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Existence Theorem for First Order Ordinary Functional Differential Equations with Periodic Boundary Conditions

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Abstract

In this paper, an existence theorem for the periodic boundary value problem of first order ordinary functional integro-differential equations is proved via a fixed point theorem in Banach algebras and under some mixed generalized Lipschitz and Caratheodory conditions.

Key words: *Fixed point theorem,Banach algebra,Lipschitz and caratheodory conditions.*

1 Statement of Problem

Let $\mathbb{R}$ be the real line and let $I_0 = [-\delta, 0]$ and $I = [0, T]$ be two closed and bounded intervals in $\mathbb{R}$. Let C be the space of continous real valued function on I_0. Given a function $\phi \in C$, we have studied the following periodic boundary value problem (In short $PBVP$) of first order ordinary functional integro-differential equations.

$$\frac{d}{dt}\left[\frac{x(t)}{f(t, x(t))}\right] = g\left(t, x_t, \int_0^t k(s, x_s)\, ds\right), a.e.\ t \in I.$$

$$x(0) = x(T)$$

$$x_0 = \phi$$

(1.1)

where $f : I \times \mathbb{R} \longrightarrow \mathbb{R} - \{0\}$ is continuous and

$k : I \times C \longrightarrow \mathbb{R}$, $g : I \times C \times \mathbb{R} \longrightarrow \mathbb{R}$, $x_t : I_0 \longrightarrow C$ is continuous Function defined by $x_t(\theta) = x(t + \theta)$ for all $\theta \in I_0$

When $f(t,x)$=1 on $I \times \mathbb{R}$.

By a solution of the $PBVP$ (1.1) we means a function

$x \in AC(I, \mathbb{R})$ that satisfies

(i) The function $t \mapsto \left(\frac{x(t)}{f(t,x(t))}\right)$ is absolutely continuous on I and

(ii) x satisfies the equations in (1.1)

where $AC(I, \mathbb{R})$ is the space of continuous functions whose first derivative exists and is absolutely continuous real-valued functions on I.

The periodic boundary value problem (1.1) is quite general in the sense that it includes several known classes of periodic boundary value problem as special cases. For example, if $f(t,x)$=1 on $I \times \mathbb{R}$.

then $PBVP$ (1.1) reduce to the $PBVP$.

$$x'(t) = g\left(t, x_t, \int_0^t k(s, x_s)\,ds\right), \qquad a.e. \quad t \in I.$$

$$x(0) = x(T)$$

(1.2)

Which further, when $g(t, x_t, y) = g(t, x_t)$ on $I \times C \times \mathbb{R}$, includes the following $PBVP$ studied in Nieto [1997,2002],

$$x'(t) = g(t, x(t)), \qquad a.e. \quad t \in I.$$

$$x(0) = x(T)$$

(1.3)

There is good deal of literature on the $PBVP$ (1.3) for different aspects of the solution. In this chapter, we discuss the $PBVP$ (1.1) for existence theory only under suitable conditions on the nonlinearities f and g involved in it.

2 Auxiliary Results

Throughout this article, let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \longrightarrow X$ is called $\mathcal{D}$ -Lipschitz if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying,

$$\|Ax - Ay\| \le \psi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha\, r, (\alpha > 0)$, A is called a Lipschitz with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called nonlinear D-contraction on X. Some times we call the function ψ a D-funtion for convenience.

An operator $T : X \longrightarrow X$ is called compact if $\overline{T(S)}$ is compact subset of x for any $S \subset X$.. Similarly $T : X \longrightarrow X$ is called totally bounded if T maps a bounded subset of X into the relatively compact subset of X. Finally $T : X \longrightarrow X$ is called compactly continuous operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true.

Theorem 2.1: (Dhage [2006]). Let $B_r(0)$ and $\overline{B_r(0)}$ be respectively open and closed balls in a Banach algebra X centered at origin O and of radius r. Let $A, B : \overline{B_r(0)} \longrightarrow X$ be two operators satisfying

(a) A is Lipschitz with the Lipschitz constant α,
(b) B is compact and continuous, and
(c) $\alpha M < 1$,

Where $M = \left\|B\left(\overline{B_r(0)}\right)\right\| := sup\left\{\|Bx\| : x \in \overline{B_r(0)}\right\}$

Then either

(i) the equation $\lambda[AxBx] = x$ has a solution for $\lambda = 1$, or
(ii) there exists on $u \in$ X such that $\|u\| = r$ $\lambda[AuBu] = u$ for some $0 < \lambda < 1$.

It is known that the theorem (2.1) is useful for proving the existence theorem for the integral equations of mixed type in Banach algebras.

3 Existence Theory

Let $B(I, \mathbb{R})$ denote the space of bounded real-valued functions on I. Let $C(I, \mathbb{R})$ denote the space of all continuos real-valued function on I. Define a norm $\|\cdot\|$ in $C(I, \mathbb{R})$ by

$$\|x\| = \sup_{t \in I} |x(t)|$$

and multiplication "." in $C(I, \mathbb{R})$ by

$$(x \cdot y)(t) = x(t) \cdot y(t) \quad \text{for} \quad t \in I.$$

Clearly $C(I, \mathbb{R})$ becomes a Banach algebra with respects to above norm and multiplication. By $L^1(I, \mathbb{R})$ we denote the set of Lebesque integrable function on I and the norm $\|\cdot\|_{L^1}$ in $L^1(I, \mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\ ds.$$

We employ a hybrid fixed point theorem of Dhage [2006] i.e. theorem (2.1) for proving the existence result for the $PBVP$ (1.1).

We give some preliminaries.

Lemma 3.1: For any $h \in L^1(I, \mathbb{R}^+)$ and $\sigma \in L^1(I, \mathbb{R})$, x is a solution to the functional equation

$$x' + h(t)\, x(t) = \sigma(t) \qquad a.e.\ \ t \in I$$
$$x(0) = x(T)$$

(3.1)

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_h(t,s)\sigma(s)\ ds\ .$$

(3.2)

where

$$G_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)}}{1-e^{-H(T)}}, 0 \le s \le t \le T \\ \dfrac{e^{H(s)-H(t)-H(T)}}{1-e^{-H(T)}}, 0 \le t \le s \le T \end{cases}$$

(3.3)

where $H(t) = \int_0^t h(s)\, ds.$

Definition 3.1 : A mapping $\beta : I \times C \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be caratheodory if

(i) $t \longmapsto \beta(t,x,y)$ is measurable for each $x \in C$ and $y \in \mathbb{R}$,

(ii) $(x,y) \longmapsto \beta(t,x,y)$ is continuous almost everywhere for $t \in I$.

Again a caratheodory function $\beta(t,x,y)$ is called L^1 – caratheodory if

(iii) for each real number $r > 0$, there exists a function $q_r \in L^1(I, \mathbb{R})$ such that $|\beta(t,x,y)| \le q_r(t) \quad a.e.\ t \in I$

for all $x \in C$ and $y \in \mathbb{R}$ with $|x| \le r$ and $|y| \le r$. Finally a caratheodory function $\beta(t,x,y)$ is called $L^1{}_X$ - caratheodory if

(iv) there exists a function $q \in L^1(I, \mathbb{R})$ such that

$$|\beta(t,x,y)| \le q(t) \qquad a.e.\ t \in I$$

for all $x \in C$ and $y \in \mathbb{R}$.

For convenience, the function q is referred to as a bound function of β.

We will use the following hypotheses in the sequel.

(A_1) The function $t \longmapsto f(t,x)$ is periodic of period T for all $x \in \mathbb{R}$

(A_2)) The function $x \mapsto \frac{x}{f(0,x)}$ is injective in $\mathbb{R}$.

(A_3) The function $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a function $\ell \in B(I, \mathbb{R})$ such that $\ell(t) > 0 \quad a.e.\ t \in I$ and

$|f(t,x) - f(t,y)| \leq \ell(t)|x - y|, \qquad a.e.\ t \in I$

for all $x, y \in \mathbb{R}$.

(A_4) The function $k : I \times C \longrightarrow \mathbb{R}$ is continuous and there exists a function $\alpha \in L^1(I, \mathbb{R}^+)$ such that

$$|k(s,y)| \leq \alpha(s)\|y\|_c \quad a.e.\ s \in I, y \in C.$$

(A_5) The function g is caratheodory on $I \times C \times \mathbb{R}$.

(A_6) There exists a function $\gamma \in L^1(I, \mathbb{R}^+)$ and a D-function $\psi \in \Psi$ such that

$$|g_h(t,x,y)| \leq \gamma(t)\psi(\|x\|_c + |y|) \quad a.e.\ t \in I$$

(3.4)

for each $x \in C$ and $y \in \mathbb{R}$.

Now consider the $PBVP$

$$\left(\frac{x(t)}{f(t,x(t))}\right)' + h(t)\left(\frac{x(t)}{f(t,x(t))}\right) = g_h\left(t, x_t, \int_0^t k(s,x_s)\,ds\right), \quad a.e.\ t \in I.$$

$$x(0) = x(T)$$

(3.5)

where

$h \in L^1(I, \mathbb{R}^+)$ is bounded and the function $g_h : I \times C \times \mathbb{R} \longrightarrow \mathbb{R}$

is defined by

$$g_h(t,x,y) = g(t,x,y) + h(t)\left(\frac{x(t)}{f(t,x(t))}\right)$$

(3.6)

Lemma 3.2 : Assume that hypotheses $(A_1) - (A_2)$ hold. Then for any bounded $h \in L^1(I, \mathbb{R}^+)$, x is a solution of the functional equation (5.8) if and only if it is a solution of the integral equation

$$x(t) = [f(t,x(t))]\left(\int_0^T G_h(t,s)\, g_h\left(s, x_s, \int_0^s k(\tau, x_\tau)\,d\tau\right) ds\right)$$

(3.7)

for all $t \in I$ where the Green's function $G_h(t,s)$ is defined by (3.3)

Theorem 3.1: Assume that hypotheses $(A_1), (A_3) - (A_6)$ hold. Suppose that there exists a real number $r > 0$ such that

$$r > \frac{F M_h \|\gamma\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r)}{1 - L M_h \|\gamma\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r)}$$

(3.8)

where $LM_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(r) < 1, F = \sup_{t\in[0,T]}|f(t,0)|$ and $L = \max_{t\in I}\ell(t)$

Then the $PBVP$ (1.1) has a solution on I.

Proof : Let $X = C(I,\mathbb{R})$. Define on open ball $\mathcal{B}_r(0)$ centered at origin O and of radius r, whose the real number are satisfies the inequality (3.8). Define two mapping A and B on $\overline{\mathcal{B}_r(0)}$ by

$$Ax(t) = f(t,x(t)), \quad t\in I$$

(3.9) and

$$Bx(t) = \int_0^T G_h(t,s)\, g_h\left(s,x_s,\int_0^s k(\tau,x_\tau)\,d\tau\right)ds, \quad t\in I$$

(3.10)

obviously A *and* B define the operators A, $B:\overline{\mathcal{B}_r(0)} \longrightarrow X$. Then the integral equation (3.7) is equivalent to the operator equation

$$Ax(t)\,Bx(t) = x(t), \quad t\in I$$

(3.11)

we shall show that the operators A *and* B satisfy all the hypotheses of Theorem (2.1) .

Step I : We first show that A is a Lipschitz on X. Let $x,y\in X$ then by (A_3)

$|Ax(t)-Ay(t)| = |f(t,x(t)) - f(t,y(t))|$

$\le \ell(t)|x-y|$

$\le L\|x-y\|$

for all $t\in I$. Taking the supremum over t we obtain

$\|Ax-Ay\| \le L\|x-y\|$

for all $x,y\in X$. So A is a Lipschitz on X with Lipschitz constant L.

Step II : Next we show that B is a completely continuous on X. Using the standard arguments as in Granns et. Al. [1991] , it is shown that B is continuous operator on X. We shall show that $B(\overline{\mathcal{B}_r(0)})$ is uniformly bounded and equicontinuous set in X. Let $x\in\overline{\mathcal{B}_r(0)}$ be arbitrary. Since g is Caratheodory, we have

$$|Bx(t)| \le \left|\int_0^T G_h(t,s)\, g_h\left(s,x_s,\int_0^s k(\tau,x_\tau)d\tau\right)ds\right|$$

$$\le \int_0^T \left|G_h(t,s)\, g_h\left(s,x_s,\int_0^s k(\tau,x_\tau)\,d\tau\right)\right|ds$$

$$\le \int_0^T G_h(t,s)\;\gamma(t)\psi\left(\|x_s\| + \int_0^s \alpha(\tau)\,\|x_\tau\|\,d\tau\right)ds$$

$$\le \int_0^T G_h(t,s)\;\gamma(t)\psi\left(\|x_s\| + \int_0^T \alpha(\tau)\,\|x_\tau\|\,d\tau\right)ds$$

$$\le M_h\int_0^T \gamma(s)\psi\left(\|x_s\| + \int_0^T \alpha(\tau)\,\|x_\tau\|\,d\tau\right)ds$$

$$\leq M_h \int_0^T \gamma(s)(1+\|\alpha\|_{L^1})\,\psi(r)\,ds$$

$$\leq M_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(r).$$

Taking the supremum over t we obtain $\|Bx\| \leq M$ for all $x \in \overline{\mathcal{B}_r(0)}$, where $M = M_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(r)$. This shows that $B\left(\overline{\mathcal{B}_r(0)}\right)$ is a uniformly bounded set in X. Next we show that $B\left(\overline{\mathcal{B}_r(0)}\right)$ is and equicontinuous set. To finish it is enough to show that $y' = (Bx)'$ is bounded on $[0,T]$.

Now for any $t \in [0,T]$, one has

$$|y'(t)| = |B'x(t)| \leq \left|\int_0^T \frac{\partial}{\partial t} G_h(t,s)\, g_h\left(s, x_s, \int_0^s k(\tau, x_\tau)\,d\tau\right) ds\right|$$

$$= \left|\int_0^T \left|(-h(t))\right|\, G_h(t,s)\, g_h\left(s, x_s, \int_0^s k(\tau, x_\tau)\,d\tau\right) ds\right|$$

$$\leq H M_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(r).$$

$$= c.$$

Where $H = \max_{t\in I} h(t)$. Hence for any $t, \tau \in [0,T]$ one has

$$|Bx(t) - Bx(\tau)| \leq c|t-\tau| \longrightarrow 0 \quad as \quad t \longrightarrow \tau.$$

This shows that $B\left(\overline{\mathcal{B}_r(0)}\right)$ is a equicontinuous set in X. Now $B\left(\overline{\mathcal{B}_r(0)}\right)$ is uniformly bounded and equicontinuous set in X, so it is compact by Arzela- Ascoli theorem. As a result B is compact and continuous operator on $\overline{\mathcal{B}_r(0)}$. Finally, by hypothesis,

$$\alpha M = L M_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(r) < 1,$$

and thus all the conditions of Theorem (2.1) are satisfied and a direct application of it yields that either the conclusion (i) or conclusion (ii) holds.

Step III: We show the conclusion (ii) is not possible. Let $u \in X$ be a solution to $PBVP$ (1.1) such that $\|u\| = r$. Then we have, for any $\lambda \in (0,1)$

$$u(t) = \lambda\left[f(t,u(t))\right]\left(\int_0^T G_h(t,s)\, g_h\left(s, u_s, \int_0^s k(\tau, u_\tau)\,d\tau\right) ds\right)$$

for $t \in I$. There fore

$$|u(t)| \leq \lambda\left|f(t,u(t))\right| \times \left(\int_0^T \left|G_h(t,s)\, g_h\left(s, u_s, \int_0^s k(\tau, u_\tau)\,d\tau\right) ds\right|\right)$$

$$\leq \lambda\left(\left|f(t,u(t)) - f(t,0)\right| + |f(t,0)|\right) \times \left(\int_0^T G_h(t,s)\left|g_h\left(s, u_s, \int_0^s k(\tau, u_\tau)\,d\tau\right)\right| ds\right)$$

$$\leq [\ell(t)|u(t)| + F] \times \left(\int_0^T M_h \left|g_h\left(s, u_s, \int_0^s k(\tau, u_\tau)\,d\tau\right)\right| ds\right)$$

$$\leq L M_h \|u\|\left(\int_0^T \gamma(s)(1+\|\alpha\|_{L^1})\,\psi(\|u\|)\,ds\right) + F M_h\left(\int_0^T \gamma(s)(1+\|\alpha\|_{L^1})\,\psi(\|u\|)\,ds\right)$$

$$\leq LM_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(\|u\|)\|u\|$$
$$+FM_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(\|u\|)$$

(3.12)

Taking the supremum in the above inequality yields

$$\|u\| \leq \frac{FM_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(\|u\|)}{1-LM_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(\|u\|)}$$

Substituting $\|u\| = r$ in above inequality yields

$$r \leq \frac{FM_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(r)}{1-LM_h \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\,\psi(r)}$$

This is a contradiction to (3.8). Hence the conclusion (ii) of Theorem (2.1) does not hold. Therefore the operator equation $AxBx = x$ and consequently the $PBVP$ (1.1) has a solution on I. This completes the proof.

4 An Example

Given the closed and bounded internals $I_0 = [-\pi, 0]$ and $I = [0, \pi]$ in $\mathbb{R}$. Consider the first order periodic boundary value problem $(PBVP)$,

$$\frac{d}{dt}\left[\frac{x(t)}{1+\frac{\sin t}{12}(|x(t)|)}\right] = -\left(\frac{x(t)}{1+\frac{\sin t}{12}(|x(t)|)}\right)$$
$$+\bar{g}\left(t, x_t\left(\frac{t}{2}\right), \int_0^t k(s, x_s)ds\right) \quad a.e.\, t \in I$$
$$x(0) = x(\pi)$$

(4.1)

Where the functions $f : I \times \mathbb{R} \longrightarrow \mathbb{R}^+ - \{0\}, \bar{g} : I \times C \times \mathbb{R} \longrightarrow \mathbb{R}$,

$k : I \times C \longrightarrow \mathbb{R}$ and $k, \bar{g}, f$ are given by

$$k(t, x_t) = \frac{x}{4\pi(1+\|x_t\|_c)}$$

$$\bar{g}(t, x, y) = \frac{p(t)x}{1+\|x_t\|_c} + |y|$$

and

$$f(t, x) = 1 + \frac{\sin t}{12}(|x|)$$

Where $p \in L^1(I, \mathbb{R})$.

Clearly the function k is continuous and it is easy to verify that f is continuous and satisfies the hypotheses $(A_1) - (A_3)$ on $I \times \mathbb{R}$ with $\ell(t) = \frac{1}{6}$ for all $t \in I$. To see this , let $x, y \in \mathbb{R}$, then we have

$$|f(t,x) - f(t,y)| = \left|1 + \frac{\sin t}{12}(|x|) - 1 - \frac{\sin t}{12}(|y|)\right|$$

$$\leq \frac{1}{6}\sin t\,(|x| - |y|)$$

$$\leq \frac{1}{6}\,(|x| - |y|).$$

Again the function $\bar{g}(t,x,y)$ is measurable in t for all $x, y \in \mathbb{R}$, and continuous in x and y almost everywhere for $t \in I$, and so $\bar{g}$ defines a Caratheodory mapping $\bar{g}\ : I \times C \times \mathbb{R} \longrightarrow \mathbb{R}$. Further more $g_1 = \bar{g}$ is also Caratheodory on $I \times C \times \mathbb{R}$, and

$$\|g_1(t,x,y)\| = \left|\frac{p(t)x(t)}{1 + \|x_t\|_c} + \int_0^{\pi - t} \frac{x(^s/_3)}{4\pi\left(1 + \|x_t(^s/_2)\|_c\right)} ds\right|$$

$$\leq \left|\frac{p(t)x(t)}{1 + \|x_t\|_c}\right| + \left|\int_0^{\pi - t} \frac{x(^s/_3)}{4\pi\left(1 + \|x_t(^s/_2)\|_c\right)} ds\right|$$

$$\leq |p(t)| + \frac{1}{4}$$

Hence the function g_1 is $L^1_{\mathbb{R}}$ – Caratheodory and satisfies all the hypotheses $(A_5) and\, (A_6)$ on $I \times C \times \mathbb{R}$ with

$\gamma(t) = |p(t)| + \frac{1}{4}$ on I and $\psi(r) = 1$ for all $r \in \mathbb{R}^+$. Therefore if $\|p\|_{L^1} < 5$ and $r = 2$, then $PBVP$ (4.1) has a solution in $\overline{B_2(0)}$ defined on I.

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