Gen. Math. Notes, Vol. 1, No. 2, December 2010, pp. 143-152

www.i-csrs.org
Available free online at http://www.geman.in

A New Method for Solving Nonlinear BVPs

Behzad Ghanbari[*,1] and Mehdi Gholami Porshokuhi[1]

[1]Department of Mathematics, Faculty of Science, Islamic Azad University, Takestan Branch, Iran
E-mail: b.ghanbary@yahoo.com,
m_gholami_p@yahoo.com

(Received 2.11.2010, Accepted 15.12.2010)

Abstract

As we know, the homotopy analysis method (HAM) provides us with a convenient way to adjust and control the convergence region and rate of the obtained series solutions. This great advantage of method is possible by finding a proper value of the so-called control parameter c_0. *In this paper, an efficient way of obtaining* c_0 *is proposed. Such value of parameter can be determined at the any order of approximation of HAM series solution, by solving of a nonlinear polynomial equation. To show the ability and efficiency of this new approach we apply this modification of HAM to some linear and nonlinear initial value problems, and obtain convergent series solutions which agree very well with their exact solutions. It is found that presented approach greatly accelerate the convergence of series solution.*

Keywords: *homotopy analysis method; boundary value problems*

---

[*] Corresponding author

1 Introduction

In 1992 Liao [1] proposed the homotopy analysis method (HAM) to get analytic approximation solution of nonlinear equations.

Unlike other existing methods, the HAM:

- Is independent of small/large physical parameters.
- Provides us a simple way to ensure the convergence of solution series.
- Provides us with great freedom to choose proper base functions.
- Is independent of small/large physical parameters.
- Provides us a simple way to ensure the convergence of solution series.

These advantages make the method to be a powerful and flexible tool in mathematics and engineering, which can be readily distinguished from existing numerically and analytically methods.

This paper is arranged in the following manner; The basic idea of HAM is described in Section 2. To illustrate the proposed method and implementation of this modification on some problems are presented in section 3. Finally, conclusions are drawn in Section 4.

1- Standard HAM

Using the concept of homotopy, Liao [1] introduced the early form of the homotopy-analysis method (HAM) for a given nonlinear differential equation

$$N[u(x)]=0, \tag{1}$$

as

$$(1-p)L[U(x;p)-u_0(x)]=-pN[U(x;p)], p\in[0,1]. \tag{2}$$

where L is an auxiliary linear operator and $u_0(x)$ is an initial guess. It is evident that, at $p=0$ and $p=1$, we respectively have $U(x;0)=u_0(x)$ and $U(x;1)=u(x)$.

In the view of HAM the solution of original equation is assumed to be expanded in the term of embedding parameter p as

$$u(x) = u_0(x) + \sum_{i=0}^{\infty} u_i(x) p^i . \tag{3}$$

AS proved by Liao [3], whereas (3) be convergent at $p = 1$, its limit must satisfy the original equation(1).

Liao [4] in 1999 further introduced more artificial degrees of freedom by using the zeroth-order deformation equation in the form of

$$(1-p)L[U(x;p) - u_0(x)] - c_0 pN[U(x;p)] = 0, p \in [0,1]. \tag{4}$$

Appling recently proposed "m th-order homotopy-derivative operator" [5]

$$D_m(\phi) = \frac{1}{m!}\frac{\partial^m \phi}{\partial p^m}\Big|_{p=0} .$$

to the both sides of (4), one reads

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hR_{m-1}(x), \tag{5}$$

where

$$R_{m-1}(x) = D_{m-1}(N[u(x;p)]) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[u(x;p)]}{\partial p^{m-1}}\Big|_{p=0} ,$$

and

$$\chi_m = \begin{cases} 1, & m > 1, \\ 0, & m \le 1. \end{cases}$$

In this way, the component solutions of $u_m, m \ge 1$, are not only dependent upon x but also the auxiliary parameter c_0 .

As we know, to find a proper convergence-control parameter c_0, to get a convergent series solution or to get a faster convergent one, there is a classic way of plotting the so- called ‘‘ c_0 -curves” or ‘‘curves for convergence-control parameter”. For example, one can consider the convergence of $u'(x)$ and $u''(x)$ of a nonlinear differential equation $N[u(x)]=0$ to find a region say R_{c_0} so that, each $c\in R_{c_0}$ gives a convergent series solution of such kind of quantities.

Such a region can be found, although approximately, by plotting the curves of these unknown quantities versus c_0.

However, it is a pity that curves for convergence-control parameter (i.e. c_0 -curves) give us only a graphically region and cannot tell us which value of $c\in R_{c_0}$ gives the fastest convergent series.

Recently in [6] a misinterpreted usage of c_0 -curves has reported.

2- proposed approach and numerical examples

To propose the idea of our approach, let’s consider solving the following type of nonlinear boundary value problems

$$u^{(n)}(x)=f\,(x,u,u',\dots,u^{(n-1)}),\quad a\le x\le b$$

subject to the two-point boundary conditions

$$u(a)=\alpha_0,u'(a)=\alpha_1,\dots,\ u^{(r)}(a)=\alpha_r,$$
$$u(b)=\beta_0,u'(b)=\beta_1,\dots,\ u^{(r)}(b)=\beta_r,$$

where $0\le r\le n-2$ is an integer, f is a polynomial in $x,u(x),u'(x),\dots,u^{(n-1)}(x),$ and $a,b,\alpha_0,\alpha_1,\dots,\alpha_r,\beta_0,\beta_1,\dots,\beta_{n-r-2}$ are real constants.

It should be noted that according to obtained results in section 2 and based on the zeroth-order deformation equation (4), m th-order HAM approximation of the solution $u(t)$, giving by

$$U_m(t) = u_0(t) + \sum_{i=0}^{m} u_i(t),$$

is also dependent upon the convergence-control parameter c_0 .

For $\theta = \frac{1}{2}(b + a)$, let's define the m th-nonlinear polynomial equation of $P_m(c_0)$ at the m th-order of HAM approximation could be constructed, as

$$P_m(c_0) = U_m^{(n)}(\theta) - f\,(\theta, u(\theta), u'(\theta), \ldots, u^{(n-1)}(\theta)) = 0. \tag{6}$$

Solving such equations determine the optimal value of c_0 , at the any order of approximation m.

To demonstrate the effectiveness of the proposed approach, we consider several examples.

Example 1. As the example, let's have the following fourth order boundary value problem involving a parameter c [7].

$$u^{(4)}(t) = (1+c)u''(t) - cu(t) + \frac{1}{2}cx^2 - 1,$$

with the boundary conditions

$$u(0) = 1,\ u'(0) = 1,$$
$$u(1) = \frac{3}{2} + \sinh(1),\ u'(1) = 1 + \cosh(1).$$

which has the exact solution in the form of

$$u(x) = 1 + \frac{1}{2}x^2 + \sinh(x).$$

We will discuss three cases: small, large and Very large values of c .

I) Small values of c. In this case, we take $c = 5$ as an example. For $c = 5$ and $m = 7$ in (6), we get into a nonlinear equation, which has the solution of $c_0 =$ -0.9085820.

II) Large values of c. In this case, we take $c = 100$ as an example. Eq (6) with $c = 100$ and $m = 7$ has the real solution of c_0=-0.3582110156.

III) Very large values of c. In this case, we take $c = 10^8$ as an example. Eq (6) with $c = 10^8$ and $m = 7$, leads to $c_0 = -5.2671251e - 7$ as proper values for c_0:

We compare the relative errors of the 7th order HAM approximations at different points in the interval $(0,1)$ using the formula

$$\delta(t) = \left| \frac{u_{appr} - u_{exact}}{u_{exact}} \right|.$$

for different cases of c in Table 1.

Table 1

Comparisons of $\delta(t)$ of 7th -order HAM solutions for different values of c_0 at different points

c	c_0	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
5	$c_0 = -0.908058227$	*1.01e-11*	*3.05e-11*	*2.54e-11*	*3.11e-11*	*1.07e-10*
	$c_0 = -1$ (HPM)	*1.53e-8*	*8.49e-8*	*1.05e-7*	*5.71e-8*	*6.86e-9*
100	c_0=-0.3582110156	*1.31e-5*	*8.00e-6*	*1.36e-5*	*5.41e-6*	*5.84e-6*
	$c_0 = -1$ (HPM)	*6.37*	*35.23*	*43.91*	*23.73*	*2.89*
	$c_0 = -5.2671251e - 7$	*3.75e-4*	*1.38e-4*	*6.11e-5*	*6.24e-5*	*1.67e-4*

10^8	$c_0 = -1$ (HPM)	*5.97e44*	*3.30e45*	*4.11e45*	*2.22e45*	*2.71e44*

Example 2. As the example, let's have the following fourth order boundary value problem involving a parameter c [8].

$$u^{(4)}(x) = cu(x)^2 + 1,\ 0 \le x \le 2$$

with the boundary conditions

$$u(0) = u'(0) = u(2) = u'(2) = 0$$

Here we will discuss following three cases of c :

I) In the first case, we take $c = 1$ as an example. For $c = 1$ and $m = 7$ in (6), we get into a nonlinear equation, which contains $c_0 = -1.04578977, -0.924614442$ as its roots.

II) In second case, we take $c = 5$ as an example. Eq (6) with $c = 5$ and $m = 7$ has the real solutions c_0=-0.954024601,-1.28093421.

III) Finally, we take $c = -12$ as an example. Eq (6) with $c = -12$ and $m = 7$, introduces $c_0 = -0.6556949$ as proper value for c_0 .

The comparisons of relative errors of the 7th order HAM for different cases of c are drawn Table in Table 2.

Table 2

Comparisons of $\delta(t)$ of 7th -order HAM solutions for different values of c_0 at different points

c	c_0	$t = 0.2$	$t = 0.6$	$t = 1.0$	$t = 1.4$	$t = 1.8$
	$c_0 = -1.04578977$	*1.11e-6*	*5.57e-7*	*6.04e-7*	*7.49e-6*	*4.25e-5*

1	c_0 = -0.92461444	*7.15e-7*	*4.31e-7*	*3.75e-7*	*2.09e-6*	*9.62e-6*
5	c_0=-0.954024601	*1.31e-4*	*1.32e-4*	*1.28e-4*	*1.25e-4*	*1.45e-4*
	c_0=-1.280934210	*5.03e-4*	*6.53e-4*	*6.80e-4*	*6.63e-4*	*1.37e-3*
-12	$c_0 = -0.65569498$	*1.06e-2*	*9.90e-3*	*9.47e-3*	*9.91e-3*	*1.05e-2*
	$c_0 = -1$ (HPM)	*6.22e1*	*6.22e1*	*6.22e1*	*6.22e1*	*6.22e1*

Example 3. As the next example, let's have the following sixth order boundary value problem [9].

$$u^{(6)}(t) = (1+c)u^{(4)}(t) - cu''(t) + cx\,,$$

with the boundary conditions

$$u(0) = 1,\ u'(0) = 1,\ u''(0) = 0,$$
$$u(1) = \frac{3}{2} + \sinh(1),\ u'(1) = 1+\cosh(1),\ u''(1) = 1+\sinh(1).$$

which has the exact solution in the form of

$$u(x) = 1 + \frac{1}{6}x^3 + \sinh(x)$$

Here, we will discuss three cases: small, large and Very large values of c.

I) Small values of c. In this case, we take $c = 5$ as an example. For this case, we get into a nonlinear equation, which has the real roots c_0 = -0.9152004169.

II) Large values of c. In this case, we take $c = 10^3$ as an example. For this case, $c = -0.105699992$ is obtained by our approach.

III) Very large values of c. In this case, we take $c = 10^8$ as an example. This value of c leads to $c_0 = -1.18\text{e-}6$ as proper values for c_0:

The comparisons of relative errors of the 7th order HAM for different cases of c are drawn Table in Table 3.

3- Conclusion

In this paper the solutions of a new method to finding the control parameter in homotopy analysis method is proposed. It is shown for obtained values of such parameter, HAM approximation series leads to exact solution of problems or produces an approximate results which are in a highly agreement with exact solution of problems. All computations were done using MAPLE with 15 digit floating point arithmetics (Digits:=15).

Table 3

Relative error $\delta(t)$ of 7^{th} -order HAM solutions for different values of c_0 at different points

c	c_0	$t=0.1$	$t=0.3$	$t=0.5$	$t=0.7$	$t=0.9$
10	$c_0=$-0.9152004169	*1.17e-12*	*1.26e-12*	*5.57e-13*	*9.69e-13*	*7.12e-13*
10^3	$c=-0.105699992$	*1.70e-5*	*1.91e-5*	*9.09e-6*	*1.38e-5*	*8.74e-6*
10^8	$c_0=-1.18e-6$	*3.73e-5*	*4.22e-5*	*2.03e-5*	*3.04e-5*	*1.91e-5*

References

[1] S. J. Liao, On the proposed homotopy analysis technique for nonlinear problems and its applications, Ph. D. dissertation, *Shanghai Jio Tong University*, (1992).

[2] S.J. Liao, General boundary element method for non-linear heat transfer problems governed by hyperbolic heat conduction equation, *Comput. Mech.* 20 (1997),397–406.

[3] S.J. Liao, Beyond Perturbation: Introduction to Homotopy Analysis Method, *Chapman & Hall/CRC Press, Boca Raton*, (2003).

[4] S.J. Liao, Numerically solving nonlinear problems by the homotopy analysis method, *Comput. Mech.*, 20(1997), 530–540.

[5] S.Liao, Notes on the homotopy analysis method: Some definitions and theorems, *Commun Nonlinear Sci Numer Simulat* ,14(2009), 983–997.

[6] A. Mehmood, S. Munawar and A. Ali, Comments to: ‘‘Homotopy analysis method for solving the MHD flow over a non-linear stretching sheet, *Commun. Nonlinear Sci. Numer. Simul.*, 14(2009) 2653–2663). *Commun Nonlinear Sci Numer Simulat*, (2010),doi:10.1016/j.cnsns.2009.12.039.

[7] M.R. Scott and H.A. Watts, SAND750198, Sandia laboratories, *Albuquerque, NM*,

[8] W. Al-Hayani , L. Casass̊, Numer. *Algorithms* , 40(2005), 67.

[9] M.A. Noor and S.T. Mohyud-Din, Homotopy perturbation method for solving sixth-order boundary value problems, *Comput. Math. Appl.*, 55(2008),2953-2972.

[10] S.N. Ha, A nonlinear shooting method for two-point boundary value problems, *Comput. Math. Appl.*, 42(2001), 1411–1420.

[11] A.A. Salama and A.A. Mansour, Fourth-order finite-difference method for third-order boundary-value problems, *Numer. Heat Transf.*, B 47(2005), 383–401.