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Homotopy Analysis Transform Method for Time-Space Fractional Gas Dynamics Equation

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Abstract

Homotopy Analysis Transform Method (HATM) is applied to tackle time-space fractional Gas dynamics equation. The proposed HATM is an elegant coupling of homotopy analysis method (HAM) and Laplace transform. The method gives an analytical solution in the form of a convergent series with easily computable components, requiring no linearization or small perturbation. The numerical solutions obtained by the proposed method indicate that the approach is easy to implement and computationally very attractive. Numerical results coupled with graphical representation explicitly reveal the complete reliability of the proposed algorithm.

Keywords: *Homotopy Analysis Transform Method (HATM), Homotopy Analysis Method (HAM), Fractional equations, nonlinear gas dynamics equation.*

1 Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last decade, fractional calculus has found applications in numerous seemingly diverse fields of science and engineering. Fractional differential equations are increasingly used to model problems in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes.

Fractional differential equation have long history. These equations have demonstrated a considerable interest both in mathematics and in applications in recent years. They have been used in modelling of many physical and chemical processes in engineering. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering [1]. In past years, it has turned out that differential equations involving derivatives of noninteger order can be adequate models for various physical phenomena [2]. Many important phenomena are well described by fractional differential equations in electromagnetics, acoustics, visco-elasticity, electro-chemistry and material science. That is because of the fact that a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus [3-7]. Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Some of the recent analytical methods for solving nonlinear problems include the homotopy analysis method HAM [8-14]. The HAM, first proposed in 1992 by Liao, has been successfully applied to solve many problems in physics and science. The HATM is a combination of the homotopy analysis method and the Laplace decomposition method. In recent years, many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform. Among these are Laplace decomposition method [15-19] and homotopy perturbation transform method [20-22].

In this paper, we consider the following nonlinear time fractional gas dynamics equation of the form :

$$\mathbf{D}_t^{\alpha}\mathbf{u} + \frac{1}{2}(u^2)_x - u + u^2 = 0, \qquad 0 < \alpha \leq 1, \tag{1.1}$$

and the fractional time-space derivatives nonlinear equation:

$$\mathbf{D}_t^{\alpha}\ u + \frac{1}{2}\mathbf{D}_x^{\beta}\ u^2 - u + u^2 = 0, \qquad 0 < \alpha, \beta \leq 1, \tag{1.2}$$

by using homotopy analysis transform method HATM [23-27], which is a generalization of the given in. We used the Caputo fractional derivative on the half axis R^+ (i.e $t \in R^+$) ${}_t^C\mathbf{D}_{0+}^{\alpha}$ for time and the Caputo fractional derivative on the half axis R^+ (i.e $t \in R^+$) ${}_t^C\mathbf{D}_{0+}^{\alpha}$ for space.

Where α and β are a parameters describing the order of the fractional derivative. The function $u(x,t)$ is the probability density function, t is the time, and x is the spatial coordinate. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$ and $\beta = 1$ the fractional gas dynamics equation reduces to the classical gas dynamics equation. The gas dynamics equations are based on the physicallaws of conservation, namely, the laws of conservation of mass, conservation of momentum, conservation of energy, and so forth. The nonlinear fractional gas dynamics has been studied previously by [28], [29] and [30].

2 Preliminaries and Notations

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper. For the finite derivative $[a, b]$ we define the following fractional integral and derivatives.

Definition 2.1: *If* $f(t) \in L_1(a, b)$, *the set of all integrable functions, and* $\alpha > 0$ *then the Riemann-Liouville fractional integral of order* α*, denoted by* J_{a+}^{α} *is defined by*

$$\mathbf{J}_{a+}^{\alpha}\mathbf{f}(\mathbf{t}) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \mathbf{f}(\tau)\, \mathbf{d}\tau \tag{2.1}$$

Definition 2.2: *For* $\alpha > 0$*, the Caputo fractional derivative of order* α*, denoted by* ${}^{C}\mathbf{D}_{a+}^{\alpha}$ *,is defined by*

$$^{C}\mathbf{D}_{a+}^{\alpha}\mathbf{f}(\mathbf{t}) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \mathbf{D}^n\mathbf{f}(\tau)\, \mathbf{d}\tau, \tag{2.2}$$

where n is such that $n - 1 < \alpha < n$ and $D = \frac{d}{\mathbf{d}\tau}$

If α is an integer, then this derivative takes the ordinary derivative

$$^{C}\mathbf{D}_{a+}^{\alpha} = \mathbf{D}^{\alpha}, \alpha = 1, 2, 3, ... \tag{2.3}$$

Finally the Caputo fractional derivative on the whole space R is defined by:

Definition 2.3: *For* $\alpha > 0$ *the Caputo fractional derivative of order* α *on the whole space , denoted by* ${}^{C}\mathbf{D}_{a+}^{\alpha}$*, is defined by*

$$^{C}\mathbf{D}_{a+}^{\alpha}\mathbf{f}(\mathbf{x}) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^x (x-\xi)^{n-\alpha-1} \mathbf{D}^n\mathbf{f}(\xi)\, \mathbf{d}\xi.$$

3 Basic Idea of Fractional Homotopy Analysis Transform Method (FHATM)

To illustrate the basic idea of the FHATM for the fractional partial differential equation as:

$$\mathbf{D}_t^{\mathbf{n}\alpha}\mathbf{u}(x,t)+\mathbf{R}\,[x]\,\mathbf{u}(x,t)+\mathbf{N}\,[x]\,\mathbf{u}(x,t)=\mathbf{g}(x,t), t\succ 0, x\in R, n-1<n\alpha\leq n, \tag{3.1}$$

where $\mathbf{D}_t^{\mathbf{n}\alpha}=\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, $R\,[x]$ is the linear operator in x, $N\,[x]$ is the general nonlinear operator in x, and $g(x,t)$ are continuous functions. For simplicity we ignore all initial and boundary conditions, which can be treated in similar way. Now the methodology consists of applying Laplace transform first on both sides of Eq. (3.1), we get

$$\mathbf{L}[\mathbf{D}_t^{\mathbf{n}\alpha}\mathbf{u}(x,t)]+\mathbf{L}[\mathbf{R}\,[x]\,\mathbf{u}(x,t)+\mathbf{N}\,[x]\,\mathbf{u}(x,t)]=\mathbf{L}[\mathbf{g}(x,t)]. \tag{3.2}$$

Now, using the differentiation property of the Laplace transform, we have

$$\mathbf{L}[\mathbf{u}(x,t)]-\frac{1}{s^{\mathbf{n}\alpha}}\sum_{k=0}^{n-1}\mathbf{s}^{(\mathbf{n}\alpha-\mathbf{k}-\mathbf{1})}\mathbf{u}^k(\mathbf{x},0)+\frac{1}{s^{\mathbf{n}\alpha}}\mathbf{L}[\ \mathbf{R}\,[x]\,\mathbf{u}(x,t)+\mathbf{N}\,[x]\,\mathbf{u}(x,t)-\mathbf{g}(x,t)]=0. \tag{3.3}$$

We define the nonlinear operator

$$N\,[\phi\,(x,t;q)] = \mathbf{L}[\phi\,(x,t;q)]-\frac{1}{s^{\mathbf{n}\alpha}}\sum_{k=0}^{n-1}\mathbf{s}^{(\mathbf{n}\alpha-\mathbf{k}-\mathbf{1})}\mathbf{u}^k(\mathbf{x},0)+\frac{1}{s^{\mathbf{n}\alpha}}\mathbf{L}[\ \mathbf{R}\,[x]\,\mathbf{u}(x,t)+\mathbf{N}\,[x]\,\mathbf{u}(x,t)-\mathbf{g}(x,t)], \tag{3.4}$$

where $q\in[0,1]$ be an embedding parameter $\phi\,(x,t;q)$ and is the real function of x,t and q. By means of generalizing the traditional homotopy methods, Liao constructed the zero order deformation equation

$$(1-q)\,\mathbf{L}\,(\phi\,(x,t;q)-u_0\,(x,t))=\mathbf{q}\hbar\mathbf{H}(\mathbf{x},\mathbf{t})\mathbf{N}\,[\mathbf{D}_t^{\alpha}\phi\,(x,t;q)]\,, \tag{3.5}$$

where $\hbar\neq\mathbf{0}$ is an auxiliary parameter, $\mathbf{H}(\mathbf{x},\mathbf{t})\neq\mathbf{0}$ is an auxiliary function, $u_0\,(x,t)$ is an initial guess of $u(x,t)$ and $\phi\,(x,t;q)$ is an unknown function. It is important that one has great freedom to choose auxiliary thing in FHATM. Obviously, when $q=0$ and $q=1$, it holds

$$\phi(\mathbf{x},\mathbf{t};\mathbf{0})=\mathbf{u}_0(\mathbf{x},\mathbf{t})\ \textbf{and}\ \phi(\mathbf{x},\mathbf{t};\mathbf{1})=\mathbf{u}(\mathbf{x},\mathbf{t}), \tag{3.6}$$

respectively. Thus, as q increases from 0 to 1, the solution varies from the initial guess $\mathbf{u}_0(\mathbf{x},\mathbf{t})$ to the solution $\mathbf{u}(\mathbf{x},\mathbf{t})$. Expanding $\phi(\mathbf{x},\mathbf{t};\mathbf{q})$ in Taylor's series with respect to q, we have

$$\phi(\mathbf{x},\mathbf{t};\mathbf{q}) = \mathbf{u}_0(\mathbf{x},\mathbf{t}) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{u}_m(\mathbf{x},\mathbf{t})\mathbf{q}^m, \tag{3.7}$$

where

$$\mathbf{u}_m(\mathbf{x},\mathbf{t}) = \frac{1}{m!}\frac{\partial^m \phi(\mathbf{x},\mathbf{t};\mathbf{q})}{\partial q^m}|_{q=0}. \tag{3.8}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(x,t)$ are selected such that the series (3.7) is convergent at $q=1$, then we have

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{u}_0(\mathbf{x},\mathbf{t}) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{u}_m(\mathbf{x},\mathbf{t}). \tag{3.9}$$

Let us define the vector

$$\overrightarrow{u_n}(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}. \tag{3.10}$$

Differentiating the zero$^{th}-$ order deformation Eq (3.5) m times with respect to q, then setting $q=0$ and dividing then by $m^!$, we have the m^{th}- order deformation equation

$$\mathbf{L}\left[\mathbf{u}_m(\mathbf{x},\mathbf{t}) - \chi_m \mathbf{u}_{m-1}(\mathbf{x},t)\right] = \hbar \mathbf{H}(\mathbf{x},\mathbf{t})\mathbf{R}_m\left(\overrightarrow{u}_{m-1}\right). \tag{3.11}$$

Applying inverse Laplace transform

$$\mathbf{u}_m(\mathbf{x},\mathbf{t}) = \chi_m \mathbf{u}_{m-1} + \hbar \mathbf{L}^{-1}[\mathbf{H}(\mathbf{x},\mathbf{t})\mathbf{R}_m(\tilde{\mathbf{u}}_{m-1})] \tag{3.12}$$

where

$$\mathbf{R}_m\left(\overrightarrow{\mathbf{u}}_{m-1}\right) = \frac{1}{(\boldsymbol{m}-\mathbf{1})!}\frac{\partial^{m-1}\mathbf{N}[\phi(\mathbf{x},\mathbf{t};\mathbf{q})]}{\partial \mathbf{q}^{m-1}}|_{q=0}, \tag{3.13}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{3.14}$$

In this way, it is easily to obtain $u_m(x,t)$ for $m \geq 1$, at $m^{th}-$ order, we have

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \sum_{\mathbf{m=0}}^{M} \mathbf{u}_m(\mathbf{x},\mathbf{t}). \tag{3.15}$$

when $M \to \infty$ we get an accurate approximation of the original equation (3.1).

4 Illustrative Examples

In this section, we use the HATM to solve the fractional time-space fractional gas dynamics equation.

Example 1. In this example, we consider the following fractional time nonlinear fractional Gas dynamics equation as [28-30]:

$$\mathbf{D}_t^{\alpha}\mathbf{u}+\frac{1}{2}(u^2)_x-u+u^2=0, \qquad t>0;\ 0<\alpha\leq 1, \tag{4.1.1}$$

with initial condition

$$\mathbf{u}(\mathbf{x},\mathbf{0})=\mathbf{e}^{-x}. \tag{4.1.2}$$

Applying the Laplace transform on both sides in Eq. (4.1.1) and after using the differentiation property of Laplace transform for fractional derivative, we get

$$s^{\alpha}L[u(x,t)]-s^{\alpha-1}u(x,0)+L[\frac{1}{2}(u^2)_x-u+u^2]=0. \tag{4.1.3}$$

On simplifying

$$L[u(x,t)]-\frac{1}{s}u(x,0)+s^{-\alpha}L[\frac{1}{2}(u^2)_x-u+u^2]=0. \tag{4.1.4}$$

We choose the linear operator as

$$\pounds[\phi(x,t;q)]=L[\phi(x,t;q)], \tag{4.1.5}$$

with the property that

$$\pounds[c]=0,\ \text{where c is constant.}$$

We now define a nonlinear operator as

$$N[\phi(x,t;q)]=L[\phi(x,t;q)]-\frac{1}{s}\mathbf{e}^{-x}+s^{-\alpha}L[\frac{1}{2}(\phi^2)_x-\phi+\phi^2]. \tag{4.1.6}$$

Using the above definition, with assumption $H(x,t)=1$, we construct the zeroth- order deformation equation

$$(1-q)\left(\phi(\mathbf{x},\mathbf{t};\mathbf{q})-\mathbf{u}_0(\mathbf{x},\mathbf{t})\right)=\mathbf{q}\hbar\mathbf{N}\left[\phi(\mathbf{x},\mathbf{t};\mathbf{q})\right]. \tag{4.1.7}$$

For $q=0$ and $q=1$, we can write

$$\phi(\mathbf{x},\mathbf{t};\mathbf{0})=\mathbf{u}_0(\mathbf{x},\mathbf{t})=\mathbf{u}(\mathbf{x},\mathbf{0}),\phi(\mathbf{x},\mathbf{t};\mathbf{1})=\mathbf{u}(\mathbf{x},\mathbf{t}). \tag{4.1.8}$$

Thus, we obtain the m^{th}- order deformation equation is given by

$$\mathbf{L}[\mathbf{u}_m(\mathbf{x},\mathbf{t})-\chi_m\mathbf{u}_{m-1}(\mathbf{x},\mathbf{t})]=\hbar\mathbf{R}_m(\overrightarrow{u}_{m-1},x,t). \tag{4.1.9}$$

Taking inverse Laplace transform of Eq. (4.1.6), we get

$$\mathbf{u}_m(\mathbf{x},\mathbf{t}) = \chi_m \mathbf{u}_{m-1}(\mathbf{x},\mathbf{t}) = \hbar \mathbf{L}^{-1}[\mathbf{R}_m(\overrightarrow{u}_{m-1}, x, t)], \tag{4.1.10}$$

where

$$\begin{aligned} \mathbf{R}_m(\overrightarrow{u}_{m-1}, x, t) &= \mathbf{L}[\mathbf{u}_{m-1}(x,t)] - \frac{(1-\chi_m)}{s}\mathbf{e}^{-x} + \frac{1}{2s^\alpha}\mathbf{L}(\frac{\partial}{\partial x}\sum_{\mathbf{i=0}}^{m-1}\mathbf{u}_i\,\mathbf{u}_{m-1-i}) - \\ &\quad \frac{1}{s^\alpha}\mathbf{L}(\mathbf{u}_{m-1}) + \frac{1}{s^\alpha}\mathbf{L}(\sum_{\mathbf{i=0}}^{m-1}\mathbf{u}_i\,\mathbf{u}_{m-1-i}). \end{aligned} \tag{4.1.11}$$

Let us take the initial approximation as

$$\mathbf{u}_0(\mathbf{x},\mathbf{t}) = \mathbf{e}^{-x}, \tag{4.1.12}$$

the other components are given by

$$\mathbf{u}_1(\mathbf{x},\mathbf{t}) = -\frac{\mathbf{e}^{-x} h\ t^\alpha}{\Gamma(\alpha+1)}, \tag{4.1.13}$$

$$\mathbf{u}_2(\mathbf{x},\mathbf{t}) = e^{-x} h\ t^\alpha[-\frac{1+h}{\Gamma(\alpha+1)} + \frac{h\ t^\alpha}{\Gamma(2\alpha+1)}], \tag{4.1.14}$$

$$\mathbf{u}_3(\mathbf{x},\mathbf{t}) = e^{-x} h\ t^\alpha[-\frac{(1+h)^2}{\Gamma(\alpha+1)} + h\ t^\alpha(\frac{2(1+h)}{\Gamma(2\alpha+1)} - \frac{h\ t^\alpha}{\Gamma(3\alpha+1)})], \tag{4.1.15}$$

$$\begin{aligned} \mathbf{u}_4(\mathbf{x},\mathbf{t}) &= e^{-x} h\ t^\alpha[-\frac{(1+h)^3}{\Gamma(\alpha+1)} + h\ t^\alpha(\frac{3(1+h)^2}{\Gamma(2\alpha+1)} + h\ t^\alpha(-\frac{3(1+h)}{\Gamma(3\alpha+1)} + \\ &\quad \frac{h\ t^\alpha}{\Gamma(4\alpha+1)}))], ... \end{aligned} \tag{4.1.16}$$

Then, the approximate solution for the nonlinear fractional equation (4.1.1) according to the HATM, we can conclude that

$$\begin{aligned} u\ (x,t) &= u_0\,(x,t) + u_1\,(x,t) + u_2\,(x,t) + u_3\,(x,t) + ... \\ &= e^{-x} - \frac{\mathbf{e}^{-x} h\ t^\alpha}{\Gamma(\alpha+1)} + e^{-x} h\ t^\alpha[-\frac{1+h}{\Gamma(\alpha+1)} + \frac{h\ t^\alpha}{\Gamma(2\alpha+1)}] + \\ &\quad e^{-x} h\ t^\alpha[-\frac{(1+h)^2}{\Gamma(\alpha+1)} + h\ t^\alpha(\frac{2(1+h)}{\Gamma(2\alpha+1)} - \frac{h\ t^\alpha}{\Gamma(3\alpha+1)})] + ... \end{aligned} \tag{4.1.17}$$

and so on then the approximate solution at $h = -1$ is given by

$$u\ (x,t) = e^{-x}(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...). \tag{4.1.18}$$

For the special case $\alpha = 1$, we obtain from (4.1.18)

$$\begin{aligned} u\ (x,t) &= e^{-x}(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{4!} + ...) \\ &= e^{t-x}, \end{aligned} \quad (4.1.19)$$

which is an exact solution to the nonlinear equation. The results for the exact solution Eq. (4.1.19) and the approximate solution Eq. (4.1.18) are obtained using the homotopy analysis transform method. Now, we calculate numerical results of the probability density function $u(x,t)$ for different time-fractional Brownian motions $\alpha = 1/3,\ 2/3,\ 1$ and for various values of t and x. The numerical results for the approximate solution (4.1.17) obtained by using HATM and the exact solution (4.1.19) for various values of of t and x are shown in Figures 1(a)–1(d) and those for different values t and α at $x = 1$ are depicted in Figure 2.

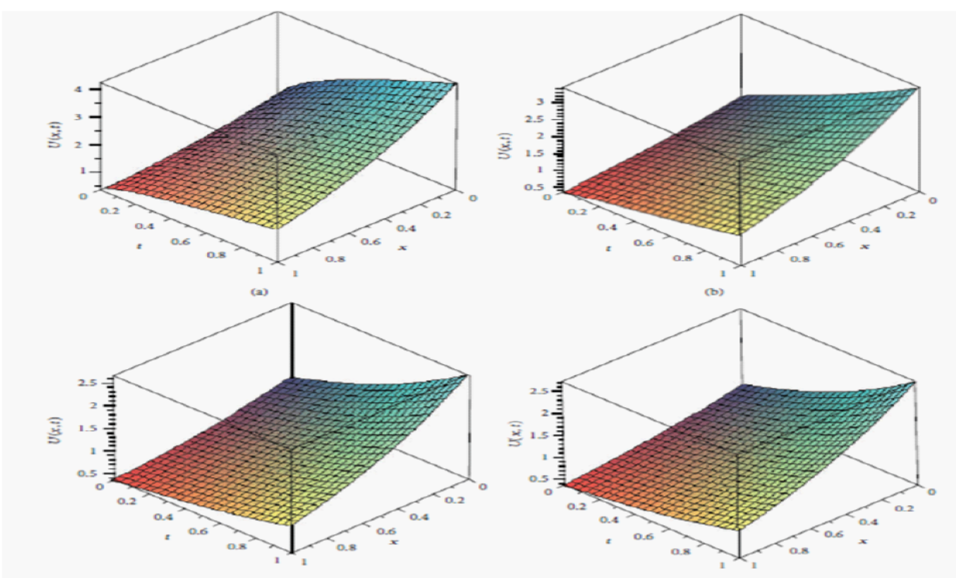
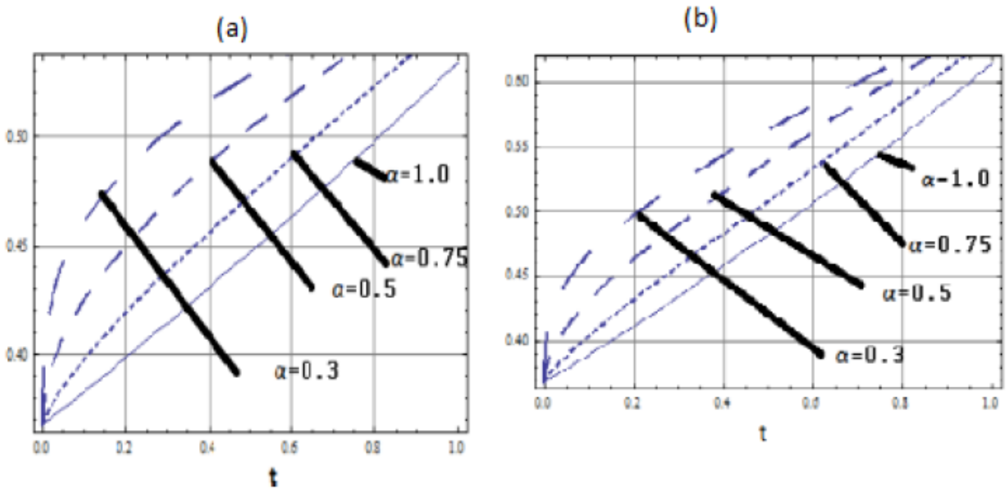


Figure 1: The surface shows solution $u(x,t)$ w.r.t. x and t when, (a) $\alpha = \frac{1}{3}$, (b) $\alpha = \frac{2}{3}$, (c) $\alpha = 1$ and (d) exact solution.



(a) the 5^{th} order approximate solution (4.1.17) (b) the 8^{th} order approximate solution (4.1.17).
Figure 2: plots of $u(x,t)$ at $x = 1, h = -0.55$ for different values of α.

However, mostly; the results given by the Adomian decomposition method and homotopy perturbation transform method converge to the corresponding numerical solutions in a rather small region. But, different from those two methods, the homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter h. So the valid region for h where the series converges is the horizontal segment of each h curve. When we choose $\alpha = 1$ then clearly, we can conclude that the obtained solution converges to the exact solution $u(x,t) = e^{t-x}$; which is an exact solution of the standard Gas dynamics equation.

Example 2. Let us consider the following fractional time-space nonlinear equation

$$\mathbf{D}_t^{\alpha}\, u + \frac{1}{2}\mathbf{D}_x^{\beta}\, u^2 - u + u^2 = 0, \qquad t > 0; \quad 0 < \alpha, \beta \leq 1, \tag{4.2.1}$$

with initial condition

$$\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{e}^{-x}. \tag{4.2.2}$$

Applying the Laplace transform on both sides in Eq. (4.1.1) and after using the differentiation property of Laplace transform for fractional derivative, we get

$$s^{\alpha}L[u(x,t)] - s^{\alpha-1}u(x,0) + L[\frac{1}{2}(u^2)_x - u + u^2] = 0. \tag{4.2.3}$$

On simplifying

$$L[u(x,t)] - \frac{1}{s}u(x,0) + s^{-\alpha}L[\frac{1}{2}(u^2)_x - u + u^2] = 0. \tag{4.2.4}$$

We choose the linear operator as

$$\pounds[\phi(x,t;q)] = L[\phi(x,t;q)], \tag{4.2.5}$$

with the property that

$$\pounds[c] = 0, \text{ where c is constant.}$$

We now define a nonlinear operator as

$$N[\phi(x,t;q)] = L[\phi(x,t;q)] - \frac{1}{s}\mathbf{e}^{-x} + s^{-\alpha}L[\frac{1}{2}\mathbf{D}_x^{\beta}(\phi^2) - \phi + \phi^2]. \tag{4.2.6}$$

Using the above definition, with assumption $H(x,t) = 1$, we construct the zeroth- order deformation equation

$$(1-q)\left(\phi(\mathbf{x},\mathbf{t};\mathbf{q}) - \mathbf{u}_0(\mathbf{x},\mathbf{t})\right) = \mathbf{q}\hbar\mathbf{N}\left[\phi(\mathbf{x},\mathbf{t};\mathbf{q})\right]. \tag{4.2.7}$$

For $q = 0$ and $q = 1$, we can write

$$\phi(\mathbf{x},\mathbf{t};\mathbf{0}) = \mathbf{u}_0(\mathbf{x},\mathbf{t}) = \mathbf{u}(\mathbf{x},\mathbf{0}), \phi(\mathbf{x},\mathbf{t};\mathbf{1}) = \mathbf{u}(\mathbf{x},\mathbf{t}). \tag{4.2.8}$$

Thus, we obtain the m^{th}- order deformation equation is given by

$$\mathbf{L}[\mathbf{u}_m(\mathbf{x},\mathbf{t}) - \chi_m\mathbf{u}_{m-1}(\mathbf{x},\mathbf{t})] = \hbar\mathbf{R}_m(\overrightarrow{u}_{m-1}, x, t). \tag{4.2.9}$$

Taking inverse Laplace transform of Eq. (4.2.6), we get

$$\mathbf{u}_m(\mathbf{x},\mathbf{t}) = \chi_m\mathbf{u}_{m-1}(\mathbf{x},\mathbf{t}) = \hbar\mathbf{L}^{-1}[\mathbf{R}_m(\overrightarrow{u}_{m-1}, x, t)], \tag{4.2.10}$$

where

$$\begin{aligned}\mathbf{R}_m(\overrightarrow{u}_{m-1}, x, t) \quad = \quad & \mathbf{L}\left[\mathbf{u}_{m-1}(\mathbf{x},\mathbf{t})\right] - \frac{(1-\chi_m)}{s}\mathbf{e}^{-x} + \frac{\mathbf{1}}{2s^{\alpha}}\mathbf{L}(\mathbf{D}_x^{\beta}\sum_{\mathbf{i}=\mathbf{0}}^{m-1}\mathbf{u}_i\,\mathbf{u}_{m-1-i}) - \\ & \frac{1}{s^{\alpha}}\mathbf{L}(\mathbf{u}_{m-1}) + \frac{1}{s^{\alpha}}\mathbf{L}(\sum_{\mathbf{i}=\mathbf{0}}^{m-1}\mathbf{u}_i\,\mathbf{u}_{m-1-i}).\end{aligned} \tag{4.2.11}$$

In order to obey both the rul of solution expression and the rule of the cofficient ergodicity, the auxiliary function can be determined uniquely $H(x,t) = 1$. Now the solution of the mth-order deformation equations (4.2.11) for m ≥ 1 become

$$\mathbf{u}_m(\mathbf{x},\mathbf{t}) = \chi_m \mathbf{u}_{m-1}(\mathbf{x},\mathbf{t}) + \hbar \mathbf{L^{-1}}[\mathbf{R}_m(\overrightarrow{u}_{m-1}, x, t)] \tag{4.2.12}$$

and so on , we substitute the initial condition (4.2.2) into the system (4.2.12) with the aid of Maple; the approximate solutions of eq. (4.2.1) take the following form

Let us take the initial approximation as

$$\begin{aligned} u_1(x,t) &= -\frac{\mathbf{e}^{-x}\ (-1+\mathbf{e}^{x}-e^{i\pi\beta})\ h\ t^{\alpha}}{\Gamma(\alpha+1)} \\ u_2(x,t) &= \frac{\begin{array}{c}\mathbf{e}^{-3x} h\ t^{\alpha}((2+3^{\beta}e^{2i\pi\beta}+e^{x}(-3+e^{x})+e^{i\pi\beta}(2+3^{\beta}-(1+2^{\beta})e^{x})) \\ h\ t^{\alpha}\Gamma(\alpha+1)-e^{x}(-1+\mathbf{e}^{x}-e^{i\pi\beta})(1+h)\Gamma(2\alpha+1))\end{array}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \\ &\ldots \end{aligned} \tag{4.2.13}$$

In this case the approximate solution for the nonlinear fractional equation (4.2.1) according to the HATM, we can conclude that

$$\begin{aligned} u\ (x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots \\ &= \mathbf{e}^{-\ x} - \frac{\mathbf{e}^{-\ x}\ (-1+\mathbf{e}^{x}-e^{i\pi\beta})\ h\ t^{\alpha}}{\Gamma(\alpha+1)} + \\ &\quad \frac{\begin{array}{c}\mathbf{e}^{-3x} h\ t^{\alpha}((2+3^{\beta}e^{2i\pi\beta}+e^{x}(-3+e^{x})+e^{i\pi\beta} \\ (2+3^{\beta}-(1+2^{\beta})e^{x}))h\ t^{\alpha}\Gamma(\alpha+1)-e^{x}(-1+\mathbf{e}^{x}-e^{i\pi\beta}) \\ (1+h)\Gamma(2\alpha+1))\end{array}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \\ &\quad \ldots \end{aligned} \tag{4.2.14}$$

And so on setting $h = -1$,we have

$$u\ (x,t) = \mathbf{e}^{-\ x} + \frac{\mathbf{e}^{-x}\ (-1+\mathbf{e}^{x}-e^{i\pi\beta})\ t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\begin{array}{c}\mathbf{e}^{-3x}\ (2+3^{\beta}e^{2i\pi\beta}+ \\ e^{x}(-3+e^{x})+e^{i\pi\beta}(2+3^{\beta}-(1+2^{\beta})e^{x}))\ t^{2\alpha}\end{array}}{\Gamma(2\alpha+1)} \tag{4.2.15}$$

Now, let $\alpha \to 1$ and $\beta \to 1$, we get the approximate solution (4.2.14) takes the following form

$$\begin{aligned} u\ (x,t) &= u_0 + u_1 + u_2 + ... \\ &= e^{-x} + \frac{t\ e^{-x}}{\Gamma(2)} + \frac{t^2 e^{-x}}{\Gamma(3)} + ... \\ &= \mathbf{e}^{-x}(1 + \frac{it}{1!} + \frac{(it)^2}{2!} + ...) \\ &= \mathbf{e}^{t-x} \end{aligned} \tag{4.2.16}$$

which is an exact solution to the nonlinear equation. The results for the exact solution Eq. (4.2.16) and the approximate solution Eq. (4.2.15) are obtained using the homotopy analysis transform method.

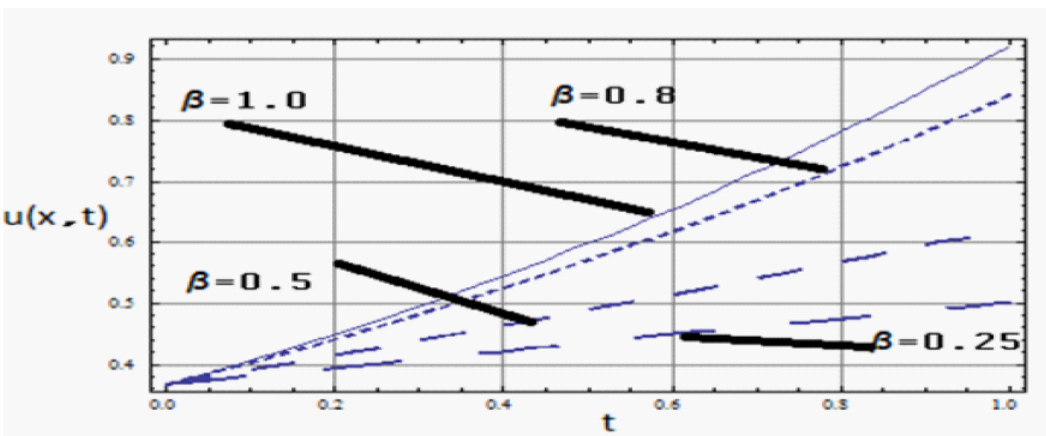


Figure 3: The 5^{th} order approximate solution (4.2.14) of $u(x,t)$ versus t at $x = 1, h = -0.55$ and $\alpha = 1$ for different values of β

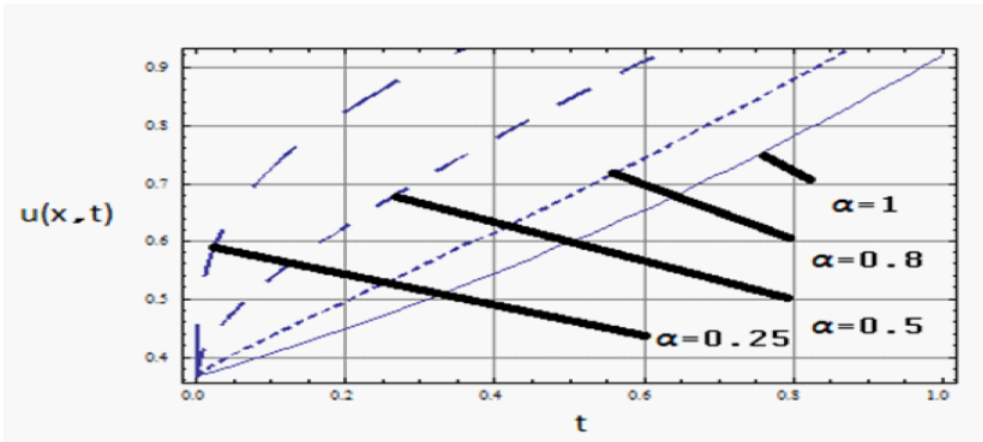


Figure 4: The 5^{th} order approximate solution (4.2.14) of $u(x, t)$ versus t at $x = 1, h = -0.55$ and $\beta = 1$ for different values of β

5 Concluding Remarks

In this paper, the homotopy analysis transform method HATM has been successfully to obtain the numerical solutions of the time-space fractional Gas Dynamics equation was initial conditions. The reliability of this method and reducation in computations give this method a wider applicability. HATM is clearly a very efficient and powerful technique for finding the numerical solutions of the proposed equation. The analytical results have been given in terms of a power series with easily computed terms. The method overcomes the difficulty in other methods because it is efficient. Three examples were investigated to demonstrate the ease and versatility of our new approach. The illustrative examples show that the method is easy to use and is an effective tool to solve fractional partial differential equations numerically. However, mostly; the results given by the Adomian decomposition method ADM homotopy perturbation transform method HPM and homotopy analayasis method HAM converge to the corresponding numerical solutions in a rather small region. But, different from those two methods, the homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter h. So the valid region for h where the series converges is the horizontal segment of each h curve. When we choose $\alpha = \beta = 1$ then clearly, we can conclude that the obtained solution $\sum_{m=0}^{\infty} u_m(x,t)$ converges to the exact solution. It, therefore provides more realistic series solutions that generally converge very rapidly in real physical problems.

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