] - s - \overline{p}} \frac{p\left\{1 - \frac{1}{1-V(\overline{p})}(V(s) - V(\overline{p}))]\right\}}{pE(V)(1-s)}.$$

Case 3. If we consider the case $\Lambda(z) = z$,the model is the $Geom/G/1$ queue with Bernoulli feedback and server setup/closedown times.

ACKNOWLEDGEMENTS.

This work is supported by the Natural Science Foundation of Chongqing Municipal Education Commission(No:KJ091104)

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