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On Some Properties of Anti-Q-Fuzzy Normal Subgroups

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Abstract

In this paper, we introduce the concept of Anti-Q-fuzzy normal subgroup and Anti- Q-fuzzy left (right) cosets of a group and discussed some of its properties.

Keywords: *Anti- fuzzy subgroup, anti- Q-fuzzy subgroup, anti- Q-fuzzy normal subgroup, anti Q-fuzzy normaliser, Anti-Q-fuzzy left (right) cosets, anti-Q-homomorphism.*

1 Introduction

The concept of fuzzy sets was initiated by Zadeh in 1965 [19]. Since then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [13] gave the idea of fuzzy subgroups in 1971. A. Solairaju and R. Nagarajan [16] introduced a new algebraic structure Q-fuzzy group in 2008. T. Priya, T. Ramachandran and K.T. Nagalakshmi [12] introduced the concept of Q-fuzzy normal subgroups. R. Biswas [1] introduced the concept of anti fuzzy subgroups of a group in 1990. Modifying his idea, we introduced a new algebraic structure anti-Q-fuzzy normal subgroups anti- Q-fuzzy left (right) cosets, Cartesian products have been discussed and some of its important properties were obtained.

2 Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.

Definition 2.1[19]: *Let X be any non empty set. A fuzzy subset* μ *of X is a function* $\mu : X \rightarrow [0, 1]$.

Definition 2.2[10]: *Let* μ *be an anti fuzzy subgroup of a group G. For any t* $\in [0,1]$*, we define the level subset of* μ *is the set,* $\mu_t = \{ x \in G \;/\; \mu(x) \leq t \}$.

Definition 2.3[1]: *A fuzzy set* μ *of a group G is called an anti fuzzy subgroup of G, if for all x, y* $\in$ *G*

(i) $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$
(ii) $\mu(x^{-1}) = \mu(x)$

Definition 2.4[14]: *An anti fuzzy subgroup* μ *of a group G is called an anti fuzzy normal subgroup of G if for all x, y* $\in$ *G,* $\mu(xyx^{-1}) = \mu(y)$ *or* $\mu(xy) = \mu(yx)$.

Definition 2.5[16]: *Let Q and G be any two sets. A mapping* $\mu : G \times Q \rightarrow [0, 1]$ *is called a Q-fuzzy set in G.*

3 Anti-Q-Fuzzy Normal Subgroups

Definition 3.1: *A Q-fuzzy set* μ *of a group G is called an Anti- Q-fuzzy subgroup of G, if for all x, y* $\in$ *G, q* $\in$ *Q,*

(i) $\mu(xy, q) \leq \max\{\mu(x,q), \mu(y,q)\}$
(ii) $\mu(x^{-1}, q) = \mu(x, q)$

Definition 3.2: *An anti- Q- fuzzy subgroup μ of a group G is called an anti-Q-fuzzy normal subgroup of G if for all x, y $\in$ G and q $\in$ Q,$\mu(xyx^{-1},q) = \mu(y, q)$ or $\mu(xy , q) = \mu(yx , q)$.*

Definition 3.3: *Let μ be an anti-Q-fuzzy subgroup of a group G. For any t $\in$[0, 1], we define the level subset of μ as, $\mu_t = \{x\in G, q \in Q / \mu(x,q) \leq t\}$.*

Theorem 3.1: *Let μ be a Q-fuzzy subset of a group G. Then μ is an anti-Q–fuzzy subgroup of G iff the level subsets μ_t, t$\in$[0, 1] are subgroups of G.*

Proof: Let μ be an anti-Q-fuzzy subgroup of G and the level subset

$$\mu_t = \{x\in G / \mu (x, q) \leq t, t\in [0, 1]\}$$

Let x, y $\in \mu_t$. Then $\mu (x, q) \leq t$ & $\mu (y, q) \leq t$

$$\begin{aligned} \text{Now } \mu (xy^{-1}, q) &\leq \max \{\mu (x, q), \mu (y^{-1}, q)\} \\ &= \max \{\mu (x, q), \mu (y, q)\} \\ &\leq \max \{t, t\} \end{aligned}$$

Therefore, $\mu (xy^{-1}, q) \leq t$, hence $xy^{-1} \in \mu_t$. Thus μ_t is a subgroup of G.

Conversely, let us assume that μ_t be a subgroup of G.

Let x, y $\in \mu_t$. Then $\mu (x, q) \leq t$ and $\mu (y, q) \leq t$

Also, $\mu (xy^{-1}, q) \leq t$, since $xy^{-1} \in \mu_t$

$$\begin{aligned} &= \max \{t , t\} \\ &= \max \{ \mu (x , q) , \mu (y , q) \} \end{aligned}$$

That is, $\mu (xy^{-1}, q) \leq \max \{\mu (x, q), \mu (y, q)\}$.

Hence μ is an anti-Q-fuzzy subgroup of G.

Definition 3.4: *Let μ be an anti- Q-fuzzy subgroup of a group G. Then $N(\mu) =\{a\in G / \mu (axa^{-1},q) = \mu (x ,q)$, for all $x \in G$, $q\in Q$ }, is called an anti-Q-fuzzy Normaliser of μ.*

Theorem 3.2: *Let μ be a Q-fuzzy subset of G. Then μ is an anti- Q- fuzzy normal subgroup of G iff the level subsets μ_t, t$\in$[0,1] are normal subgroups of G.*

Proof: Let μ be an anti-Q- fuzzy normal subgroup of G and the level subsets μ_t, t$\in$[0,1], is a subgroup of G. Let x$\in$G and a$\in \mu_t$, then $\mu (a, q) \leq t$.

Now, μ (xax^{-1},q) = μ (a, q) $\leq$ t,

Since μ is an anti-Q-fuzzy normal subgroup of G, μ (xax^{-1}, q) $\leq$ t.

Therefore, xax^{-1} $\in$ μ_t . Hence μ_t is a normal subgroup of G.

Theorem 3.3: *Let μ be an anti- Q-fuzzy subgroup of a group G. Then*

- *i. N(μ) is a subgroup of G.*
- *ii. μ is an anti- Q-fuzzy normal $\Leftrightarrow$ N(μ) = G.*
- *iii. μ is an anti Q-fuzzy normal subgroup of the group N(μ).*

Proof:

(i) Let a, b $\in$ N(μ) then μ (axa^{-1}, q) = μ (x, q), for all x $\in$ G.

μ (bxb^{-1}, q) = μ (x, q), for all x $\in$ G.

$$\begin{aligned}\text{Now } \mu(abx(ab)^{-1}, q) &= \mu(abxb^{-1}a^{-1}, q) \\ &= \mu(bxb^{-1}, q) \\ &= \mu(x, q)\end{aligned}$$

Thus we get, μ (abx(ab)$^{-1}$, q) = μ (x , q) $\Rightarrow$ ab $\in$ N(μ)

Therefore, N(μ) is a subgroup of G.

(ii) Clearly N(μ) $\subseteq$ G , μ is an anti- Q-fuzzy normal subgroup of G.

Let a $\in$ G, then μ (axa^{-1}, q) = μ (x, q).

Then a $\in$ N(μ) $\Rightarrow$ G $\subseteq$ N(μ).

Hence N(μ) = G.

Conversely, let N(μ) = G.

Clearly μ (axa^{-1}, q) = μ (x, q), for all x $\in$ G and a $\in$ G.

Hence μ is an anti- Q – fuzzy normal subgroup of G.

(iii) From (2), μ is an anti- Q-fuzzy normal subgroup of a group N(μ).

Definition 3.5: *Let μ be a Q-fuzzy subset of G and let ${}_xf : G \times Q \rightarrow G \times Q$ [$f_x : G \times Q \rightarrow G \times Q$] be a function defined by ${}_xf(a , q) = (xa , q)$ [$f_x(a , q) = (ax , q)$]. A Q-fuzzy left (right) coset*

${}_x\mu$ (μ_x) is defined to be ${}_xf(\mu)$ ($f_x(\mu)$).

It is easily seen that $({}_x\mu)(y , q) = \mu (x^{-1}y , q)$ and $(\mu_x)(y , q) = \mu(yx^{-1} , q)$, for every (y , q) in G x Q.

Theorem 3.4 [6]: *Let μ be a Q-fuzzy subset of G. Then the following conditions are equivalent for each x, y in G.*

(i) $\mu (xyx^{-1} , q) \geq \mu (y , q)$
(ii) $\mu (xyx^{-1} , q) = \mu (y , q)$
(iii) $\mu (xy , q) = \mu (yx ,q)$
(iv) ${}_x\mu = \mu_x$
(v) ${}_x\mu_x^{-1} = \mu$

Proof: Straight forward.

Theorem 3.5: *If μ is an anti- Q-fuzzy subgroup of G, then $g\mu g^{-1}$ is also an anti-Q-fuzzy subgroup of G, for all $g \in G$ and $q \in Q$.*

Proof: Let μ be an anti- Q-fuzzy subgroup of G.

Then **(i)**
$$\begin{aligned}(g\mu g^{-1}) (xy , q) &= \mu (g^{-1}(xy)g , q)\\ &= \mu(g^{-1}(xgg^{-1}y)g , q)\\ &= \mu ((g^{-1}xg)(g^{-1}yg) , q)\\ &\leq \max\{ \mu (g^{-1}xg, q) , \mu (g^{-1}yg , q)\}\\ &\leq \max\{g\mu g^{-1}(x , q) , g\mu g^{-1}(y , q) \},\end{aligned}$$
for all x , y in G and $q \in Q$.

(ii)
$$\begin{aligned}g\mu g^{-1}(x , q) &= \mu(g^{-1}xg , q)\\ &= \mu ((g^{-1}xg)^{-1} , q)\\ &= \mu (g^{-1}x^{-1}g , q)\\ &= g\mu g^{-1}(x^{-1} , q)\end{aligned}$$
, for all x , y in G and $q \in Q$.

Hence $g\mu g^{-1}$ is an anti- Q-fuzzy subgroup of G.

Theorem 3.6: *If μ is an anti- Q-fuzzy normal subgroup of G, then $g\mu g^{-1}$ is also an anti- Q-fuzzy normal subgroup of G , for all $g \in G$ and $q \in Q$.*

Proof: Let μ be an anti- Q-fuzzy normal subgroup of G. then $g\mu g^{-1}$ is a subgroup of G.

Now $g\mu g^{-1}(xyx^{-1},q) = \mu(g^{-1}(xy\, x^{-1})\, g, q)$
$= \mu(xy\, x^{-1}, q)$
$= \mu(y, q)$
$= \mu(g\, y\, g^{-1}, q)$
$= g\mu g^{-1}(y,q)$.

Thus $g\mu g^{-1}$ is also an anti- Q-fuzzy normal subgroup of G

Theorem 3.7: *The intersection of any two anti –Q-fuzzy subgroups of G is also an anti –Q-fuzzy subgroup of G.*

Proof: Let λ and μ be two anti-Q-fuzzy subgroups of G.

Then $(\lambda\cap\mu)\, (xy^{-1}, q) = \min\, (\lambda\, (xy^{-1}, q),\mu\, (xy^{-1}, q))$
$\leq \min\{\max\{\lambda(x , q) , \lambda\, (y ,q)\}, \max\{\mu(x , q) , \mu\, (y ,q)\}\}$
$\leq \max\{ \min\{\lambda(x , q),\mu(x ,q)\},\min\{\lambda(y,q),\mu(y ,q)\}\}$
$= \max \{ (\lambda\cap\mu)\, (x , q) , (\lambda\cap\mu)\, (y , q)\}$

Thus $(\lambda\cap\mu)(xy^{-1}, q) \leq \max\{(\lambda\cap\mu)(x, q),(\lambda\cap\mu)(y,q)\}$

Therefore $\lambda\cap\mu$ is an anti Q-fuzzy subgroup of G.

Remark: If μ_i , $i\in\Delta$ is an anti- Q-fuzzy subgroup of G, then $\cap\, \mu_i$ is an anti- Q- $i\in\Delta$ fuzzy subgroup of G.

Theorem 3.8: *The intersection of any two anti-Q-fuzzy normal subgroups of G is also an anti- Q-fuzzy normal subgroup of G.*

Proof: Let λ and μ be two anti- Q-fuzzy normal subgroups of G. According to theorem 3.7, $\lambda\cap\mu$ is an anti-Q-fuzzy subgroup of G.

Now for all x, y in G, we have

$(\lambda\cap\mu)\, (xyx^{-1} , q) = \max\, (\lambda(xyx^{-1} , q) , \mu(xyx^{-1} , q))$
$= \max\, (\lambda\, (y , q) , \mu(y , q))$
$=(\lambda\cap\mu)\, (y , q)$

Hence $\lambda\cap\mu$ is an anti- Q-fuzzy normal subgroup of G.

Remark: If μ_i , $i\in\Delta$ are anti-Q-fuzzy normal subgroup of G, then $\cap\, \mu_i$ is an anti Q-fuzzy normal subgroup of G. $i\in\Delta$

Definition 3.6: *The mapping $f: G \times Q \rightarrow H \times Q$ is said to be a group Q-homomorphism if*

(i) f: G → H is a group homomorphism
(ii) f(xy , q) = (f(x)f(y) , q) , for all x , y ∈ G and q ∈ Q.

Definition 3.7: *The mapping f: G x Q → H x Q is said to be a group anti-Q-homomorphism if*

(i) f: G → H is a group homomorphism
(ii) f(xy , q) = (f(y)f(x) , q) , for all x , y ∈ G and q ∈ Q.

Theorem 3.9: *Let f: G x Q → H x Q be a group anti-Q- homomorphism.*

(i) *If μ is an anti- Q-fuzzy normal subgroup of H, Then f^{-1}(μ) is an anti- Q-fuzzy normal Subgroup of G.*
(ii) *If f is an epimorphism and μ is an anti- Q-fuzzy normal subgroup of G, then f(μ) is an anti-Q-fuzzy normal subgroup of H.*

Proof:

(i) Let f: G x Q → H x Q be a group anti- Q- homomorphism and let μ be an anti- Q-fuzzy normal subgroup of H. Now for all x, y ∈ G, we have

$$
\begin{aligned}
f^{-1}(\mu)(xyx^{-1}, q) &= \mu(f(xyx^{-1}, q)) \\
&= \mu(f(x)^{-1}f(y)f(x), q) \\
&= \mu(f(y), q) \\
&= f^{-1}(\mu)(y, q)
\end{aligned}
$$

Hence f^{-1}(μ) is an anti-Q-fuzzy normal subgroup of G.

(ii) Let μ be an anti-Q-fuzzy normal subgroup of G. Then f(μ) is an anti Q fuzzy subgroup of H.

Now, for all u, v in H, we have

$$
\begin{aligned}
f(\mu)(uvu^{-1}, q) &= \inf_{f(y)=uvu^{-1}} \mu(y, q) = \inf_{f(x)=u;\ f(y)=v} \mu(xyx^{-1}, q) \\
&= \inf_{f(y)=v} \mu(y, q) = f(\mu)(v, q), \text{ (since f is an epimorphism)}
\end{aligned}
$$

Hence f(μ) is an anti-Q-fuzzy normal subgroup of H.

Definition 3.8: *Let λ and μ be two Q-fuzzy subsets of G. The product of λ and μ is defined to be the Q-fuzzy subset λμ of G is given by* $\lambda\mu(x, q) = \inf_{yz = x} \max(\lambda(y, q), \mu(z, q))$ *, x ∈ G.*

Theorem 3.10: *If λ & μ are anti-Q-fuzzy normal subgroups of G, then $\lambda\mu$ is an anti-Q-fuzzy normal subgroup of G.*

Proof: Let λ & μ be two anti-Q-fuzzy normal subgroups of G.

(i) $\lambda\mu(xy, q) = \inf_{x_1y_1 = x,\ x_2y_2 = y} \max\{\lambda(x_1y_1, q), \mu(x_2y_2, q)\}$

$\leq \inf_{x_1y_1 = x,\ x_2y_2 = y} \max\{\max\{\lambda(x_1,q),\lambda(y_1,q)\}, \max\{\mu(x_2,q),\mu(y_2,q)\}\}$

$\leq \max\{\inf_{x_1y_1 = x,\ x_2y_2 = y} \max\{\lambda(x_1,q),\lambda(y_1, q)\}, \inf \max\{\mu(x_2, q), \mu(y_2, q)\}\}$

$\lambda\mu(xy, q) \leq \max\{\lambda\mu(x, q), \lambda\mu(y, q)\}$

(ii) $\lambda\mu(x^{-1}, q) = \inf_{(yz)^{-1} = x^{-1}} \max\{\mu(z^{-1}, q), \lambda(y^{-1}, q)\}$

$= \inf_{x = yz} \max\{\mu(z, q), \lambda(y, q)\}$

$= \inf_{x = yz} \max\{\lambda(y, q), \mu(z, q)\}$

$= \lambda\mu(x, q).$

Hence λμ is an anti-Q-fuzzy normal subgroup of G.

4 Cartesian Product of Anti Q-Fuzzy Normal Subgroups

Theorem 4.1: *If μ & δ are two anti-Q-fuzzy subgroups of a group G, then $\mu \times \delta$ is also an anti-Q-fuzzy subgroup of the group G x G.*

Proof: Let μ & δ be two anti-Q-fuzzy subgroups of a group G.

Let $(x_1, y_1), (x_2, y_2) \in G \times G$ & $q \in Q$

Then $(\mu \times \delta)\{((x_1, y_1)(x_2, y_2)^{-1}, q)\} = (\mu \times \delta)\{((x_1, y_1)(x_2^{-1}, y_2^{-1}), q)\}$

$= (\mu \times \delta)\{((x_1x_2^{-1}, y_1 y_2^{-1}), q)\}$

$= \max\{\mu(x_1x_2^{-1}, q), \delta(y_1 y_2^{-1}, q)\}$

$= \max\{\mu(x_1,q), \mu(x_2^{-1},q), \delta(y_1, q), \delta(y_2^{-1}, q)\}$

$= \max\{\mu(x_1,q), \mu(x_2,q), \delta(y_1, q), \delta(y_2,q)\}$

$= \max\{(\mu \times \delta)((x_1, y_1), q), (\mu \times \delta)((x_2, y_2), q)\}$

$\therefore(\mu \times \delta)$ is an anti-Q-fuzzy subgroup of G x G.

Theorem 4.2: *If μ & δ are two anti-Q-fuzzy normal subgroups of a group G, then $\mu \times \delta$ is also an anti-Q-fuzzy normal subgroup of the group G x G.*

Proof: Straight forward.

5 Conclusion

In this article we have discussed anti-Q-fuzzy normal subgroups, anti-Q-fuzzy normaliser and anti-Q-fuzzy normal subgroups under anti Q- homomorphism. Interestingly, it has been observed that anti-Q-fuzzy concept adds another dimension to the defined anti-fuzzy normal subgroups. This concept can further be extended for new results.

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