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The Group of Extensions of a Topological Local Group

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Abstract

In this paper we prove that the set of extensions of a topological local group is a group.

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1 Introduction

Let H and G be topological groups, H abelian. By a topological extensions of H by G, we mean a short exact sequence
$\varepsilon : 1 \longrightarrow H \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1$ with π an open continuous homomorphism and H a closed normal subgroup of E. A cross-section of a topological group extension (E, π) of H by G is a continuous map $u : G \to E$ such that $\pi u(x) = x$ for each $x \in G$. The set of all extensions of H by G with a continuous cross-section, denoted by $Ext_c(G, H)$, with the Bair-sum is a group [3].

In this paper we show a similar result for topological local groups [5]. In section 1 we give some definitions which will be needed in sequel. In section 2, we introduce the pull-back and the push-out of a topological local extension and prove that the class of topological local extensions is a group.

We use the following notations:

- " 1 " is the identity element of X.
- "$\leq$" : $G \leq H$, G a sublocal group (subgroup) of a local group (group) H.
- $D = \{(x, y) \in X \times X; xy \in X\}$ where X is a local group.

2 Primary Definitions

We recall the following definition from [6]:
A *local group* $(X, .)$ is like a group except that the action of group is not necessarily defined for all pairs of elements, The associative law takes the following form: if $x.y$ and $y.z$ are defined, then if one of the products $(x.y).z$, $x.(y.z)$ is defined, so is the other and the two products are equal. It is assumed that each element of X has an inverse.

Definition 2.1. [5] Let X be a local group. If there exist:

a) a distinguished element $e \in X$, identity element,

b) a continuous product map $\varphi : D \to X$ defined on an open subset

$$(e \times X) \cup (X \times e) \subset D \subset X \times X.$$

c) a continuous inversion map $\nu : X \to X$
satisfying the following properties:

  (*i*) *Identity*: $\varphi(e, x) = x = \varphi(x, e)$ for every $x \in X$

  (*ii*) *Inverse*: $\varphi(\nu(x), x) = e = \varphi(x, \nu(x))$ for every $x \in X$

  (*iii*) *Associativity*: If (x, y), (y, z), $(\varphi(x, y), z)$ and $(x, \varphi(y, z))$ all belong to D, then

$$\varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$$

then X is called a *topological local group.*

Example 2.2. Let X be a Hausdorff topological space and $\triangle_X$ be the diagonal of X, $a \in X$ and $D = (\{a\} \times X) \cup (X \times \{a\}) \cup \triangle_X$. Define $\varphi : D \longrightarrow X$ by:

$$\varphi(x, y) = \begin{cases} x & , y = a, \\ y & , x = a, \\ a & , x = y, \end{cases} \quad .$$

Now X, by the action of φ, is a local group.

If $x \in X$, $x \neq a$, we have $\varphi(x,a) = x$. If U is a neighborhood of x, then $\varphi^{-1}(U) = U \times \{a\}$. There are two cases;

1) $a \in U$: since X is Hausdorff, there are disjoint neighborhood U_1, U_2 containing a, x, respectively. Then $x \in U_2 \cap U$ and $a \notin U_2 \cap U = V$ and $\varphi^{-1}(V) = V \times \{a\}$. Hence, $\varphi(V \times \{a\}) \subset U$. So φ is continuous.

2) $a \notin U$: $\varphi^{-1}(U) = U \times \{a\}$.

If $x = a$ and W is a closed neighborhood of a in X then $\varphi^{-1}(W) = \triangle_X \cup (W \times \{a\}) \cup (\{a\} \times W)$. Hence, φ is continuous. Therefore, $\varphi : D \to X$, $(x,y) \mapsto xy$ and $X \to X$, $x \mapsto x^{-1}$ are continuous. So X is a topological local group.

Definition 2.3. A *sublocal group* of X is a subset $Y \subseteq X$ such that $e \in Y$, $Y = Y^{-1}$ and if $x, y \in Y$ and $x * y^{-1} \in X$ then $x * y^{-1} \in Y$.

A *subgroup* of a local group X is a subset $H \subseteq X$ such that $e \in H$, $H \times H \subseteq D$ and for all $x, y \in H$, $x * y \in H$.

Definition 2.4. A continuous map $f : (X, .) \to (X', *)$ of topological local groups, is called a *homomorphism* if:

1. $(f \times f)(D) \subseteq D'$ where $D' = \{(x', y') \in X' \times X', x' * y' \in X'\}$;
2. $f(e) = e'$ and $f(x^{-1}) = (f(x))^{-1}$;
3. if $x.y \in X$ then $f(x) * f(y)$ exists in X' and $f(x.y) = f(x) * f(y)$.

With these morphisms topological local groups form a category which contains the subcategory of topological groups.

Definition 2.5. A homomorphism of topological local groups $f : X \to X'$ is called *strong* if for every $x, y \in X$, the existence of $f(x)f(y)$ implies that $xy \in X$.

A morphism is called a monomorphism (epimorphism) if it is injective (surjective).

We denote the product of p copies of X by X^p.

Lemma 2.6. [1, Lemma 2.5] *Let U be a symmetric neighborhood of the identity in a topological local group X. There is a neighborhood U_0 of identity in U such that for every $x, y \in U_0$, $xy \in U$.*

Definition 2.7. Let X, Y be topological local groups and U is a symmetric neighborhood in X. The continuous map $f : U \to Y$ is an *open continuous local homomorphism* of X onto Y if

1. there exists a symmetric neighborhood U_0 in U such that if $x_1, x_2 \in U_0$, then $x_1x_2 \in U$;
2. $f(x_1x_2) = f(x_1)f(x_2) \quad x_1, x_2 \in U_0$;
3. for every symmetric neighborhood W, $W \subseteq U_0$, $f(W)$ is open in Y.

The map f is called an *open continuous local isomorphism* of X to Y if U_0 can be chosen so that $f|_{U_0}$ is one to one.

Definition 2.8. *A topological local group extension* of a topological local group Y by a topological local group X is a triple (E, π, η) where E is a topological local group, π is an open continuous local homomorphism of E to X, and η is an open continuous local isomorphism of Y onto the kernel of π [2].

Remark. If (E, π, η) is a topological local group extension of N by X, with π a strong homomorphism and $N = ker\pi$, then N is a closed normal topological subgroup of E.

3 The Group of Topological Local Extensions

It is known that the set of extensions of a group is an abelian group [3]. We show that the class of topological local group extensions with the Bair-sum forms a group.

Definition 3.1. Let $\varepsilon_1 = (E_1, \pi_1, \eta_1)$ and $\varepsilon_2 = (E_2, \pi_2, \eta_2)$ be topological local extensions of an abelian topological group C by a topological local group X. If there exists a strong isomorphism σ of E_1 onto E_2 such that $\sigma \circ \eta_1(n) = \eta_2(n)$ and $\pi_1 = \pi_2 \circ \sigma$.

$$\begin{array}{llccccccccc} \varepsilon_1 : & 1 & \longrightarrow & C & \longrightarrow & E_1 & \xrightarrow{\pi_1} & X & \longrightarrow & 1 \\ & & & \| & & \downarrow{\scriptstyle \sigma} & & \| & & \\ \varepsilon_2 : & 1 & \longrightarrow & C & \longrightarrow & E_2 & \xrightarrow{\pi_2} & X & \longrightarrow & 1 \end{array}$$

The ε_1 and ε_2 are *equivalent*, $\varepsilon_1 \equiv \varepsilon_2$.

Lemma 3.2. *Let $\varepsilon = (E, \pi, \eta)$ be an extension of an abelian topological group C by a topological local group X. If $\gamma : X' \to X$ is a strong homomorphism, then there exists an extension $\varepsilon_\gamma = (E', \pi', \eta')$ of C by a topological local group X', such that the following diagram commutes.*

$$\begin{array}{llccccccccc} \varepsilon_\gamma : & 0 & \longrightarrow & C & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & X' & \longrightarrow & 1 \\ & & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \gamma} & & \\ \varepsilon : & 0 & \longrightarrow & C & \xrightarrow{\iota} & E & \xrightarrow{\pi} & X & \longrightarrow & 1 \end{array} \tag{3.1}$$

where $E' = \{(e, x') | \pi(e) = \gamma(x'), e \in E, x' \in X'\}$.

Proof. The maps π and γ are strong local homomorphisms. We consider

$$E' = \{(e, x')|\pi(e) = \gamma(x'), e \in E, x' \in X'\};$$

E' is a sublocal group of $E \oplus X'$. By [5, Proposition 2.22], E' is a topological local group. We define

$$\pi' : E' \to X',\ \pi'(e, x') = x', \quad \sigma : E' \to E,\ \sigma(e, x') = e, \quad \eta' : C \to ker\pi \oplus \{1_{X'}\},$$

$\eta'(n) = (\eta(n), 1_{X'})$.
Since π is onto then so is π'.

Let V_1 is a neighborhood of the identity in X. By Lemma 2.6, there is a symmetric neighborhood V_0 in V_1 such that $\pi(e_1).\pi(e_2), \gamma(x'_1).\gamma(x'_2) \in V_1$ for $\pi(e_1), \pi(e_2),\ \gamma(x'_1), \gamma(x'_2) \in V_0$ which $\pi(e_1) = \gamma(x'_1),\ \pi(e_2) = \gamma(x'_2)$.
Since π and γ are strong homomorphisms, if $\pi(e_1)\pi(e_2) = \gamma(x'_1)\gamma(x'_2)$, $\pi(e_1e_2) = \gamma(x'_1x'_2)$, then $(e_1e_2, x'_1x'_2)$ is defined in E'. We define an action on E' by

$$(e_1, x'_1).(e_2, x'_2) := (e_1e_2, x'_1x'_2).$$

Now π' is a local homomorphism, since π' is onto.

$$\pi'((e_1, x'_1).(e_2, x'_2)) = \pi'(e_1e_2, x'_1x'_2) = x'_1x'_2 = \pi'(e_1, x'_1).\pi'(e_2, x'_2),$$

Since π and γ are strong homomorphisms. Therefore, π' is strong. Similarly σ is a strong homomorphism.

Now, we show that π' is continuous. For every $x' \in X'$, there is a symmetric neighborhood $V_{x'}$ of x'. It is enough to show that $\pi'^{-1}(V_{x'})$ is open in E'. There exists a symmetric neighborhood V of $\gamma(x')$ in X such that $V_{x'} \subseteq \gamma^{-1}(V)$. Since π is onto, then there exists $e \in E$ such that $\gamma(x') = \pi(e)$. Since π is continuous, so there is a symmetric neighborhood V_e of e such that $V_e \subset \pi^{-1}(V)$. Now $V_e \oplus V_{x'}$ is a symmetric open set in $E \oplus X'$. Therefore, $V_{(e,x')} = [V_e \oplus V_{x'}] \cap E'$ is an open set in E' and $(e, x') \in V_{(e,x')} \subset \pi'^{-1}(V_{x'})$. So $\pi'^{-1}(V_{x'})$ is an open set in E'.

We have $\pi'(V_{(e,x')}) = \pi'((V_e \oplus V_{x'}) \cap E') = V'_{x'}$ where $V'_{x'}$ is a symmetric neighborhood of x' and $V'_{x'} \subseteq V_{x'}$. Then, π' is an open continuous map. We will have $\sigma\eta' = \eta$ and η' is a local isomorphism.

The diagram (3.1) commutes, since $\gamma\pi'(e, x') = \gamma(x') = \pi(e) = \pi\sigma(e, x')$, i.e. $\gamma\pi' = \pi\sigma$. Suppose $\varepsilon'' = (E'', \pi'', \eta'')$ is an extension of C by X', such that the following diagram commutes

$$\begin{array}{lccccccccc}
\varepsilon'' : & 0 & \longrightarrow & C & \longrightarrow & E'' & \xrightarrow{\pi''} & X' & \longrightarrow & 1 \\
 & & & \| & & \downarrow{\scriptstyle \sigma''} & & \downarrow{\scriptstyle \gamma} & & \\
\varepsilon : & 0 & \longrightarrow & C & \longrightarrow & E & \xrightarrow{\pi} & X & \longrightarrow & 1
\end{array}$$

Let $\sigma' : E'' \to E'$, $\sigma'(e'') = (\sigma''(e''), \pi''(e''))$. Then, $\pi'\sigma' = \pi''$ and $\sigma'\eta'' = \eta'$. Now by the five lemma [3], $(1_C, \sigma', 1_{X'}) : \varepsilon'' \to \varepsilon_\gamma$, ε'' and ε_γ are equivalent. $\square$

Note 3.3. As in Lemma 3.2, if there exists $\varepsilon_\gamma \xrightarrow{(Id_C,\sigma,\gamma)} \varepsilon$, then ε_γ is a *pullback* of ε.

Lemma 3.4. *Let $\varepsilon = (E, \pi, \eta)$ and $\varepsilon_1 = (E_1, \pi_1, \eta_1)$ be extensions of two abelian topological groups C , C_1 by topological local groups X, X_1, respectively. Assume $\alpha_1, \sigma_1, \gamma_1$ are strong homomorphisms of ε_1 to ε. Suppose $\gamma_1 = \gamma : X_1 \to X$. Then we have*

$$\varepsilon_1 \xrightarrow{(\alpha_1,\sigma',Id_X)} \varepsilon_\gamma \xrightarrow{(Id_C,\sigma,\gamma)} \varepsilon.$$

Proof. By assumptions and Lemma 3.2, we have the following commutative diagrams:

$$\begin{array}{lccccccccc} \varepsilon_1 : & 0 & \longrightarrow & C_1 & \longrightarrow & E_1 & \xrightarrow{\pi_1} & X_1 & \longrightarrow & 1 \\ & & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \gamma_1} & & \\ \varepsilon : & 0 & \longrightarrow & C & \longrightarrow & E & \xrightarrow{\pi} & X & \longrightarrow & 1 \end{array} \qquad \begin{array}{lccccccccc} \varepsilon_\gamma : & 0 & \longrightarrow & C & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & X_1 & \longrightarrow & 1 \\ & & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \gamma} & & \\ \varepsilon : & 0 & \longrightarrow & C & \longrightarrow & E & \xrightarrow{\pi} & X & \longrightarrow & 1 \end{array}$$

where $\sigma' : E_1 \to E'$, $\sigma'(e_1) = (\sigma_1(e_1), \pi_1(e_1))$. Then, $\sigma_1 = \sigma \circ \sigma'$.
So, the diagram $(\alpha_1, \sigma', Id_{X_1}) : \varepsilon_1 \to \varepsilon_\gamma$ is commutative, $\pi'\sigma' = \pi_1$ and $\sigma'\eta_1 = \eta'$. □

Lemma 3.5. *Let $\varepsilon = (E, \pi, \eta)$ be an extension of an abelian topological group C by a topological local group X. If $\alpha : C \to C'$ is a continuous homomorphism of topological local groups, then there exists an extension ${}_\alpha\varepsilon = (K, \pi', \eta')$ of abelian topological group C' by a topological local group X such that the following diagram commutes.*

$$\begin{array}{lccccccccc} \varepsilon : & 0 & \longrightarrow & C & \xrightarrow{\iota} & E & \xrightarrow{\pi} & X & \longrightarrow & 1 \\ & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \sigma} & & \| & & \\ {}_\alpha\varepsilon : & 0 & \longrightarrow & C' & \xrightarrow{\iota'} & K & \xrightarrow{\pi'} & X & \longrightarrow & 1 \end{array} \qquad (3.2)$$

where $K = \frac{C' \oplus E}{H}$, $H = \{(-\alpha(n), \iota(n)) | n \in C\}$ and σ a strong local homomorphism.

Proof. Suppose

$$H = \{(-\alpha(n), \iota(n)) | n \in C\}.$$

Then, H is a subgroup of $C' \oplus E$. By [5, Proposition 2.22], $C' \oplus E$ is a topological local group. The map ι is injective and $\iota(C) \equiv ker\pi$. Then, $\iota(C)$ is a closed subgroup of E and α a homomorphism of topological groups. So H is a closed topological subgroup of $C' \oplus E$. Since $-\alpha(C)$ is an open subgroup of C' then $-\alpha(C)$ is a closed topological subgroup of C'. Note that H is a normal subgroup , since for every $(n', e) \in C' \oplus E$,

$$(n', e)(-\alpha(n), \iota(n)) = (n' - \alpha(n), e.\iota(n))) = (-\alpha(n) + n', \iota(n).e) = (-\alpha(n), \iota(n))(n', e)$$

Since $(-\alpha(n), \iota(n)) \in H$ and $\iota(n) \in H$ and by [5, Defintion 3.1] and $H \trianglelefteq E$, then $\iota(n).e$ is defined. So, $(n', e)H = H(n', e)$.

$$K = \frac{C' \oplus E}{H}, \qquad \sigma : E \to \frac{C' \oplus E}{H}, \;\; e \mapsto (0, e)H$$

K, $\frac{C' \oplus E}{H}$ are topological local groups [5, Lemma 1.8, Defintion 3.8].
Let V_1 be a neighborhood of the identity in E. By Lemma 2.6, there is a symmetric neighborhood V_0 in V such that $e_1 e_2 \in V_1$ for $e_1, e_2 \in V_0$. We define an action on K by

$$((n'_1, e_1)H).((n'_2, e_2)H) =: (n'_1 n'_2, e_1 e_2)H. \qquad \text{for } e_1, e_2 \in V_0$$

Now $\sigma(e_1 e_2) = (0, e_1 e_2)H = ((0, e_1)H)((0, e_2)H) = \sigma(e_1)\sigma(e_2) \in (0 \oplus V_1)H$ for $e_1, e_2 \in V_0$. Then σ is a strong homomorphism. We define

$$\iota' : C' \to \frac{C' \oplus E}{H}, \;\; \iota(n') = (n', 1_E)H, \qquad \pi' : ((n', e)H) \mapsto \pi(e) \qquad \eta' : C' \to ker\pi',$$

$$n \mapsto (0, \eta(n))H$$

We show that π' is an onto continuous strong homomorphism. For every $x \in X$, since π is onto, then there is $e \in E$, such that $\pi(e) = x$. We can write $\pi(e) = \pi'((n', e)H)$ for each $n' \in C'$. Then, π' is onto. If $((n'_1, e_1)H).((n'_2, e_2)H)$ is defined in $\frac{C' \oplus E}{H}$, then

$$\pi'(((n'_1, e_1)H).((n'_2, e_2)H)) = \pi'((n'_1 n'_2, e_1 e_2)H) = \pi(e_1 e_2) = \pi(e_1)\pi(e_2)$$

and

$$\pi'((n'_1, e_1)H).\pi'((n'_2, e_2)H) = \pi(e_1)\pi(e_2).$$

where $e_1, e_2 \in V_0$. So, π' is a local homomorphism.

Since π is strong and π' onto, we have

$$\pi'((n'_1, e_1)H).\pi'((n'_2, e_2)H) = \pi(e_1)\pi(e_2) = \pi(e_1.e_2) = \pi'((n_1 n_2, e_1 e_2)H),$$

where $e_1, e_2 \in V_0$. Now, we show that π' is an open continuous map. It is enough to show that for every $x \in X$, there is a symmetric neighborhood V_x such that $\pi'^{-1}(V_x)$ is open in K. Since π is open, onto and continuous, then there is $e \in E$ with $\pi(e) = x$ and a symmetric neighborhood V_e of e such that $\pi(V_e) = V_x$, so $V_e \subseteq \pi^{-1}(V_x)$. Then, $C' \oplus V_e$ is open in $C' \oplus E$. Suppose

$$H' = \{(-\alpha(n), \iota(n)) | \iota(n) \in V_e, n \in C'\},$$

Then, H' is a normal subgroup of $C' \oplus V_e$. So $H' = H \cap (C' \oplus V_e)$ and by [4, Theroem 17.2, p.94], H' is closed in $C' \oplus V_e$. Since $\frac{C' \oplus V_e}{H'}$ is open in $\pi'^{-1}(V_x)$ then π' is continuous.

We have $\pi'(\frac{C' \oplus V_e}{H'}) = \pi(V_e) = V_x$. So, π' is open. Hence, the diagram (3.2) is commutative $\pi'\sigma = \pi$, $\sigma\eta = \eta'$ and uniqueness of ${}_\alpha\varepsilon$ is similar to Lemma 3.2. □

Remark. As in Lemma 3.4, if there exists $\varepsilon \xrightarrow{(\alpha,\sigma,Id_{X_1})} {}_\alpha\varepsilon$, then ${}_\alpha\varepsilon$ is called a *pushout* of ε.

Note 3.6. As in Lemma 3.4, we will have the factorization of $\varepsilon_1 \xrightarrow{(\alpha_1,\sigma_1,\gamma_1)} \varepsilon$ with $\alpha = \alpha_1 : C_1 \to C$:

$$\varepsilon_1 \xrightarrow{(\alpha,\sigma,Id_{X_1})} {}_\alpha\varepsilon_1 \xrightarrow{(Id_{C_1},\sigma',\gamma_1)} \varepsilon .$$

Note 3.7. Consider

$$\varepsilon_1 \xrightarrow{(\alpha_1,\sigma_1,\gamma_1)} \varepsilon \xrightarrow{(\alpha_2,\sigma_2,\gamma_2)} \varepsilon_2$$

By Lemmas 3.2 and 3.5, there exist unique ε_{γ_1} and ${}_{\alpha_2}\varepsilon$ between ε_1, ε and ε, ε_2,respectively. Then

$$\varepsilon_1 \xrightarrow{(\alpha_1,\sigma_1',Id_{X_1})} \varepsilon_{\gamma_1} \xrightarrow{(Id_C,\sigma_1'',\gamma_1)} \varepsilon \xrightarrow{(\alpha_2,\sigma_2'',Id_X)} {}_{\alpha_2}\varepsilon \xrightarrow{(Id_{C_2},\sigma_2'',\gamma_2)} \varepsilon_2$$

Therefore, we have $\varepsilon_{\gamma_1} \longrightarrow {}_{\alpha_2}(\varepsilon_{\gamma_1})$ and $({}_{\alpha_2}\varepsilon)_{\gamma_1} \longrightarrow {}_{\alpha_2}\varepsilon$, since they are unique up to equivalent extensions. Then, ${}_{\alpha_2}(\varepsilon_{\gamma_1}) = ({}_{\alpha_2}\varepsilon)_{\gamma_1}$.

Let $\varepsilon_1 = (E_1, \pi_1, \eta_1)$ and $\varepsilon_2 = (E_2, \pi_2, \eta_2)$ be topological local extensions of an abelian topological group C_1 , C_2 by topological local group X_1, X_2, respectively. Suppose

$$\varepsilon_1 \oplus \varepsilon_2 : \qquad 0 \longrightarrow C_1 \oplus C_2 \xrightarrow{(\iota_1,\iota_2)} E_1 \oplus E_2 \xrightarrow{(\pi_1,\pi_2)} X_1 \oplus X_2 \longrightarrow 1 \tag{3.3}$$

Now we define an action in $Ext(X, C)$. Let $\varepsilon_1, \varepsilon_2 \in Ext(X, C)$, then $\varepsilon_1 + \varepsilon_2 = {}_{P_C}(\epsilon_1 \oplus \varepsilon_2)_{\Delta_X}$ where $P_C : C \oplus C \to C$, $P_C(c_1, c_2) = c_1$ is the projection map and $\Delta_X : X \to X \times X$, $\Delta(x) = (x, x)$ is the diagonal map. we have

$${}_{P_C}(\varepsilon_1 \oplus \varepsilon_2) : \qquad 0 \longrightarrow C \longrightarrow \frac{C \oplus E_1 \oplus E_2}{H} \longrightarrow X \oplus X \longrightarrow 1$$

$${}_{P_C}(\varepsilon_1 \oplus \varepsilon_2)_{\Delta_X} : \qquad 0 \longrightarrow C \longrightarrow E' \longrightarrow X \longrightarrow 1$$

where E' is a sublocal group of $\frac{C \oplus E_1 \oplus E_2}{H} \oplus X$, similar to Lemma 3.2.

Theorem 3.8. *Let C be an abelian topological group and X a topological local group. The set $Ext(X,C)$ of all equivalence classes of extensions of C by X is an abelian group under the binary operation:*

$$\varepsilon_1 + \varepsilon_2 =_{P_C} (\varepsilon_1 \oplus \varepsilon_2)_{\Delta_X}.\ \varepsilon_1, \varepsilon_2 \in Ext(X,C) \tag{3.4}$$

The class of the fibered extension $C \rightarrowtail C \oplus X \twoheadrightarrow X$ is the zero element of this group, while the inverse of any ε is the extension ${}_{-1_C}\varepsilon$. For, $i = 1,2$, and the homomorphisms $\alpha_i, \alpha : C \to C'$ and the strong homomorphisms $\gamma_i, \gamma : X \to X'$ one has

$$_{\alpha}(\varepsilon_1 + \varepsilon_2) \equiv \ {}_{\alpha}\varepsilon_1 + \ {}_{\alpha}\varepsilon_2, \qquad (\varepsilon_1 + \varepsilon_2)_\gamma = \varepsilon_{1\gamma} \ + \varepsilon_{2\gamma} \tag{3.5}$$

$$_{(\alpha_1+\alpha_2)}\varepsilon = \ {}_{\alpha_1}\varepsilon + \ {}_{\alpha_2}\varepsilon, \qquad \varepsilon_{(\gamma_1+\gamma_2)} \ = \varepsilon_{\gamma_1} + \ \varepsilon_{\gamma_2}. \tag{3.6}$$

Proof. Let ε_1 and ε_2 be two topological local extensions of an abelian topological group C by a topological local group X. We clearly have

$$_{(\alpha_1 \oplus \alpha_2)}(\varepsilon_1 \oplus \varepsilon_2) = \ {}_{\alpha_1}\varepsilon_1 \oplus \ {}_{\alpha_2}\varepsilon_2, \qquad (\varepsilon_1 \oplus \varepsilon_2)_{(\gamma_1 \oplus \gamma_2)} = \varepsilon_{1\gamma_1} \oplus \ \varepsilon_{2\gamma_2}, \tag{3.7}$$

By Lemma 3.2, for $\alpha : C \to C'$ and $P_C : C \oplus C \to C$, we have

$$\alpha P_C = P_{C'}(\alpha \oplus \alpha) : C \oplus C \to C',$$

and similarly for $\gamma : X' \to X$ and $\Delta_X : X \to X \oplus X$;

$$\Delta_X \gamma = (\gamma \oplus \gamma)\Delta_{X'} : X' \to X \oplus X.$$

Now we prove (3.5) and (3.6)

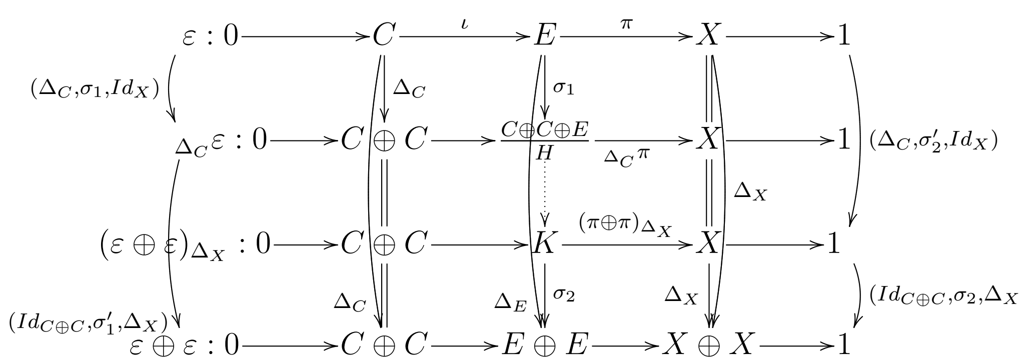
$$_{\alpha}(\varepsilon_1 + \varepsilon_2) \equiv_{\alpha P_C} (\varepsilon_1 \oplus \varepsilon_2)_{\Delta_X} \equiv_{P_{C'}(\alpha\oplus\alpha)} (\varepsilon_1 \oplus \varepsilon_2)_{\Delta_X} \equiv_{P_{C'}} ({}_{\alpha}\varepsilon_1 \oplus {}_{\alpha}\varepsilon_2)_{\Delta_X} \equiv_{\alpha} \varepsilon_1 + {}_{\alpha}\varepsilon_2.$$

$$(\varepsilon_1 + \varepsilon_2)_\gamma \equiv_{P_A} (\varepsilon_1 \oplus \varepsilon_2)_{\Delta_X \gamma} \equiv_{P_A} (\varepsilon_1 \oplus \varepsilon_2)_{(\gamma\oplus\gamma)\Delta_{X'}} \equiv_{P_A} (\varepsilon_{1\gamma} \oplus \varepsilon_{2\gamma})_{\Delta_{X'}} \equiv \varepsilon_{1\gamma} + \varepsilon_{2\gamma}.$$

For (3.6), it is enough to show that

$$_{\Delta_C}\varepsilon \equiv (\varepsilon \oplus \varepsilon)_{\Delta_X}, \qquad \varepsilon_{P_X} \equiv_{P_C} (\varepsilon \oplus \varepsilon). \tag{3.8}$$

Since $(\Delta_C, \Delta_E, \Delta_X) : \varepsilon \to \varepsilon \oplus \varepsilon$, then there exist ${}_{\Delta_C}\varepsilon$, $(\varepsilon \oplus \varepsilon)_{\Delta_X}$ between ε, $\varepsilon \oplus \varepsilon$ and ε, $\varepsilon \oplus \varepsilon$, respectively.



Hence $\Delta_E = \sigma_1' \circ \sigma_1 = \sigma_2 \circ \sigma_2'$. So there exists $\sigma' : \frac{C\oplus C\oplus E}{H} \to K$, $((c_1, c_2), e) + H \mapsto (\sigma_1'(((c_1, c_2), e) + H), {}_{\Delta_C}\pi(((c_1, c_2), e) + H))$ such that:

$$\begin{aligned} H = \{(-\Delta_C(c), \iota(c)) | c \in C\} \trianglelefteq C \oplus C \oplus E; \\ K = \{(e_1, e_2, x) | \pi \oplus \pi(e_1, e_2) = \Delta_X(x)\} \leq E \oplus E \oplus X; \\ \sigma_1' : ((c_1, c_2), e) + H) \mapsto \iota \oplus \iota(c_1, c_2) + \Delta_E(e); \\ {}_{\Delta_C}\pi : ((c_1, c_2), e) + H \mapsto \pi(e). \end{aligned}$$

Now we show that σ' is an isomorphism. It is enough to prove that $Id_c \circ_{\Delta_C} \pi = (\pi \oplus \pi)_{\Delta_X} \circ \sigma'$. Then by the five lemma [3], σ' is an isomorphism.
We have $Id_C({}_{\Delta_C}\pi((c_1, c_2), e)) = Id_C(\pi(e))$ and
$(\pi\oplus\pi)_{\Delta_X}(\sigma'((c_1, c_2), e)+H) = (\pi\oplus\pi)_{\Delta_X}(\sigma_1'(((c_1, c_2), e)+H), {}_{\Delta_C}\pi(((c_1, c_2), e)+H)) = {}_{\Delta_C}\pi(((c_1, c_2), e) + H) = \pi(e)$. Then, ${}_{\Delta_C}\varepsilon \equiv (\varepsilon \oplus \varepsilon)_{\Delta_X}$. Similarly, we have $\varepsilon_{P_X} \equiv_{P_C} (\varepsilon \oplus \varepsilon)$ by $(P_C, P_E, P_X) : \varepsilon \oplus \varepsilon \to \varepsilon$.

For $\alpha_i : C \to C'$ and $\gamma_i : X' \to X$, $i = 1, 2$, we define

$$\alpha_1 + \alpha_2 : \qquad C \xrightarrow{\Delta_C} C \oplus C \xrightarrow{\alpha_1\oplus\alpha_2} C' \oplus C' \xrightarrow{P_{C'}} C'$$

By (3.8), then (3.6) holds:

$${}_{\alpha_1}\varepsilon +_{\alpha_2} \varepsilon \equiv_{P_{C'}} ({}_{\alpha_1}\varepsilon \oplus {}_{\alpha_2}\varepsilon)_{\Delta_X} \equiv_{P_{C'}} ({}_{\alpha_1\oplus\alpha_2}(\varepsilon \oplus \varepsilon))_{\Delta_X} \equiv {}_{P_{C'}(\alpha_1\oplus\alpha_2)\Delta_C}\varepsilon \equiv {}_{\alpha_1+\alpha_2}\varepsilon.$$

Similarly, $\varepsilon_{\gamma_1} + \varepsilon_{\gamma_2} = \varepsilon_{\gamma_1+\gamma_2}$.

Now we show that $Ext(X, C)$ is a group. we clearly have

$$(\Delta_X \oplus Id_X)\Delta_X = (Id_X \oplus \Delta_X)\Delta_X, \tag{3.9}$$

and

$$P_C(P_C \oplus Id_C) = (Id_C \oplus P_C)P_C : C \oplus C \oplus C \to C \tag{3.10}$$

$$\begin{aligned} \varepsilon_1 + (\varepsilon_2 + \varepsilon_3) &= \varepsilon_1 + {}_{P_C}(\varepsilon_2 \oplus \varepsilon_3)_{\Delta_X} = {}_{P_C}(\varepsilon_1 \oplus {}_{P_C}(\varepsilon_2 \oplus \varepsilon_3)_{\Delta_X})_{\Delta_X} \\ &= {}_{P_C(Id_C\oplus P_C)}(\varepsilon_1 \oplus (\varepsilon_2 \oplus \varepsilon_3))_{(Id_X\oplus\Delta_X)\Delta_X}. \end{aligned}$$

Similarly

$$(\varepsilon_1 + \varepsilon_2) + \varepsilon_3 = {}_{P_C(P_C\oplus Id_C)}((\varepsilon_1 \oplus \varepsilon_2) \oplus \varepsilon_3)_{(\Delta_X\oplus Id_X)\Delta_X}.$$

By (3.9), (3.10), $E_1 \oplus (E_2 \oplus E_3) \equiv (E_1 \oplus E_2) \oplus E_3$, Note 3.7 and the uniqueness of lemmas 3.2, 3.5, we obtain

$${}_{P_C(Id_C\oplus P_C)}(\varepsilon_1 \oplus (\varepsilon_2 \oplus \varepsilon_3))_{(Id_X\oplus\Delta_X)\Delta_X} \equiv {}_{P_C(P_C\oplus Id_C)}((\varepsilon_1 \oplus \varepsilon_2) \oplus \varepsilon_3)_{(\Delta_X\oplus Id_X)\Delta_X}.$$

Hence, $(\varepsilon_1 + \varepsilon_2) + \varepsilon_3 \equiv \varepsilon_1 + (\varepsilon_2 + \varepsilon_3)$.
Suppose $\tau_C : C_1 \oplus C_2 \to C_2 \oplus C_1$, $\tau_C(c_1, c_2) = (c_2, c_1)$ is an isomorphism and $(\tau_C, \tau_E, \tau_X) : \varepsilon_1 \oplus \varepsilon_2 \to \varepsilon_2 \oplus \varepsilon_1$.

We can obtain ${}_{\tau_C}(\varepsilon_1 \oplus \varepsilon_2) \equiv (\varepsilon_2 \oplus \varepsilon_1)_{\tau_X}$. It is easy to show that $P_C \tau_C = P_C$ and $\tau_X \Delta_X = \Delta_X$. Thus,

$$\varepsilon_1 + \varepsilon_2 = {}_{P_C}(\varepsilon_1 \oplus \varepsilon_2)_{\Delta_X} = {}_{P_C \tau_C}(\varepsilon_1 \oplus \varepsilon_2)_{\Delta_X} \equiv {}_{P_C}(\varepsilon_2 \oplus \varepsilon_1)_{\tau_X \Delta_X} = {}_{P_C}(\varepsilon_2 \oplus \varepsilon_1)_{\Delta_X} = \varepsilon_2 + \varepsilon_1.$$

So, $Ext(X, C)$ is abelian.
For every $\varepsilon \in Ext(X, C)$, there is the commutative diagram:

$$\begin{array}{lccccccccc}
\varepsilon : & 0 & \longrightarrow & C & \xrightarrow{\iota} & E & \xrightarrow{\pi} & X & \longrightarrow & 1 \\
 & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \sigma} & & \| & & \\
\varepsilon_0 : & 0 & \longrightarrow & C & \longrightarrow & C \oplus X & \longrightarrow & X & \longrightarrow & 1
\end{array}$$

where $\sigma(e) = (0, \pi(e))$, then $\varepsilon_0 = {}_{0_C}\varepsilon$ where $0_C : C \to C$ is a zero homomorphism. Therefore,

$$\varepsilon + \varepsilon_0 = {}_{Id_c}\varepsilon + {}_{0_C}\varepsilon = {}_{(Id_C + 0_C)}\varepsilon \equiv {}_{Id_c}\varepsilon = \varepsilon$$

Hence, ε_0 is the zero element of $Ext(X, C)$.
By (3.6), and

$$\varepsilon + {}_{-Id_C}\varepsilon = {}_{Id_c}\varepsilon + {}_{-Id_C}\varepsilon = {}_{(}Id_C - Id_C)\varepsilon \equiv {}_{0_c}\varepsilon = \varepsilon_0$$

Then, ${}_{-Id_C}\varepsilon$ is the inverse element of ε of $Ext(X, C)$. Therefore, $Ext(X, C)$ is a group. □

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