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Cone Metric Space and Some Fixed Point Results for Pair of Expansive Mappings

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Abstract

The purpose of this work is to extend and generalize some fixed point theorems for Expansive type mappings in complete cone metric spaces. Our theorems improve and generalize of the results [1] and [3].

Keywords: *Complete cone metric space, common fixed point, expansive type contractive mapping, Non-Normal cones.*

1 Introduction

Very recently, Huang and Zhang [1] introduce the concept of cone metric spaces. They have proved some fixed point Theorems for contractive mappings using normality of the cone. The results in [1] were generalized by Sh. Rezapour and Hamlbarani [2] omitted the assumption of normality on the cone, which is a milestone in cone metric space. Many authors have studied fixed point theorem in such spaces, see for instance [4]. [5], [8]. [11] and [13]. In sequel, the authors [7], [8] and [12] introduced a new class of multifunction and obtained a unique fixed point. These results are also generalized by [15] with normal constant K = 1.Also we have observed the recent work of fixed point s for non-explosive map in cone metric spaces see ([6], [10]). Recently, Azam introduced cone rectangular metric spaces in [14].

The purpose of this paper is to analyse the existence and uniqueness of fixed points for pair of expansive mappings defined on a complete metric space. we generalize the results of [3].

2 Preliminary Notes

First we recall the definition of cone metric spaces and some properties of theirs [1].

Definition: 2.1 [1]: *Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if:*

(i) P is closed, non-empty and $P \neq \{0\}$;
(ii) $a, b \in R$, $a, b \geq 0$ $x, y \in \boldsymbol{P} \Rightarrow ax+by \in \boldsymbol{P}$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For given a cone $P \subset$ E , we define a Partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in \boldsymbol{P}$. We shall write x $\ll$ y to denote $x \leq y$ but $x \neq y$ to denote $y - x \in p^0$, where p^0 stands for the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ .The least positive number K satisfying the above is called the normal constant of P. The least positive number satisfying the above is called the normal constant P. The cone P is called regular if every increasing sequence which is bounded from above is convergent .that is , if $\{x_n\}n \geq 1$ *is a* sequence such that $x_1 \leq x_2 \leq \ldots.. x_n \leq \ldots.. \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_{n-x}\| \longrightarrow 0$ $(n \longrightarrow \infty)$. Equivalently the cone p is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 2.2[2]**:**

(i) Every regular cone is normal
(ii) For each k > 1, there is a normal cone with normal constant K > k.

In following we always suppose E is a Banach space. P is a cone in E with $intP \neq \phi$and $\leq$ is partial ordering with respect to P.

Definition 2.3[1]: *Let X be a non – empty set. Suppose the mapping $d: X \times X \rightarrow E$ satisfies the following condition:*

(i) $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. It is obvious that cone metric spaces generalize metric space.

Example 2.3 [1]: Let $E = R^2$, $P = \{(x,y) \in E: x,y \geq 0\}$, $X = R$ and $d: X \times X \to E$,

On defined by $d(x,y) = (|x - y|, \propto |x - y|)$ where $\propto \geq 0$ is a constant. Then (X, d) is a cone metric space.

Example: 2.4: Let E= l^1, P $= \{\{x_n\}n \geq 1 \in E: x_n \geq 0, for\ all\ n\}$ (X, d) a metric space and $d: X \times X \to E$, defined by $d(x,y) = \left\{\frac{d(x,y)}{2^n}\right\}$ $n \geq 1$.Then (X, d) is a cone metric space.

Definition 2.5 [1]: *Let (X, d) be a cone metric space,* $x \in X$ *and* $\{x_n\}_{n \geq 1}$ *a sequence in X. Then,*

(i) $\{x_n\}_{n \geq 1}$ *converges to x whenever for every* $c \in E$ *with* $0 \ll c$, *there is a natural number N. such that* $d(x_n, x) \ll c$ *for all* $n \geq N$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$, $(n \to \infty)$.
(ii) $\{x_n\}_{n \geq 1}$ *is said to be a Cauchy sequence if for every* $c \in E$ *with* $o \ll c$, *there is a natural number N such that* $d(x_n, x_m) \ll c$ *for all* $n, m \geq N$.
(iii) *(X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent.*

Definition 2.6[1]: *Let* (X, d) *be a cone metric space, P be a cone in real Banach space, if*
(i) $a \in P$ *and* $a \ll c$ *for some* $k \in [0,1]$, *then* $a = 0$.
(ii) $u \leq v, v \ll w$, *then* $u \ll w$.

3 Main Results

In this section we shall prove some fixed point theorems for pair of expansive type contractive mappings by using omitting the assumption of normality of the theorems 2.1, 2.3, 2.5, 2.6 of [3].

Theorem 3.1: *Let* (X, d) *be a cone metric space and suppose* $f_1, f_2: X \to X$ *be any two onto mapping satisfies the contractive condition*

$$d(f_1x, f_2y) \geq Kd(x,y) \tag{1}$$

for all $x, y \in X$,where $K > 1$ is a constant. Then f_1 and f_2 have a unique common fixed point in X.

Proof: If $f_1 x = f_2 y$ then

$$0 \geq Kd(x,y) \Rightarrow 0 = d(x,y) \Rightarrow x = y.$$

Thus, f_1 is one to one. Define $G_1 = f_1^{-1}$

$$d(x,y) \geq K\, d\left(f_1^{-1}x,\ f_1^{-1}y\right)$$
$$= d(G_1 x, G_1 y).$$

So, $d(G_1 x, G_1 y) \leq hd(x,y)$ where $h = \frac{1}{K} < 1$. By theorem 2.3 in [2]. G_1 Has a unique fixed point x^* in $X.i.e.\ G_1 x^* = x^* \Rightarrow f_1^{-1}x^* = x^* \Rightarrow x^* = f_1 x$.

Therefore, x^* is a fixed point of f_1. Similarly it can be established that $x^* = f_2 x$.

Hence $x^* = f_1 x = f_2 x$. Thus x^* is the common fixed point of pair maps f_1 and f_2 .

Corollary 3.2: *Let (X,d) be a cone metric space and suppose $f_1, f_2 : X \to X$ be any two onto mapping satisfying the condition*

$$d\left(f_1^{2n+1}x, f_2^{2n+2}y\right) \geq Kd(x,y) \qquad (2)$$

for all $x, y \in X$,where $K > 1$ is a constant. Then f_1 and f_2 have a common fixed point in X.

Theorem 3.1: *Let (X,d) be a cone metric space and suppose $f_1, f_2 : X \to X$ be any two continuous and onto mapping satisfying the condition*

$$d(f_1 x, f_2 y) \geq K[d(f_1 x, x) + d(f_2 y, y)] \qquad (3)$$

for all $x, y \in X$,where $\frac{1}{2} < K \leq 1$ is a constant. Then f_1 and f_2 have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in$X$. Since f_1 and f_2 be onto (surjective), there exist $x_0 \in X$ and $x_1 \in X$ such that

$$f_1(x_1) = x_0 \text{ and } f_2(x_2) = x_1.$$

In this way, we define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ by

$$x_{2n} = f_1 x_{2n+1} \text{ for } n = 0, 1, 2, 3\ldots\ldots\ldots \text{ and}$$
$$x_{2n+1} = f_2 x_{2n+2} \text{ for } n = \text{o}, 1, 2, 3\ldots\ldots\ldots\ldots\ldots$$

Note that, if $x_{2n} = x_{2n+1}$, for some $n \geq 1$, then x_{2n} is fixed point of f_1 and f_2.

Now putting $x = x_{2n+1}$ and $y = x_{2n+2}$, we have

$$d(x_{2n,}\, x_{2n+1}) = d(f_1 x_{2n+1}, f_2 x_{2n+2})$$

$$d(x_{2n,}\, x_{2n+1}) \geq K[d(f_1 x_{2n+1},\ x_{2n+1}) + d(f_2 x_{2n+2},\ x_{2n+2})]$$

$$= K[d(x_{2n},\ x_{2n+1}) + d(x_{2n+1},\ x_{2n+2})]$$

$$(1-K) d(x_{2n,}\, x_{2n+1}) \geq \text{K}\ d(x_{2n+1},\ x_{2n+2})$$

$$d(x_{2n,}\, x_{2n+1}) \geq \frac{1}{1-K} d(x_{2n+1},\ x_{2n+2})$$

$$\Rightarrow d(x_{2n+1,}\, x_{2n+2}) \leq h d(x_{2n},\ x_{2n+1})\,. \tag{4}$$

Where $h = \frac{1-K}{K}$, $0 \leq h \leq 1$.

In general

$$d(x_{2n,}\, x_{2n+1}) \leq h\, d(x_{2n-1}, x_{2n}) \leq \ldots\ldots\ldots \leq h^{2n} d(x_{0,}\, x_1).$$

So for $n < m$, we have

$$d(x_{2n,}\, x_{2m}) \leq d(x_{2n,}\, x_{2n+1}) + \ldots\ldots\ldots + d(x_{2m-1,}\, x_{2m})$$

$$\leq (h^{2n} + h^{2n+1} + \ldots\ldots\ldots + h^{2m-1})\ d(x_0, x_1)$$

$$\leq \frac{h^{2n}}{1-h} d(x_0, x_1) \tag{5}$$

Let $0 \leq c$ be given, choose a natural number N_1 such that $\frac{h^{2n}}{1-h} d(x_0, x_1) \leq c$, for all $n \geq N_1$. Thus $d(x_{2n,}\, x_{2m}) \leq c$, for $n < m$. Therefore $\{x_{2n}\}$ is a Cauchy sequence in (X, d). since (X, d) is a complete cone metric space, there exist $x^* \in X$ such that $x_{2n} \rightarrow x^*\ as\ n \rightarrow \infty$. If f_1 is a continuous, then

$$d(f_1 x^*, x^*) \leq d(f_1 x_{2n+1,}\ f_1 x^*) + d(f_1 x_{2n+1,}\, x^*) \rightarrow 0\ as\ n \rightarrow \infty.$$

Since $x_{2n} \rightarrow x^*$ and $f_1 x_{2n+1,} \rightarrow f_1 x^*\ as\ n \rightarrow \infty$. Therefore $d(f_1 x^*, x^*) = 0$.

This implies that $f_1 x^* = x^*$. hence x^* is a fixed point of f_1.

Similarly, it can be established that $f_2x^* = x^*$, Therefore $f_1x^* = x^* = f_2x^*$.

Thus x^* is the common fixed point of pair of maps f_1 and f_2.This completes the proof.

Theorem 3.2: *Let* (X, d) *be a cone metric space and suppose* $f_{1,} f_2 : X \to X$ *be any two continuous and onto mapping satisfying the condition*

$$d(f_1x, f_2y) \geq Kd(x, y) + Ld(f_2y, x) \tag{6}$$

for all $x, y \in X$, where $L \geq 0, K > 1$ is a constant. Then f_1 and f_2 have a common fixed point in X.

Proof: Let x_0 be an arbitrary point in$X$. Since f_1 and f_2 be onto (surjective), there exist $x_0 \in X$ and $x_1 \in X$ such that

$$f_1(x_1) = x_0 \text{ and } f_2(x_2) = x_1.$$

In this way, we define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ by

$$x_{2n} = f_1x_{2n+1} \text{ for } n = 0, 1, 2, 3\ldots\ldots\ldots \text{ and}$$

$$x_{2n+1} = f_2x_{2n+2} \text{ for } n = \text{o}, 1, 2, 3\ldots\ldots\ldots\ldots\ldots$$

Note that, if $x_{2n} = x_{2n+1}$, for some $n \geq 1$, then x_{2n} is fixed point of f_1 and f_2.

Now putting $x = x_{2n+1}$ and $y = x_{2n+2}$, we have

$$\begin{aligned} d(x_{2n,} x_{2n+1}) &= d(f_1x_{2n+1}, f_2x_{2n+2}) \\ &\geq Kd(x_{2n+1}, x_{2n+2}) + Ld(f_2x_{2n+2}, x_{2n+1}) \\ &= Kd(x_{2n+1}, x_{2n+2}) + Ld(x_{2n+1}, x_{2n+1}) \\ &\geq Kd(x_{2n+1}, x_{2n+2}) \end{aligned}$$

$$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{K} d(x_{2n,} x_{2n+1}), \tag{7}$$

where $h = \frac{1}{K}$, $0 \leq h \leq 1$

So for $n < m$, we have

$$d(x_{2n,} x_{2m}) \leq d(x_{2n,} x_{2n+1}) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + d(x_{2m-1,} x_{2m})$$

$$\leq (h^{2n} + h^{2n+1} + \dots\dots\dots\dots\dots + h^{2m-1})\, d(x_0, x_1)$$

$$\leq \frac{h^{2n}}{1-h} d(x_0, x_1) \qquad (8)$$

Let $0 \leq c$ be given, choose a natural number N_1 such that $\frac{h^{2n}}{1-h} d(x_0, x_1) \leq c$, for all $n \geq N_1$. Thus $d(x_{2n,} x_{2m}) \leq c$, for $n < m$. Therefore $\{x_{2n}\}$ is a Cauchy sequence in (X, d). since (X, d) is a complete cone metric space, there exist $x^* \in X$ such that $x_{2n} \to x^*$ *as* $n \to \infty$. If f_1 is a continuous, then

$$d(f_1x^*, x^*) \leq dd(f_1x_{2n+1,}\ f_1x^*) + d(f_1x_{2n+1,}\ x^*) \to 0 \ as\ n \to \infty.$$

Since $x_{2n} \to x^*$ and $f_1x_{2n+1,} \to f_1x^*$ *as* $n \to \infty$. Therefore $d(f_1x^*, x^*) = 0$.

This implies that $f_1x^* = x^*$. hence x^* is a fixed point of f_1.

Similarly, it can be established that $f_2x^* = x^*$.Therefore $f_1x^* = x^* = f_2x^*$.

Thus x^* is the common fixed point of pair of maps f_1 and f_2.This completes the proof.

Theorem 3.3: *Let (X, d) be a cone metric space and suppose $f_{1,} f_2 : X \to X$ be any two continuous and onto mapping satisfying the condition*

$$d(f_1x, f_2y) \geq Kd(x,\ y) + Ld(x,\ f_1x) + \mathrm{M}\, d(y, f_2y) \qquad (9)$$

for all $x, y \in X$,where $K \geq -1, L \geq 1$ *and* $\mathrm{M} < are$ constant, with $K + L + M > 1$. Then f_1 and f_2 have a common fixed point in *X*.

Proof: Let x_0 be an arbitrary point in X. Since f_1 and f_2 be onto (surjective), there exist $x_0 \in X$ and $x_1 \in X$ such that

$$f_1(x_1) = x_0 \ \text{ and } \ f_2(x_2) = x_1.$$

In this way, we define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ by

$x_{2n} = f_1x_{2n+1}$ for n = 0, 1, 2, 3……… and
$x_{2n+1} = f_2x_{2n+2}$ for n =o, 1, 2, 3…………

Note that, if $x_{2n} = x_{2n+1}$, for some $n \geq 1$, then x_{2n} is fixed point of f_1 and f_2.

Now putting $x = x_{2n+1}$ and $y = x_{2n+2}$, we have

$$d(x_{2n,} x_{2n+1}) = d(f_1x_{2n+1}, f_2x_{2n+2})$$

$$d(x_{2n,}\, x_{2n+1}) \geq Kd(x_{2n+1}, x_{2n+2}) + Ld(x_{2n+1},\ f_1x_{2n+1}) + Md(x_{2n+2},\ f_2x_{2n+2})$$

$$\geq Kd(x_{2n+1}, x_{2n+2}) + Ld(x_{2n+1}, x_{2n}) + Md(x_{2n+2},\ x_{2n+1})$$

$$= Ld(x_{2n}, x_{2n+1}) + (K+M)d(x_{2n+1}, x_{2n+2})$$

$$(1-L)d(x_{2n,}\, x_{2n+1}) \geq (\mathrm{K}+M)\ d(x_{2n+1},\ x_{2n+2})$$

$$d(x_{2n,}\, x_{2n+1}) \geq \frac{K+M}{1-L}\, d(x_{2n+1},\ x_{2n+2})$$

$$\Rightarrow d(x_{2n+1,}\, x_{2n+2}) \leq \frac{1-L}{K+M}\, d(x_{2n,}\, x_{2n+1})$$

$$\Rightarrow d(x_{2n+1,}\, x_{2n+2}) \leq h\, d(x_{2n},\ x_{2n+1})\,.$$

Where $h = \frac{1-L}{K+M}$, $0 \leq h \leq 1$.

In general

$$d(x_{2n,}\, x_{2n+1}) \leq h\, d(x_{2n-1}, x_{2n}) \leq \ldots\ldots\ldots \leq h^{2n}d(x_{0,}\, x_1).$$

So for $n < m$, we have

$$d(x_{2n,}\, x_{2m}) \leq d(x_{2n,}\, x_{2n+1}) + \ldots\ldots\ldots + d(x_{2m-1,}\, x_{2m})$$

$$\leq (h^{2n} + h^{2n+1} + \ldots\ldots\ldots\ldots + h^{2m-1})\, d(x_0, x_1)$$

$$\leq \frac{h^{2n}}{1-h} d(x_0, x_1) \qquad (10)$$

Let $0 \leq c$ be given, choose a natural number N_1 such that $\frac{h^{2n}}{1-h}\, d(x_0, x_1) \leq c$, for all $n \geq N_1$. Thus $d(x_{2n,}\, x_{2m}) \leq c$, for $n < m$. Therefore $\{x_{2n}\}$ is a Cauchy sequence in (X, d). since (X, d) is a complete cone metric space, there exist $x^* \in X$ such that $x_{2n} \to x^*\ as\ n \to \infty$. If f_1 is a continuous, then

$$d(f_1x^*, x^*) \leq d(f_1x_{2n+1,}\ f_1x^*) + d(f_1x_{2n+1,}\, x^*) \to 0\ as\ n \to \infty.$$

Since $x_{2n} \to x^*$ and $f_1x_{2n+1,} \to f_1x^*\ as\ n \to \infty$. Therefore $d(f_1x^*, x^*) = 0$.

This implies that $f_1x^* = x^*$., Hence x^* is a fixed point of f_1.

Similarly, it can be established that $f_2x^* = x^*$. Therefore $f_1x^* = x^* = f_2x^*$. Thus x^* is the common fixed point of pair of maps f_1 and f_2. This completes the proof.

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