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A New Seventh and Eighth-Order Ostrowski's Type Schemes for Solving Nonlinear Equations with their Dynamics

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Abstract

The present article is devoted to the developement of the new seventh and eighth order iterative methods for finding simple root of nonlinear equations. The proposed methods are improvement of the existing sixth order method. Numerical examples are also presented to support the theoretical results. Finally, we have compared new methods with some existing methods by basins of attraction and observed that the proposed scheme is more efficient.

Keywords: *Iterative method, Newton method, order of convergence, computational efficiency, basin of attraction.*

1 Introduction

Solving nonlinear equations arise in many branches of science and engineering, is one of the most important problems in numerical analysis. The Newton's method is well known and most likely used method for solving nonlinear equations. The iterative step of Newton's method is given by [15]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, 3, ...$$

provided $f'(x_n) \neq 0$. It's a second order one point method and evaluated of one derivative and one function for each iteration. Multipoint iteration methods have overcome the theoretical limit of one point method regarding the convergence order of computational efficiency and become the most powerfull tool to find the roots of nonlinear equation, boundary value problem and system of nonlinear equations etc. The maximum attainable computational efficiency of multi-point without method is closely related to the hypothesis given by Kung and Traub[10]. Kung and Traub have conjecture that the convegence order of any multipoint method without memory with n-evaluation is not larger than 2^{n-1}. A number of modification of Newton's method with improved rate of convergence are reported by previous researcher in [1, 3, 4, 13, 7, 17, 11] and there in. Some of the scheme developed from Newton method by some authors are given below:
In [14] Zhao et. al presented eighth order iterative formula as defined by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)} G(\mu_n), \\ x_{n+1} &= z_n - H(\nu_n) \frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]}, \end{aligned} \tag{1}$$

where $\mu_n = \frac{f(y_n)}{f(x_n),}$ $\nu_n = \frac{f(z_n)}{f(x_n)}$, $G(\mu_n)$ and $H(\nu_n)$ are real valued function. In [5] Babajee et. al. Presented a eighth order method as defined by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \left(1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right), \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)} \left(1 - \frac{f(y_n)}{f(x_n)}\right)^{-2}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \frac{\left(1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2 + 5\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \frac{f(z_n)}{f(y_n)}\right)}{\left(1 - \frac{f(y_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)}\right)^2}, \end{aligned} \tag{2}$$

In[16] Lofti et. al. proposed a eighth order method as defined by,

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)} g(s_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} h(t_n), \end{aligned} \tag{3}$$

where $s_n = \frac{f(y_n)}{f(x_n)}$, $t_n = \frac{f(z_n)}{f(x_n)}$, $g(s_n)$ and $h(t_n)$ are weight functions.

The main goal and motivation behind this work is to develop new numerical methods with better computational efficiency with a fixed number of functional evaluations per iteration using weight function technique. In view of above, in this paper we have developed three point methods of order seven and optimal order eight. We have also studied the dynamics of these methods and compare it with the dynamics of methods given by Zhao et. al.[14], Babajee et. al.[5], Lofti et. al. [16] and Mirzaee and Hamzeh [7] using fractal pictures.
The dynamical behaviour of iterative methods for some quadratic and cubic polynomials gives valuable information about the convergence and stability of the methods. In order to do this, we recall some basic concept [5, 8, 6, 9, 2] and we shortly present them. Let $R : C \to C$ be a rational map on a Riemann sphere for $z \in C$ define its orbit as the set $orb(z) = \{z, R(z), R^2(z), ..., R^n(z)....\}$ are subsequently, a point z_0 is a fixed point of R if $R(z_0) = z_0$. A periodic point z_0 of period m is such that $R^m(z_0) = z_0$ where m is the smallest such integer and also a point z_0 is called attracting if $|R'(z_0)| \leq 1$ repelling if $|R'(z_0)| \geq 1|$, and neutral if $|R'(z_0)| = 1|$. The Julia set of a non-linear map $R(z)$ denoted $J(R)$ is the closure of the set of its repelling periodic points, the complement of $J(R)$ is the Fatou set $F(R)$, where the basin of attraction of the different roots lie. Let us consider the function $f : D \subseteq R \to R$ a scaler function and has a simple root on the interval D. That is $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ in this neighbourhood of α for detailed one can [15, 10, 12].

Definition 1.1. *Let $\alpha \in R$, $x_n \in R$, $n = 0, 1, 2, ...$ then the sequence $\{x_n\}$ is said to converge to α if*

$$Lim_{n\to\infty}|x_n - \alpha| = 0$$

If in addition, there exist a constant $C > 0$, an integer $n_0 > 0$, and $p \geqslant 0$. s. t. for all $n > n_0$

$$|x_{n+1} - \alpha| \geqslant C|x_n - \alpha|^p,$$

then $\{x_n\}$ is said to convergence to root α with order at least p. If $p = 2$ or 3, the convergence is said to be quadratic or cubic, respectively. Here $e_n = x_n - \alpha$ is the error at the n^{th} iterate and the relation

$$e_{n+1} = Ce_n^p + O(e_n^{p+1}), \tag{4}$$

is called the error equation. the value of p is called the order of convergence.

Definition 1.2. *The computational efficiency of an iterative method of order p requiring n function evaluation per iteration is most frequently calculated by Ostrowski-Traub's efficiency index, which is defined by*

$$E = n\sqrt{p}.$$

Definition 1.3. *Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α. Then computational order of convergence of methods are approximated by*

$$COC \approx \frac{ln|(e_{n+1})(e_n)^{-1}|}{ln|(e_n)(e_{n-1})^{-1}|},$$

Now we summarize contents of the paper. In section.1, seventh and eighth order scheme are developed and its convergence analysis are studied. In section.2, the efficiency of the new proposed method is discussed and is compared with the existing methods of similar nature. Some numerical example are considered in section 3 to show that the convergence behavior of methods and to certify the theoretical results. Section 4 includes the concluding remarks.

1.1 Iterative Methods and Convergence Analysis

Consider the method presented by Mirzaee and Hamzee [7]

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)f(x_n)(2f(y_n) - f(x_n))}{f'(x_n)[4f(y_n)f(x_n) - 2f(y_n)^2 - f(x_n)^2]}. \end{aligned} \tag{5}$$

This mehtod includes three function and one derivative evaluation per iteration. The order of convergence of this method is sixth and efficiency index of above method is 1.565.

Method 1: We proposed the improved scheme by using weight function in the second and last step in method [7] given as follows

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} A(t_n),$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} (B(t_n) * H(u_n)). \tag{6}$$

where $t_n = \frac{f(y_n)}{f(x_n)}$, $u_n = \frac{f(z_n)}{f(y_n)}$. The following theorem demonstrates its convergence analysis.

Theorem 1.4. *Let $\alpha \in D$ be simple zero of a sufficiently differentiable function $f : D \subset R \to R$ for an open interval D which contains x_0 as an initial approximation of α. Then the three-step iteration* (6) *has seventh order convergence if*

$A(0) = 1,\ A'(0) = 1,\ A''(0) = 4,\ A^{(3)}(0) = 30,\ B(0) = 1,\ H(0) = 1,\ B'(0) = 2,$ $H'(0) = 1,\ B''(0) = 12$ *and its error equation is given by*

$$e_{n+1} = \frac{1}{24} c_2^2 c_3 (48c_3 + c_2^2(-144 + 4B^{(3)}[0] - A^{(4)}[0]))e^7 + O[e]^8.$$

Proof: Let α be the simple root of $f(x)$ and $f'(\alpha) \neq 0$. We denote the error equation at n^{th} iteration has $e_n = x_n - \alpha$. We apply the Taylor expansion in each term involved in (6) around the simple root, First we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5], \tag{7}$$

where $c_i = \frac{f^{(i)}(\alpha)}{i! f'(\alpha)}$, $i = 1, 2, 3...$ and we also have

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4], \tag{8}$$

From (7) and (8) it can be found that

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + ... + O(e_n^9). \tag{9}$$

By considering(9) and the first step of (6) , we obtain

$$y_n - \alpha = c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2 c_3 + 3c_4)e_n^4 ... O(e_n^9).$$

We also have

$$\begin{aligned} f(y_n) \quad &= f'(\alpha)[c_2 e_n^2 + \{-2c_2^2 + 2c_3\}e_n^3 \\ &+\{5c_2^3 - 7c_2 c_3 + 3c_4\}e_n^4 + \{2c_2^2(-2c_2^2 + 2c_3) \\ &-2(4c_2^4 - 10c_2^2 c_3 + 3c_3^2 + 5c_2 c_4 - 2c_5)\}e_n^5 ... O(e_n^9)], \end{aligned} \tag{10}$$

Now
$$\frac{f(y_n)}{f(x_n)} = c_2e_n + \{-3c_2^2 + 2c_3\}e_n^2 + \{8c_2^3 - 10c_2c_3 + 3c_4\}e_n^3 + O(e_n^9). \quad (11)$$

In the same way by considering (11) and second step of (6), we obtain

$$(z_n - \alpha) = \{1 - A(0)\}e_n + c_2\{A(0) - A'(0)\}e_n^2 + \{2c_3(A(0) - A'(0)) - \frac{1}{2}c_2^2\{4A(0) - 8A'(0) + A''(0))\}e_n^3 + ... + O(e_n^4). \quad (12)$$

To get maximum possible order of convergence the coefficient of e_n, e_n^2 and e_n^3 in above equation must vanish and by equating the coefficient of e_n, e_n^2 and e_n^3 to zero, we get $A(0) = 1$ and $A'(0) = 1$ $A''(0) = 4$. Additionally, we have

$$\begin{aligned} f(z_n) = f'(\alpha)[&\{c_2c_3 + c_2^3(5 - \frac{1}{6}A^{(3)}(0))\}e_n^4 \\ &\{-2c_3^2 - 2c_2c_4 - c_2^2c_3(-32 + A^{(3)}(0)) + c_2^4(-36 + \frac{5}{3}A^{(3)}(0) + \frac{1}{24}A^{(4)}(0))\}e_n^5 \\ &+\{-7c_3c_4 + c_2(-3c_5 - 2c_3^2(-33 + A^{(3)}(0)) - \frac{3}{2}c_2^2c_4(-32 + A^{(3)}(0)) \\ &+\frac{1}{3}c_2^3c_3(-786 + 37A^{(3)}(0) - A^{(4)}(0)) - \frac{1}{120}c_2^5(-20400 + 1240A^{(3)}(0) \\ &-65A^{(4)}(0) + A^{(4)}(0))\}e_n^6 ... + O(e_n^9). \end{aligned} \quad (13)$$

Considering (10) and (13) we obtained

$$\begin{aligned} \frac{f(z_n)}{f(y_n)} = &\{-c_3 + c_2^2(5 - \frac{1}{6}A^{(3)}(0)\}e_n^2 \\ &+\{-2c_4 - \frac{2}{3}c_2c_3(-30 + A^{(3)}(0)) + c_2^3(-26 + \frac{4}{3}A^{(3)}(0) - \frac{1}{24}A^{(4)}(0))\}e_n^3 \\ &+\{-3c_5 + c_3^2(19 - \frac{2}{3}A^{(3)}(0)) - c_2c_4(-29 + A^{(3)}(0)) + \frac{1}{12}c_2^2c_3(86A^{(3)}(0) \\ &-3(520 + A^{(4)}(0))) - \frac{1}{120}c_2^4(-11160 + 820A^{(3)}(0) - 55A^{(4)}(0) + A^{(5)}(0))\}e_n^4 \\ &... + O(e_n^9) \end{aligned} \quad (14)$$

and similarly

$$\begin{aligned} \frac{f(z_n)}{f'(x_n)} = &\{-c_2c_3 + c_2^3(5 - \frac{1}{6}A^{(3)}(0)\}e_n^4 \\ &+\{-2c_3^2 - 2c_2c_4(-34 + A^{(3)}(0)) + c_2^4(-46 + 2A^{(3)}(0) - \frac{1}{24}A^{(4)}(0))\}e_n^5 \\ &+\{-7c_3c_4 + c_2(-3c_5 + c_3^2(73 - 2A^{(3)}(0)) + \frac{1}{2}c_2^2c_4(104 - 3A^{(3)}(0)) \\ &+\frac{1}{6}c_2^3c_3(89A^{(3)}(0) - 2(1035 + A^{(4)}(0))) + c_2^5(262 - \frac{43}{3}A^{(3)}(0) \\ &+\frac{5}{8}A^{(4)}(0) - \frac{1}{120}A^{(5)}(0))\}e_n^6 ... + O(e_n^9). \end{aligned} \quad (15)$$

Finally, Taylor expansion for the simple root in the last step of (6) by using (11), (13) and (15) becomes

$$\begin{aligned}
&(z_n-\alpha)-\frac{f(z_n)}{f'(x_n)}B(t_n)H(u_n)\\
&=\{\frac{1}{6}(-1+B(0)H(0))c_2(6c_3+c_2^2(-30+A^{(3)}(0)))\}e_n^4\\
&+\{(2(-1+B(0)H(0)c_3^2+2(-1+B(0)H(0))c_2c_4\\
&+c_2^2c_3(32+H(0)B'(0)+B(0)H(0)(-34+A^{(3)}(0))-A^{(3)}(0))\\
&+\frac{1}{24}c_2^4(-864+4H(0)B'(0)(-30+A^{(3)}(0))+40A^{(3)}(0)-A^{(4)}(0)\\
&+B(0)H(0)(1104-48A^{(3)}(0)+A^{(4)}(0)))\}e_n^5\\
&+\{7(-1+B(0)H(0)c_3c_4+c_2(3(-1+B(0)H(0))c_5+c_3^2(66\\
&+4H(0)B'(0)-B(0)(H'(0)+H(0)(73-2A^{(3)}(0))-2A^{(3)}(0))\\
&+\frac{1}{2}c_2^2c_4(96+4H(0)B'(0)-3A^{(3)}(0)+B(0)H(0)(-104+3A^{(3)}(0))\\
&+\frac{1}{6}c_2^3c_3(-1572+74A^{(3)}(0)+H(0)(3B''(0)+B'(0)(-282+8A^{(3)}(0)))\\
&-2A^{(4)}(0)+B(0)(-2H'(0)(-30+A^{(3)}(0))+H(0)(2070-89A^{(3)}(0)\\
&+2A^{(4)}(0))))+\frac{1}{360}c_2^5(-3(-20400+1240A^{(3)}(0)+5H(0)-2B''(0)\\
&(-30+A^{(3)}(0))+B'(0)(-1464+60A^{(3)}(0)-A^{(4)}(0)))-65A^{(4)}(0)\\
&+A^{(5)}(0))+B(0)(-10H'(0)(-30+A^{(3)}(0))^2+3H(0)(-31440\\
&+1720A^{(3)}(0)-75A^{(4)}(0)+A^{(5)}(0)))))\}e_n^6\\
&+\{6(-1+B(0)H(0)c_4^2+10(-1+B(0)H(0))c_3c_5+\frac{2}{3}(66+12H(0)\\
&-2A^{(3)}(0)+B(0)(-3+H(0)(-75+2A^{(3)}(0))))+2c_2(2(-1+B(0)\\
&H(0))c_6+c_3c_4(98+14H(0)-3A^{(3)}(0)+B(0)(-2+H(0)(-110\\
&+3A^{(3)}(0)))))+\frac{1}{6}c_2^3c_4(B(0)(4(-30+A^{(3)}(0))+H(0)(-3000\\
&+130A^{(3)}(0)-3A^{(4)}(0)))+3(752+H(0)(268-8A^{(3)}(0))-36A^{(3)}(0)\\
&+A^{(4)}(0)\}e_n^7+O(e_n^8).
\end{aligned}\tag{16}$$

To get the maximum possible order of convergence the coefficient of e_n^4, e_n^5 and e_n^6 in above equation must vanish and by equating the coefficient of e_n^4, e_n^5 and e_n^6 to zero we get $A^{(3)}(0)=30$, $B(0)=1$, $H(0)=1$, $B'(0)=2$, $H'(0)=1$, $B''(0)=12$. Putting all these value in the equation (14), we get the error equation for the method (6) as

$$e_{n+1} = \frac{1}{24}c_2^2c_3(48c_3 + c_2^2(-144 + 4B^{(3)}(0) - A^{(4)}(0)))e^7 + O(e)^8.$$

This proves the results.

Method 2: It is remarkable that for attaining an optimal three step method with four evaluations the order of convergence should be eighth but the order of the scheme (6) is one unit lower, so it would be interesting from practical and analytical point of view to increase the order of convergence from seven to eight without using any new function evaluation per full iteration. To serve the purpose, we consider

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)}A(t_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}(B(t_n) * H(u_n) * G(s_n)), \end{aligned} \tag{17}$$

where t_n, u_n are defined as Method 1 and $s_n = \frac{f(z_n)}{f(x_n)}$. Clearly iteration class (17) requires three function evaluations and one derivative function evaluation. Now we give the following theorem with proof.

Theorem 1.5. *Assume that the function $f : D \subset R \to R$ for an open interval D has a simple root $\alpha \in D$. Let $f(x)$ be sufficiently smooth in the interval D and the initial guess x_0 is sufficiently close to α. Then the order of convergence of the new method defined by* (17) *is eighth if* $A(0) = 1$, $A'(0) = 1$, $A''(0) = 4$, $A^{(3)}(0) = 30$, $A^{(4)}(0) = 0$, $B(0) = 1$, $H(0) = 1$, $B'(0) = 2$, $H'(0) = 1$, $B''(0) = 12$, $B^{(3)}(0) = 36$, $G(0) = 1$, $G'(0) = 2$,*and its error equation is*

$$\begin{aligned} e_{n+1} \;=\; & \frac{1}{120}c_2c_3(480c_2^2c_3 - 120c_2c_4 + 60c_3^2(-2 + H''(0)) + c_2^4(5B^{(4)}(0) \\ & -A^{(5)}(0)))e_n^8 + O(e_n^9). \end{aligned} \tag{18}$$

Proof: Using Taylor's series and symbolic computation, we can have the similar relation as the proof of the theorem 1. So we only give the following expressions

$$\frac{f(z_n)}{f(x_n)} = \{-c_2c_3\}e_n^3 + \{3c_2^2c_3 - 2c_3^2 - 2c_2c_4 + c_2^4(14 - \frac{1}{24}A^{(4)}(0))\}e_n^4$$
$$+\{-5c_2^2c_4 - 7c_3c_4 + c_2(9c_3^2 - 3c_5) - \frac{1}{3}c_2^3c_3(-315 + A^{(4)}(0)) + c_2^5(-154$$
$$+\frac{7}{12}A^{(4)}(0) - \frac{1}{120}A^{(5)}(0))\}e_n^5$$
$$+\{6c_3^3 - 6c_1^20c_3c_5 + c_2(26c_3c_4 - 4c_6) + c_2^2(7c_5 - c_3^2(304 + A^{(4)}(0)))$$
$$-\frac{1}{2}c_2^3c_4(-318 + A^{(4)}(0)) + \frac{1}{24}c_2^4c_3(131A^{(4)}(0) - 2(16788 + A^{(5)}(0)))$$
$$-\frac{1}{720}c_2^6(-730080 + 3420A^{(4)}(0) - 102A^{(5)}(0) + A^{(6)}(0))\}e_n^6$$
$$+\{23c_3^2c_4 - 17c_4c_5 - 13c_3c_6 + c_2(18c_4^2 + 34c_3c_5 - 5c_7 + c_3^3(398$$
$$-\frac{1}{4}A^{(4)}(0))) + c_2^2(9c_6 - 3c_3c_4(-309 + A^{(4)}(0))) + \frac{1}{3}c_2^3(-2c_5(-318$$
$$+A^{(4)}(0)) + c_3^2(-15171 + 61A^{(4)}(0) - A^{(5)}(0))) + \frac{1}{8}c_2^4c_4(-16360$$
$$+64A^{(4)}(0) - A^{(5)}(0)) - \frac{1}{120}c_2^5c_3(-1263240 + 6100A^{(4)}(0) - 193A^{(5)}(0)$$
$$+2A^{(6)}(0) + c_2^7(-5179 + \frac{88}{3}A^{(4)}(0) - \frac{41}{30}A^{(5)}(0) + \frac{1}{36}A^{(6)}(0) - \frac{A^{(7)}(0)}{5040})\}e_n^7$$
$$+O(e_n^8),$$

and

$$e_{n+1} = (z_n - \alpha) - \frac{f(z_n)}{f'(x_n)}(B(t_n) * H(u_n) * G(s_n)) = \{(-1 + G(0))c_2c_3\}e_n^4$$
$$+\{-48c_2^2c_3 + 48c_3^2 + 48c_2c_4 + c_2^4(-336 + A^{(4)}(0))\}e_n^5$$
$$+\frac{1}{120}\{(-1 + G(0))(360c_2^2c_4 - 840c_3c_4 + 360c_2(2c_3^2 - c_5) - 40c_2^3c_3(-324$$
$$+A^{(4)}(0)) + c_2^5(-16800 + 65A^{(4)}(0) - A^{(5)}(0)))\}e_n^6$$
$$+\{-2(-1 + G(0))(2c_3^3 - 3c_4^2 - 5c_3c_5) - 4(-1 + G(0))c_2(4c_3c_4 - c_6)$$
$$+\frac{1}{2}(-1 + G(0))c_2^3c_4(-328 + A^{(4)}(0)) + A^{(4)}(0))) + \frac{1}{24}c_2^4c_3(122A^{(4)}(0)$$
$$-2(15360 + A^{(5)} + G(0)(30576 + 4B^{(3)}(0) - 123A^{(4)}(0) + 2A^{(5)}(0)))$$
$$+\frac{1}{720}(-1 + G(0))c_2^6(-619200 + 3000A^{(4)}(0) - 96A^{(5)}(0) + A^{(6)}(0)))e_n^7$$
$$+... + O(e_n^9). \quad (19)$$

This is clearly shows that the weight function in (17) must be chosen as stated in the theorem to make it optimal. Now, we have the following error equation

$$e_{n+1} = \frac{1}{120}c_2c_3(480c_2^2c_3 - 120c_2c_4 + 60c_3^2(-2 + H''(0)) + c_2^4(5B^{(4)}(0)$$
$$- A^{(5)}(0)))e_n^8 + O(e_n^9). \quad (20)$$

2 Numerical Reports

To get the accuracy of methods, it is necessary to study the numerical results of the presented methods and schemes available in the literature. In order to illustrate the convergence behavior of new methods and to check the validity of the theoretical results we employ Method 1 and Method 2 to solve some nonlinear equations. For comparison in numerical experiment we test presented method with existing eighth-order methods presented in equation(18)(ZWG) of [14], equation(11)(BCST) of [5], equation(5)(LCTAZ) of [16]. In Table 1 we have mention seven nonlinear test functions with their roots. As shown in Table 3, the proposed methods are giving a better accuracy to the eighth-order methods discribed in ZWG, BCST and LCTAZ. All computations were done using Mathematica 9. We have used the stopping criteria for computer program: $|f(x_n)| < 10^{-125}$. For numerical testing we consider the following functions along with the mentioned weight function.

Table 1: Functions and their roots

$f(x)$	α
$f_1(x) = x^2 - e^x - 3x + 2$	$\alpha_1 \approx 0.2575302854...$
$f_2(x) = x^3 + 4x^2 - 10$	$\alpha_2 \approx 1.3652300134...$
$f_3(x) = Sin^2 x - x^2 + 1$	$\alpha_3 \approx 1.4044916482...$
$f_4(x) = e^{-x} + sinx - 1$	$\alpha_4 \approx 2.0768312743...$
$f_5(x) = (1 + cosx)(e^x - 2)$	$\alpha_5 \approx 0.6931478055...$
$f_6(x) = x^5 - sinx$	$\alpha_6 \approx 0.9610369414...$
$f_7(x) = x^5 - x^2 + 7x - 41$	$\alpha_7 \approx 1.9878112719...$

Table 2: Some forms of weight functions

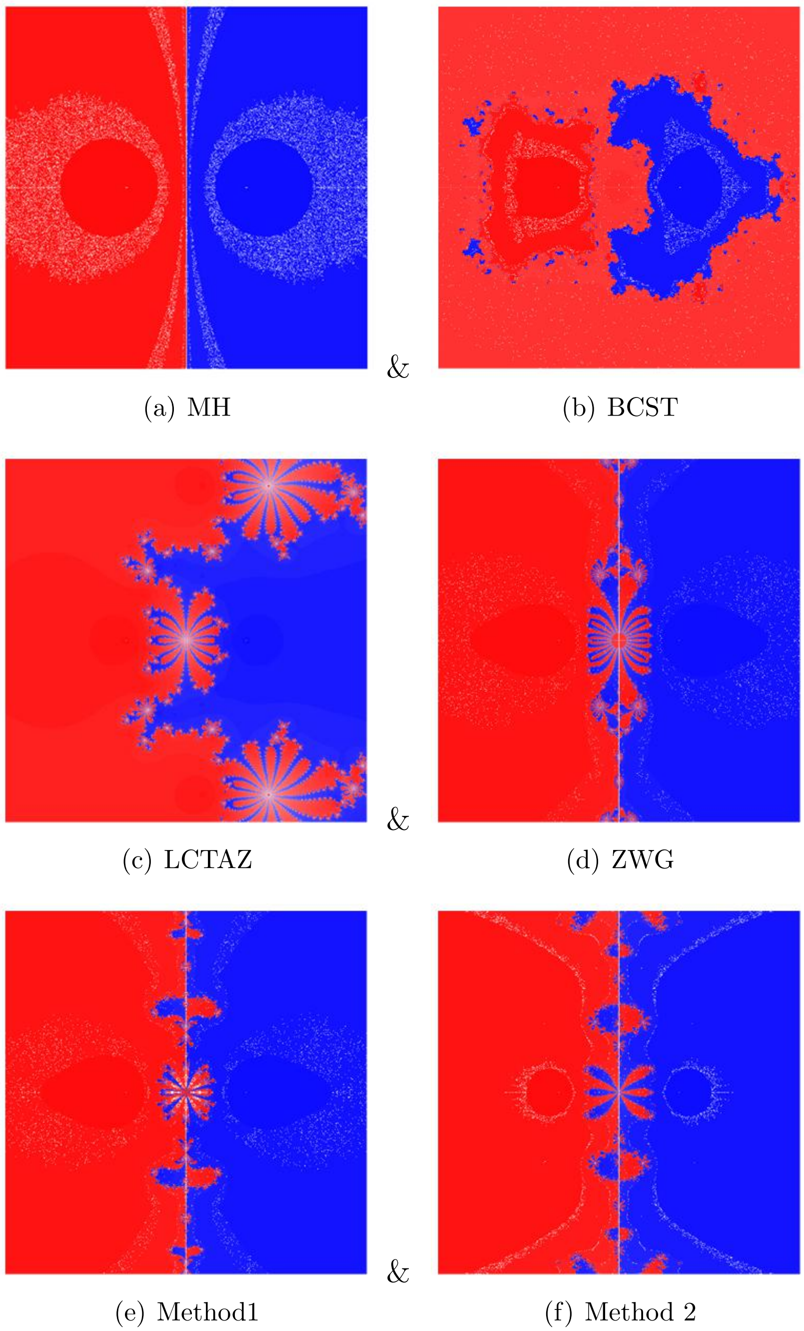
Method 1	$A(t_n)$	$B(t_n)$	$H(u_n)$	
forms	$\frac{1-t_n}{1-2t_n}$	$\frac{1-t_n}{1-3t_n}$	e^{u_n}	
Method 2	$A(t_n)$	$B(t_n)$	$H(u_n)$	$G(s_n)$
forms	$1 + t_n + 2t_n^2 + 5t_n^3$	$1 + 2t_n + 6t_n^2 + 6t_n^3$	$\frac{1}{1-u_n}$	$\frac{1}{1-2s_n}$

Table 3: Numerical comparison of different methods

	n	TNFE	$\lvert f(x_n)\rvert$	COC
$f_1(x) = x^2 - e^x - 3x + 2, x_0 = 0$				
MH	3	12	4.2661×10^{-284}	6
BCST	3	12	7.8554×10^{-601}	8
LCTAZ	3	12	1.22442×10^{-628}	8
ZWG	3	12	5.6014×10^{-640}	8
Method 1	3	12	1.7375×10^{-432}	7
Method 2	3	12	5.6656×10^{-646}	8
$f_2(x) = x^3 + 4x^2 - 10, x_0 = 1.4$				
MH	3	12	1.3963×10^{-412}	6
BCST	3	12	3.3903×10^{-884}	8
LCTAZ	3	12	8.1291×10^{-900}	8
ZWG	3	12	2.0753×10^{-921}	8
Method 1	3	12	2.5909×10^{-615}	7
Method 2	3	12	1.8773×10^{-1013}	8
$f_3(x) = (sinx)^2 - x^2 + 1, x_0 = 1.3$				
MH	3	12	1.4130×10^{-270}	6
BCST	3	12	1.8917×10^{-467}	8
LCTAZ	3	12	6.4120×10^{-406}	8
ZWG	3	12	1.5358×10^{-426}	8
Method 1	3	12	2.0502×10^{-327}	7
Method 2	3	12	1.3592×10^{-486}	8
$f_4(x) = e^{-x} + sinx - 1, x_0 = 2$				
MH	3	12	6.1367×10^{-308}	6
BCST	3	12	1.1709×10^{-624}	8
LCTAZ	3	12	9.0715×10^{-563}	8
ZWG	3	12	2.2817×10^{-584}	8
Method 1	3	12	2.2836×10^{-450}	7
Method 2	3	12	1.5961×10^{-679}	8
$f_5(x) = (1 + cosx)(e^x - 2), x_0 = 0.5$				
MH	3	12	3.6539×10^{-239}	6
BCST	3	12	2.8311×10^{-551}	8
LCTAZ	3	12	2.3369×10^{-549}	8
ZWG	3	12	2.0304×10^{-571}	8
Method 1	3	12	9.7337×10^{-405}	7
Method 2	3	12	9.9590×10^{-602}	8
$f_6(x) = x^5 - sinx, x_0 = 1$				
MH	3	12	2.6556×10^{-245}	6
BCST	3	12	7.0322×10^{-553}	8
LCTAZ	3	12	1.9786×10^{-462}	8
ZWG	3	12	1.5024×10^{-484}	8
Method 1	3	12	7.1331×10^{-414}	7
Method 2	3	12	5.5247×10^{-592}	8
$f_7(x) = x^5 - x^2 + 7x - 41, x_0 = 1.97$				
MH	3	12	3.5859×10^{-407}	6
BCST	3	12	9.1147×10^{-848}	8
LCTAZ	3	12	5.0920×10^{-803}	8
ZWG	3	12	7.2698×10^{-824}	8
Method 1	3	12	3.0280×10^{-575}	7
Method 2	3	12	9.6242×10^{-886}	8

3 Fractal Pictures for Basin of Attractions

Here we investigate the comparison of some high order simple root finder method in the complex plane using basin of attraction.

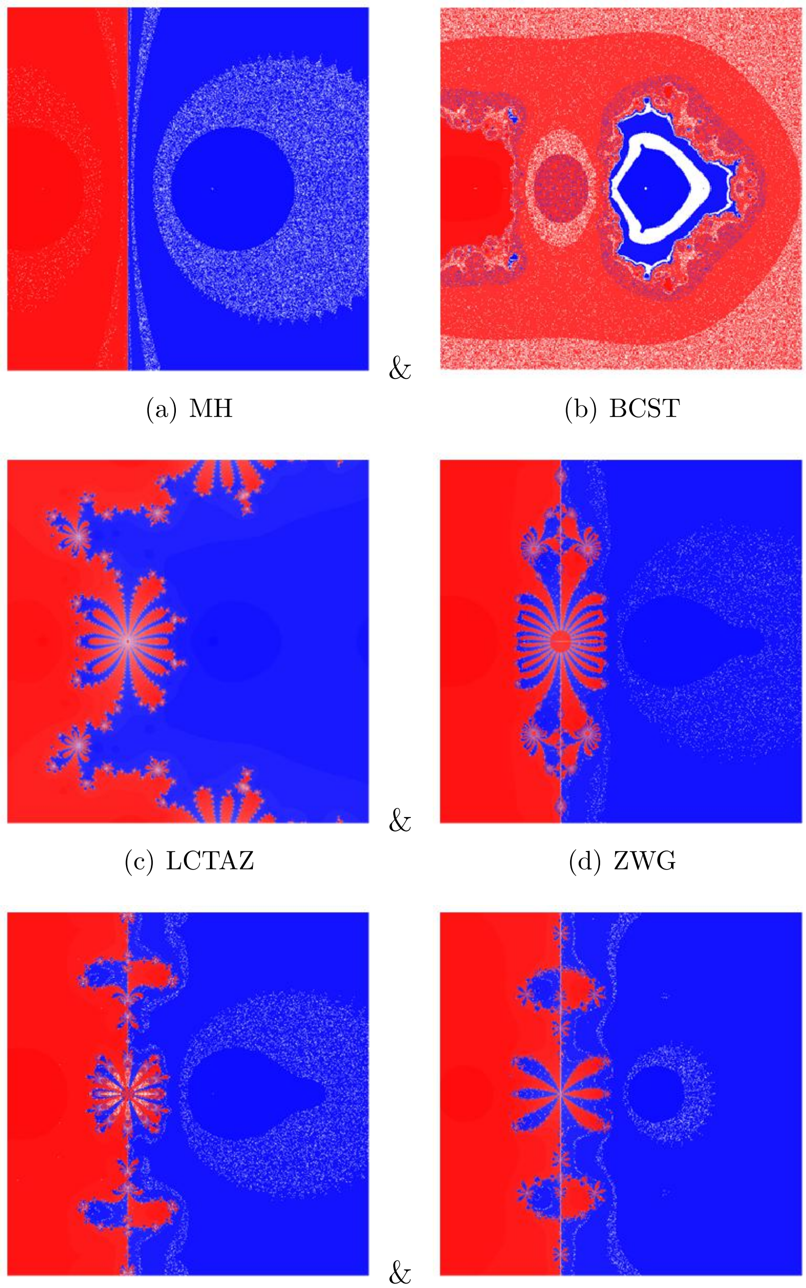


(a) MH & (b) BCST

(c) LCTAZ & (d) ZWG

(e) Method1 & (f) Method 2

Figure 1: Basin of attraction for the polynomial $z^2 - 1$.

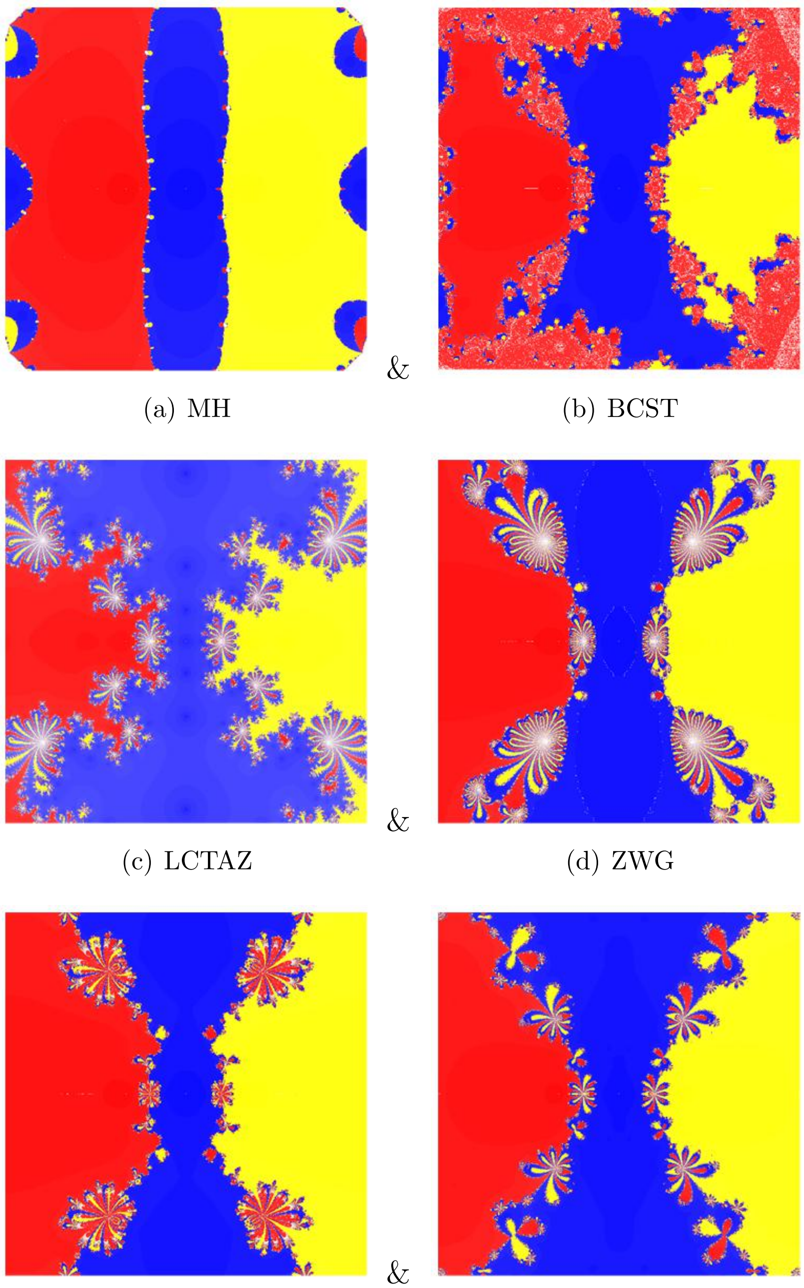


(a) MH & (b) BCST

(c) LCTAZ & (d) ZWG

(e) Method 1 & (f) Method 2

Figure 2: Basin of attraction for the polynomial $z^2 + 2 * z - 1$.



(a) MH (b) BCST

(c) LCTAZ (d) ZWG

(e) Method 1 (f) Method 2

Figure 3: Basin of attraction for the polynomial $z^3 - z$.

It is known that the corresponding fractal of an iterative root-finding method is a boundary set in the complex plane, which is characterized by the iterative methods applied to a fixed polynomial $f(z)$. We compare proposed Method 1(6) and Method 2(17) to the iteratives methods presented equation(18) of [14] denoted by ZWG, equation(11) of [5] denoted by BCST, equation(5) of [16] denoted by LCTAZ by using the basin of attraction for three complex polynomial $p_1(z) = z^2 - 1$ with two root of unity; $p_2(z) = z^2 + 2z - 1$ with root -1.46771+0.226699I, -0.453398I, 1.46771+0.226699I and $p_3(z) = z^3 - z$ with -1, 0, 1. In our numerical experiment, we have used a grid of 400*400 points in a rectangle $D = [-3, 3] * [-3, 3] \subset C$ and assign a color to each point $z_0 \in D$. We have assigned white color for divergent points. Figures 1 shows that the method MH(eqn:8) of [7], Method 1(6) and Method 2(17) without memory have very little diverging points compared to other methods. Figures 2 shows that the basin of attraction for $P_2(z)$ and we can see that the method MH(eqn:8) of [7], Method 1(6) and Method 2(17) are slightly better than the schemes of method ZWG(eqn:18) of [14], method BCST(eqn:11) of [5], method LCTAZ(eqn:5) of [16] and convergence speed of our methods are faster than other methods. In fighure 3, the dynamical behavior of method MH(eqn:8) of [7], Method 1(6) and Method 2(17) are best to the method ZWG(eqn:18) of [14], method BCST(eqn:11) of [5] and method LCTAZ(eqn:5) of [16]. From the whole, we have seen that our method are better than the other.

4 Conclusion

It is widely know that many problems in different scientific fields are reduced to solve single valued non-linear equations. During the last few years numerous papers devoted to the mentioned iterative methods have appeared in several journals. In the forgoing study, we have proposed a three step seven and optimal eight order without memory methods. These methods use four functional evaluations per iteration. A comparison of computational efficiencies of the new schemes with existing schemes are given in the table. Theoretical order of convergence and the analysis of computational efficiency are verified in the considered examples. The dynamical and numerical results have confirmed the robust and efficient character of the proposed algorithm.

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