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A Method to Find the Solution of the Linear Octonionic Equation $\alpha(x\alpha) = (\alpha x)\alpha = \alpha x\alpha = \rho$

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Abstract

Because of the non-commutativity of quaternions and octonions, it is rather diffucult to find an explicit method for the solution of the equation with one unknown such that $\alpha(x\alpha) = (\alpha x)\alpha = \alpha x\alpha = \rho$ in quaternions and octonions. In this note, firstly we consider the linear octonionic equation with one unknown of the form:

$$\alpha(x\alpha) = (\alpha x)\alpha = \alpha x\alpha = \rho, with 0 \neq \alpha \in \mathbf{O},$$

secondly present a method, which is reduce this octonionic equation to an equation with the left and right coefficients to a real system of eight equations to find the solutions of this equation, and finally reach the solution of the linear octonionic equation from this real system. We end the paper with one example to illustrate the results of this paper.

Keywords: *he octonionic equations; The system of real equations.*

1 Introduction

A well-known property of **H** or **O** is that the multiplication in **H** or **O** is noncommutative. Therefore, not all results about complex numbers can be extended into quaternions or octonions. When working with quaternions and octonions, one often meets with various linear and nonlinear equations in these number systems. This is one of the challenging topics in the theory of quaternions and octonions. Recently regularity results for quaternionic equations have been investigated by many authors: we provide below some references but the list is by no means complete.

In the article [5] they describe the set of solutions of the equation $x\alpha = x + \beta$ over an algebraic division ring. The author of the paper [6] classifies solutions of the quaternionic equation $ax + xb = c$. In the article [8] they solve linear equations of the forms $ax = xb$ and $ax = \overline{x}b$ in the real Cayley–Dickson algebras (quaternions, octonions, sedenions), and establish a form of roots of such equations. In the article [3] they consider the equations of the forms $ax = xb$ and $ax = \overline{x}b$ for some generalizations of quaternions and octonions. In [1], the author discusses how to find one for some particular quadratic polynomials that include $x^2 + bx + xc + d$ as a particular case. In the article [7], the $\alpha x\beta + \gamma x\delta = \rho$ linear quaternionic equation with one unknown, $\alpha x\beta + \gamma x\delta = \rho$, is solved. In [4], the quaternionic equation $ax + xb = c$ is studied. In [2], Bolat and İpek obtained some results on the solutions of the $\alpha x\beta + \gamma x\delta = \rho$ linear matrix quaternionic equation with one unknown.

In this note, firstly we consider the linear octonionic equation with one unknown of the form:

$$\alpha(x\alpha) = (\alpha x)\alpha = \alpha x\alpha = \rho, with 0 \neq \alpha \in \mathbf{O}, \tag{1}$$

secondly present a method which is reduce this octonionic equation to an equation with the left and right coefficients to a real system of eight equations to find the solutions of this equation, and finally reach the solutions of the linear octonionic equation from this real system. We end the paper with one example to illustrate the results of this paper.

2 Some Preliminaries

In this section, we introduce some definitions, notations and basic properties which we need to use in the presentations and proofs of our main results.

Let $\mathbf{O}$ be the octonion algebra over the real number field $\mathbb{R}$. In that case, $\mathbf{O}$ is an eight-dimensional non-associative but alternative division algebra over its center field $\mathbb{R}$ and the canonical basis of $\mathbf{O}$ is

$$e_0 = 1,\ e_1 = i,\ e_2 = j,\ e_3 = k,\ e_4 = e,\ e_5 = ie,\ e_6 = je,\ e_7 = ke. \tag{2}$$

The multiplication rules for the basis of $\mathbf{O}$ are listed in the following table

$\times$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

$Table : The multiplication table for the basis of \mathbf{O}.$

By the equation (2), all elements of $\mathbf{O}$ take the form

$$a = a_0e_0 + a_1e_1 + a_2e_2 + a_3\ e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7,$$

with real coefficients $\{a_i\}$. Also, the octonion a can be written in the form $a = a_0e_0 + \overrightarrow{a}$, where $\overrightarrow{a} = a_1e_1 + a_2e_2 + a_3\ e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$. The conjugate of a is defined by

$$\overline{a} = a_0e_0 - a_1e_1 - a_2e_2 - a_3\ e_3 - a_4e_4 - a_5e_5 - a_6e_6 - a_7e_7$$

and the octonions a and b satisfy $\overline{(ab)} = \overline{b}\overline{a}$.

Also, it is well known by the Cayley-Dickson process that any $a \in \mathbf{O}$ can be written as

$$a = a' + a''e,$$

where $a', a'' \in \mathbf{H}$ and

$$\mathbf{H} = \left\{a = a_0 + a_1i + a_2j + a_3k : i^2 = j^2 = k^2 = -1,\ ijk = -1,\ a_s \in \mathbb{R},\ s = 0,1,2,3\right\}$$

the real quaternion division algebra. The addition and multiplication for any $a = a' + a''e,\ b = b' + b''e \in \mathbf{O}$ are defined respectively by

$$\begin{aligned} a+b &= (a' + a''e) + (b' + b''e) \\ &= (a' + b') + (a'' + b'')\,e \end{aligned}$$

and

$$\begin{aligned} ab &= (a' + a''e)\,(b' + b''e) \\ &= \left(a'b' - \overline{b''}a''\right)\left(b''a' + a''\overline{b'}\right)e, \end{aligned} \tag{3}$$

where $\overline{b'}$ and $\overline{b''}$ denote the conjugates of the quaternions b' and b''.

The real and the imaginary parts of a are given by

$$\frac{a+\overline{a}}{2} = a_0 e_0$$

and

$$\frac{a-\overline{a}}{2} = \sum_{k=1}^{7} a_k e_k,$$

respectively.

The product of an octonion with its conjugate, $\overline{a}a = a\overline{a}$, is always a non-negative real number:

$$\overline{a}a = \sum_{k=0}^{7} a_k^2.$$

Using this, the norm of an octonion can be defined as

$$\|a\| = \sqrt{\overline{a}a}.$$

This norm agrees with the standard Euclidean norm on $\mathbb{R}^8$.

The existence of a norm on $\mathbf{O}$ implies the existence of inverses for every nonzero element of $\mathbf{O}$. The inverse of $a \neq 0$ is given by

$$a^{-1} = \frac{\overline{a}}{\|a\|^2}$$

and it satisfies $a^{-1}a = aa^{-1} = 1$.

For $k \in \mathbb{R}$, the octonion $k.a$ is the octonion

$$k.a = \sum_{i=0}^{7} (ka_i)\, e_i.$$

Finally, the scalar product of the octonions $a, b \in \mathbf{O}$ is

$$\langle a, b\rangle = \sum_{i=0}^{7} a_i b_i. \tag{4}$$

3 Main Results

In this section, we first give the following lemma, and then using this lemma and the properties, which we give for the octonions in the previous section, we then present the solution of the octonionic equation (1) by the formula (10) in Theorem 3. Furthermore, we give one example to illustrate the results of this paper.

For any octonions $\alpha, \beta \in \mathbf{O}$, the following equality is true:

$$\alpha\beta = \beta\alpha - 2\overrightarrow{\beta}\,\overrightarrow{\alpha} - 2\left\langle \overrightarrow{\beta}, \overrightarrow{\alpha} \right\rangle. \tag{5}$$

Proof. Let $\alpha = \sum_{k=0}^{7} \alpha_k e_k$, $\beta = \sum_{k=0}^{7} \beta_k e_k \in \mathbf{O}$. Then, we have

$$\begin{aligned}
\alpha\beta &= (\sum_{k=0}^{7} \alpha_k e_k)(\sum_{k=0}^{7} \beta_k e_k) \\
&= (\beta_0\alpha_0 - \beta_1\alpha_1 - \beta_2\alpha_2 - \beta_3\alpha_3 - \beta_4\alpha_4 - \beta_5\alpha_5 - \beta_6\alpha_6 - \beta_7\alpha_7)\, e_0 \qquad (6) \\
&+ (\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_5\beta_4 + \alpha_4\beta_5 - \alpha_6\beta_7 + \alpha_7\beta_6)\, e_1 \\
&+ (\alpha_2\beta_0 + \alpha_3\beta_1 + \alpha_0\beta_2 - \alpha_1\beta_3 - \alpha_6\beta_4 - \alpha_7\beta_5 + \alpha_4\beta_6 + \alpha_5\beta_7)\, e_2 \\
&+ (\alpha_0\beta_3 + \alpha_3\beta_0 - \alpha_2\beta_1 + \alpha_1\beta_2 - \alpha_7\beta_4 + \alpha_6\beta_5 - \alpha_5\beta_6 + \alpha_4\beta_7)\, e_3 \\
&+ (\alpha_4\beta_0 + \alpha_5\beta_1 + \alpha_6\beta_2 + \alpha_7\beta_3 + \alpha_0\beta_4 - \alpha_1\beta_5 - \alpha_2\beta_6 - \alpha_3\beta_7)\, e_4 \\
&+ (\alpha_5\beta_0 - \alpha_4\beta_1 + \alpha_7\beta_2 - \alpha_6\beta_3 + \alpha_1\beta_4 + \alpha_0\beta_5 + \alpha_3\beta_6 - \alpha_2\beta_7)\, e_5 \\
&+ (\alpha_0\beta_6 + \alpha_6\beta_0 + \alpha_5\beta_3 - \alpha_3\beta_5 + \alpha_2\beta_4 - \alpha_4\beta_2 + \alpha_1\beta_7 - \alpha_7\beta_1)\, e_6 \\
&+ (\alpha_0\beta_7 + \alpha_7\beta_0 - \alpha_4\beta_3 + \alpha_3\beta_4 - \alpha_5\beta_2 + \alpha_2\beta_5 + \alpha_6\beta_1 - \alpha_1\beta_6)\, e_7,
\end{aligned}$$

$$\begin{aligned}
\beta\alpha &= (\sum_{k=0}^{7} \beta_k e_k)(\sum_{k=0}^{7} \alpha_k e_k) \\
&= (\beta_0\alpha_0 - \beta_1\alpha_1 - \beta_2\alpha_2 - \beta_3\alpha_3 - \beta_4\alpha_4 - \beta_5\alpha_5 - \beta_6\alpha_6 - \beta_7\alpha_7)\, e_0 \qquad (7) \\
&+ (\beta_1\alpha_0 + \beta_0\alpha_1 - \beta_3\alpha_2 + \beta_2\alpha_3 + \beta_4\alpha_5 - \beta_5\alpha_4 + \beta_7\alpha_6 - \beta_6\alpha_7)\, e_1 \\
&+ (\beta_0\alpha_2 - \beta_1\alpha_3 + \beta_2\alpha_0 + \beta_3\alpha_1 + \beta_4\alpha_6 - \beta_5\alpha_7 - \beta_6\alpha_4 - \beta_7\alpha_5)\, e_2 \\
&+ (\beta_3\alpha_0 + \beta_0\alpha_3 + \beta_1\alpha_2 - \beta_2\alpha_1 + \beta_4\alpha_7 - \beta_5\alpha_6 + \beta_6\alpha_5 - \beta_7\alpha_4)\, e_3 \\
&+ (\beta_0\alpha_4 - \beta_1\alpha_5 - \beta_2\alpha_6 - \beta_3\alpha_7 + \beta_4\alpha_0 + \beta_5\alpha_1 + \beta_6\alpha_2 + \beta_7\alpha_3)\, e_4 \\
&+ (\beta_0\alpha_5 + \beta_1\alpha_4 - \beta_2\alpha_7 + \beta_3\alpha_6 - \beta_4\alpha_1 + \beta_5\alpha_0 - \beta_6\alpha_3 + \beta_7\alpha_2)\, e_5 \\
&+ (\beta_0\alpha_6 + \beta_1\alpha_7 + \beta_2\alpha_4 - \beta_3\alpha_5 - \beta_4\alpha_2 + \beta_5\alpha_3 + \beta_6\alpha_0 - \beta_7\alpha_1)\, e_6 \\
&+ (\beta_0\alpha_7 - \beta_1\alpha_6 + \beta_2\alpha_5 + \beta_3\alpha_4 - \beta_4\alpha_3 - \beta_5\alpha_2 + \beta_6\alpha_1 + \beta_7\alpha_0)\, e_7,
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\beta}\,\overrightarrow{\alpha} &= (\sum_{k=1}^{7} \beta_k e_k)(\sum_{k=1}^{7} \alpha_k e_k) \\
&= (-\beta_1\alpha_1 - \beta_2\alpha_2 - \beta_3\alpha_3 - \beta_4\alpha_4 - \beta_5\alpha_5 - \beta_6\alpha_6 - \beta_7\alpha_7)\, e_0 \qquad (8) \\
&+ (-\beta_3\alpha_2 + \beta_2\alpha_3 + \beta_4\alpha_5 - \beta_5\alpha_4 + \beta_7\alpha_6 - \beta_6\alpha_7)\, e_1 \\
&+ (\beta_1\alpha_3 + \beta_3\alpha_1 + \beta_4\alpha_6 + \beta_5\alpha_7 - \beta_7\alpha_5 - \beta_6\alpha_4)\, e_2 \\
&+ (\beta_1\alpha_2 - \beta_2\alpha_1 + \beta_6\alpha_5 - \beta_5\alpha_6 - \beta_7\alpha_4 + \beta_4\alpha_7)\, e_3
\end{aligned}$$

$$\begin{aligned}
&+(-\beta_3\alpha_7+\beta_7\alpha_3-\beta_1\alpha_5+\beta_5\alpha_1-\beta_2\alpha_6+\beta_6\alpha_2)\,e_4\\
&+(\beta_3\alpha_6-\beta_6\alpha_3-\beta_4\alpha_1+\beta_1\alpha_4+\beta_7\alpha_2-\beta_2\alpha_7)\,e_5\\
&+(-\beta_3\alpha_5+\beta_5\alpha_3-\beta_4\alpha_2+\beta_2\alpha_4-\beta_7\alpha_1+\beta_1\alpha_7)\,e_6\\
&+(\beta_3\alpha_4-\beta_4\alpha_3+\beta_2\alpha_5-\beta_5\alpha_2-\beta_1\alpha_6+\beta_6\alpha_1)\,e_7,
\end{aligned}$$

and

$$\left\langle \overrightarrow{\beta}, \overrightarrow{\alpha} \right\rangle = \beta_1\alpha_1+\beta_2\alpha_2+\beta_3\alpha_3+\beta_4\alpha_4+\beta_5\alpha_5+\beta_6\alpha_6+\beta_7\alpha_7. \tag{9}$$

Thus, from $(6), (7), (8)$ and (9) we reach the result of the Lemma.

Let $\alpha =_{i=0}^{7} \alpha_i e_i \in \mathbf{O}-\{\mathbf{0}\}$ and $\rho =_{i=0}^{7} \rho_i e_i \in \mathbf{O}$. Then, the linear octonionic equation (1) has the unique solution such that

$$\begin{aligned}
x \;=\; &\varpi\,[(\mu_0\rho_0+2\alpha_0\ (\alpha_1\rho_1+\alpha_2\rho_2+\alpha_3\rho_3+\alpha_4\rho_4+\alpha_5\rho_5+\alpha_6\rho_6+\alpha_7\rho_7))\,e_0 \qquad (10)\\
&+(-2\,\alpha_0\alpha_1\rho_0+\mu_1\rho_1-2\,\alpha_1\alpha_2\rho_2-2\,\alpha_1\alpha_3\rho_3-2\,\alpha_1\alpha_4\rho_4-2\,\alpha_1\alpha_5\rho_5-2\,\alpha_1\alpha_6\rho_6-2\alpha_1\alpha_7\rho_7)\,e_1\\
&+(-2\,\alpha_0\alpha_2\rho_0-2\,\alpha_1\alpha_2\rho_1+\mu_2\rho_2-2\,\alpha_2\alpha_3\rho_3-2\,\alpha_2\alpha_4\rho_4-2\,\alpha_2\alpha_5\rho_5-2\,\alpha_2\alpha_6\rho_6-2\,\alpha_2\alpha_7\rho_7)\,e_2\\
&+(-2\,\alpha_0\alpha_3\rho_0-2\,\alpha_1\alpha_3\rho_1-2\,a_2a_3\rho_2+\mu_3\rho_3-2\,a_3a_4\rho_4-2\,a_3a_5\rho_5-2\,a_3a_6\rho_6-2\,a_3a_7\rho_7)\,e_3\\
&+(-2\,\alpha_0\alpha_4\rho_0-2\,\alpha_1\alpha_4\rho_1-2\,a_2a_4\rho_2-2\,a_3a_4\rho_3+\mu_4\rho_4-2\,a_4a_5\rho_5-2\,a_4a_6\rho_6-2\,a_4a_7\rho_7)\,e_4\\
&+(-2\,\alpha_0\alpha_5\rho_0-2\,\alpha_1\alpha_5\rho_1-2\,a_2a_5\rho_2-2\,a_3a_5\rho_3-2\,a_4a_5\rho_4+\mu_5\rho_5-2\,\alpha_5\alpha_6\rho_6-2\,a_5a_7\rho_7)\,e_5\\
&+(-2\,\alpha_0\alpha_6\rho_0-2\,\alpha_1\alpha_6\rho_1-2\,a_2a_6\rho_2-2\,a_3a_6\rho_3-2\,a_4a_6\rho_4-2\,\alpha_5\alpha_6\rho_5+\mu_6\rho_6-2\,a_6a_7\rho_7)\,e_6\\
&+(-2\,\alpha_0\alpha_7\rho_0-2\,\alpha_1\alpha_7\rho_1-2\,a_2a_7\rho_2-2\,a_3a_7\rho_3-2\,a_4a_7\rho_4-2\,a_5a_7\rho_5-2\,a_6a_7\rho_6+\mu_7\rho_7)\,e_7].
\end{aligned}$$

where $\varpi = (_{i=0}^{7}\alpha_i^2)^{-\frac{1}{2}}$, $\mu_0 = -_{i=0}^{7}\left(\alpha_i^2+2\alpha_0^2\right)$, $\mu_k =_{i=0}^{7}\left(\alpha_i^2-2\alpha_k^2\right)$, $k = 1, ..., 7$.

Proof. We now will deal with reducing the octonionic equation (1) to an equation with the left and right coefficients to a real system of eight equations. From Lemma 3, the term $\alpha(x\alpha)$ in the left side of the equation (1) is obtained such that

$$\begin{aligned}
\alpha(x\alpha) &= \alpha\left(\alpha x - 2\overrightarrow{\alpha}\,\overrightarrow{x} - 2\left\langle \overrightarrow{\alpha}, \overrightarrow{x}\right\rangle\right)\\
&= \alpha\,(\alpha x) - 2\alpha\left(\overrightarrow{\alpha}\,\overrightarrow{x}\right) - 2\alpha\left\langle \overrightarrow{\alpha}, \overrightarrow{x}\right\rangle\\
&= \alpha\left[\alpha\left(x_0e_0+\overrightarrow{x}\right)\right] - \alpha\left(\overrightarrow{\alpha}\,\overrightarrow{x}\right) - 2\alpha\left\langle \overrightarrow{\alpha}, \overrightarrow{x}\right\rangle\\
&= \alpha\,(\alpha x_0e_0) + \alpha\left(\alpha\overrightarrow{x}\right) - 2\alpha\left(\overrightarrow{\alpha}\,\overrightarrow{x}\right) - 2\alpha\left\langle \overrightarrow{\alpha}, \overrightarrow{x}\right\rangle\\
&= (\alpha\alpha)\,x_0e_0 + \alpha\left[\left(\alpha_0e_0+\overrightarrow{\alpha}\right)\overrightarrow{x}\right] - 2\alpha\left(\overrightarrow{\alpha}\,\overrightarrow{x}\right) - 2\alpha\left\langle \overrightarrow{\alpha}, \overrightarrow{x}\right\rangle\\
&= \alpha^2x_0 + \alpha\,\overrightarrow{x}\,\alpha_0 - \alpha\left(\overrightarrow{\alpha}\,\overrightarrow{x}\right) - 2\alpha\left\langle \overrightarrow{\alpha}, \overrightarrow{x}\right\rangle. \qquad (11)
\end{aligned}$$

From the equalities $(3), (4)$ and (11) we have

$$\begin{aligned}
\alpha(x\alpha) \;=\; &\left[\left(-_{i=0}^{7}\alpha_i^2\right)x_0 + 2\alpha_0^2x_0 - 2\alpha_0{}_{i=1}^{7}\alpha_ix_i\right]e_0\\
&+\left[\left(_{i=0}^{7}\alpha_i^2\right)x_1 + 2\alpha_0\alpha_1x_0 - 2\alpha_1{}_{i=1}^{7}\alpha_ix_i\right]e_1\\
&+\left[\left(_{i=0}^{7}\alpha_i^2\right)x_2 + 2\alpha_0\alpha_2x_0 - 2\alpha_2{}_{i=1}^{7}\alpha_ix_i\right]e_2\\
&+\left[\left(_{i=0}^{7}\alpha_i^2\right)x_3 + 2\alpha_0\alpha_3x_0 - 2\alpha_3{}_{i=1}^{7}\alpha_ix_i\right]e_3
\end{aligned}$$

$$
\begin{aligned}
&+\left[\left({}_{i=0}^{7}\alpha_i^2\right)x_4+2\alpha_0\alpha_4x_0-2\alpha_4{}_{i=1}^{7}\alpha_ix_i\right]e_4\\
&+\left[\left({}_{i=0}^{7}\alpha_i^2\right)x_5+2\alpha_0\alpha_5x_0-2\alpha_5{}_{i=1}^{7}\alpha_ix_i\right]e_5\\
&+\left[\left({}_{i=0}^{7}\alpha_i^2\right)x_6+2\alpha_0\alpha_6x_0-2\alpha_6{}_{i=1}^{7}\alpha_ix_i\right]e_6\\
&+\left[\left({}_{i=0}^{7}\alpha_i^2\right)x_7+2\alpha_0\alpha_7x_0-2\alpha_7{}_{i=1}^{7}\alpha_ix_i\right]e_7,
\end{aligned}
$$

where $\alpha={}_{i=0}^{7}\,\alpha_ie_i$. Therefore, we obtain that the equation (1) is equivalent to the real system

$$
\begin{aligned}
\left(-{}_{i=0}^{7}\alpha_i^2\right)x_0+2\alpha_0^2x_0-2\alpha_0{}_{i=1}^{7}\alpha_ix_i &= \rho_0\\
\left({}_{i=0}^{7}\alpha_i^2\right)x_1+2\alpha_0\alpha_1x_0-2\alpha_1{}_{i=1}^{7}\alpha_ix_i &= \rho_1\\
\left({}_{i=0}^{7}\alpha_i^2\right)x_2+2\alpha_0\alpha_2x_0-2\alpha_2{}_{i=1}^{7}\alpha_ix_i &= \rho_2\\
\left({}_{i=0}^{7}\alpha_i^2\right)x_3+2\alpha_0\alpha_3x_0-2\alpha_3{}_{i=1}^{7}\alpha_ix_i &= \rho_3\\
\left({}_{i=0}^{7}\alpha_i^2\right)x_4+2\alpha_0\alpha_4x_0-2\alpha_4{}_{i=1}^{7}\alpha_ix_i &= \rho_4\\
\left({}_{i=0}^{7}\alpha_i^2\right)x_5+2\alpha_0\alpha_5x_0-2\alpha_5{}_{i=1}^{7}\alpha_ix_i &= \rho_5\\
\left({}_{i=0}^{7}\alpha_i^2\right)x_6+2\alpha_0\alpha_6x_0-2\alpha_6{}_{i=1}^{7}\alpha_ix_i &= \rho_6\\
\left({}_{i=0}^{7}\alpha_i^2\right)x_7+2\alpha_0\alpha_7x_0-2\alpha_7{}_{i=1}^{7}\alpha_ix_i &= \rho_7,
\end{aligned}
$$

or in matrix-vector form

$$Ax=b, \tag{12}$$

where A is the form

$$
A=\begin{bmatrix}
\mu_0 & -2\,\alpha_0\alpha_1 & -2\,\alpha_0\alpha_2 & -2\,\alpha_0\alpha_3 & -2\,\alpha_0\alpha_4 & -2\,\alpha_0\alpha_5 & -2\,\alpha_0\alpha_6 & -2\,\alpha_0\alpha_7\\
2\,\alpha_0\alpha_1 & \mu_1 & -2\,\alpha_1\alpha_2 & -2\,\alpha_1\alpha_3 & -2\,\alpha_1\alpha_4 & -2\,\alpha_1\alpha_5 & -2\,\alpha_1\alpha_6 & -2\,\alpha_1\alpha_7\\
2\,\alpha_0\alpha_2 & -2\,\alpha_1\alpha_2 & \mu_2 & -2\,\alpha_2\alpha_3 & -2\,\alpha_2\alpha_4 & -2\,\alpha_2\alpha_5 & -2\,\alpha_2\alpha_6 & -2\,\alpha_2\alpha_7\\
2\,\alpha_0\alpha_3 & -2\,\alpha_1\alpha_3 & -2\,\alpha_2\alpha_3 & \mu_3 & -2\,\alpha_3\alpha_4 & -2\,\alpha_3\alpha_5 & -2\,\alpha_3\alpha_6 & -2\,\alpha_3\alpha_7\\
2\,\alpha_0\alpha_4 & -2\,\alpha_1\alpha_4 & -2\,\alpha_2\alpha_4 & -2\,\alpha_3\alpha_4 & \mu_4 & -2\,\alpha_4\alpha_5 & -2\,\alpha_4\alpha_6 & -2\,\alpha_4\alpha_7\\
2\,\alpha_0\alpha_5 & -2\,\alpha_1\alpha_5 & -2\,\alpha_2\alpha_5 & -2\,\alpha_3\alpha_5 & -2\,\alpha_4\alpha_5 & \mu_5 & -2\,\alpha_5\alpha_6 & -2\,\alpha_5\alpha_7\\
2\,\alpha_0\alpha_6 & -2\,\alpha_1\alpha_6 & -2\,\alpha_2\alpha_6 & -2\,\alpha_3\alpha_6 & -2\,\alpha_4\alpha_6 & -2\,\alpha_5\alpha_6 & \mu_6 & -2\,a_6a_7\\
2\,\alpha_0\alpha_7 & -2\,\alpha_1\alpha_7 & -2\,\alpha_2\alpha_7 & -2\,\alpha_3\alpha_7 & -2\,\alpha_4\alpha_7 & -2\,\alpha_5\alpha_7 & -2\,a_6a_7 & \mu_7
\end{bmatrix},
$$

the octonion $x={}_{i=0}^{7}\,x_ie_i$ is denoted $x=[x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7]^T$, called the vector representation of x and $b=[\rho_0,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7]^T$. If we first compute the determinant of the coefficient matrix A, we get

$$|A|=\left({}_{i=0}^{7}\alpha_i^2\right)^8$$

and because $0\neq\alpha\in\mathbf{O}$, $\det A\neq 0$. Thus, A is invertible. Now, let be

$\varpi = ({}_{i=0}^{7}\alpha_i^2)^{-\frac{1}{2}}$. Then $\varpi = |A|^{-\frac{1}{4}}$. Hence, for

$$A^{-1} = \varpi \begin{bmatrix} \mu_0 & 2\,\alpha_0\alpha_1 & 2\,\alpha_0\alpha_2 & 2\,\alpha_0\alpha_3 & 2\,\alpha_0\alpha_4 & 2\,\alpha_0\alpha_5 & 2\,\alpha_0\alpha_6 & 2\,\alpha_0\alpha_7 \\ -2\,\alpha_0\alpha_1 & \mu_1 & -2\,\alpha_1\alpha_2 & -2\,\alpha_1\alpha_3 & -2\,\alpha_1\alpha_4 & -2\,\alpha_1\alpha_5 & -2\,\alpha_1\alpha_6 & -2\,\alpha_1\alpha_7 \\ -2\,\alpha_0\alpha_2 & -2\,\alpha_1\alpha_2 & \mu_2 & -2\,\alpha_2\alpha_3 & -2\,\alpha_2\alpha_4 & -2\,\alpha_2\alpha_5 & -2\,\alpha_2\alpha_6 & -2\,\alpha_2\alpha_7 \\ -2\,\alpha_0\alpha_3 & -2\,\alpha_1\alpha_3 & -2\,\alpha_2\alpha_3 & \mu_3 & -2\,\alpha_3\alpha_4 & -2\,\alpha_3\alpha_5 & -2\,\alpha_3\alpha_6 & -2\,\alpha_3\alpha_7 \\ -2\,\alpha_0\alpha_4 & -2\,\alpha_1\alpha_4 & -2\,\alpha_2\alpha_4 & -2\,\alpha_3\alpha_4 & \mu_4 & -2\,\alpha_4\alpha_5 & -2\,\alpha_4\alpha_6 & -2\,\alpha_4\alpha_7 \\ -2\,\alpha_0\alpha_5 & -2\,\alpha_1\alpha_5 & -2\,\alpha_2\alpha_5 & -2\,\alpha_3\alpha_5 & -2\,\alpha_4\alpha_5 & \mu_5 & -2\,\alpha_5\alpha_6 & -2\,\alpha_5\alpha_7 \\ -2\,\alpha_0\alpha_6 & -2\,\alpha_1\alpha_6 & -2\,\alpha_2\alpha_6 & -2\,\alpha_3\alpha_6 & -2\,\alpha_4\alpha_6 & -2\,\alpha_5\alpha_6 & \mu_6 & -2\,a_6a_7 \\ -2\,\alpha_0\alpha_7 & -2\,\alpha_1\alpha_7 & -2\,\alpha_2\alpha_7 & -2\,\alpha_3\alpha_7 & -2\,\alpha_4\alpha_7 & -2\,\alpha_5\alpha_7 & -2\,a_6a_7 & \mu_7 \end{bmatrix},$$

the equation (12) has the unique solution $x = A^{-1}b$. If we multiply $A^{-1}b$, we get the vector representation of the octonion x such that

$$x = \varpi \begin{bmatrix} \mu_0\rho_0+2\,\alpha_0\alpha_1\rho_1+2\,\alpha_0\alpha_2\rho_2+2\,\alpha_0\alpha_3\rho_3+2\,\alpha_0\alpha_4\rho_4+2\,\alpha_0\alpha_5\rho_5+2\,\alpha_0\alpha_6\rho_6+2\,\alpha_0\alpha_7\rho_7 \\ -2\,\alpha_0\alpha_1\rho_0+\mu_1\rho_1-2\,\alpha_1\alpha_2\rho_2-2\,\alpha_1\alpha_3\rho_3-2\,\alpha_1\alpha_4\rho_4-2\,\alpha_1\alpha_5\rho_5-2\,\alpha_1\alpha_6\rho_6-2\alpha_1\alpha_7\rho_7 \\ -2\,\alpha_0\alpha_2\rho_0-2\,\alpha_1\alpha_2\rho_1+\mu_2\rho_2-2\,\alpha_2\alpha_3\rho_3-2\,\alpha_2\alpha_4\rho_4-2\,\alpha_2\alpha_5\rho_5-2\,\alpha_2\alpha_6\rho_6-2\,\alpha_2\alpha_7\rho_7 \\ -2\,\alpha_0\alpha_3\rho_0-2\,\alpha_1\alpha_3\rho_1-2\,a_2a_3\rho_2+\mu_3\rho_3-2\,a_3a_4\rho_4-2\,a_3a_5\rho_5-2\,a_3a_6\rho_6-2\,a_3a_7\rho_7 \\ -2\,\alpha_0\alpha_4\rho_0-2\,\alpha_1\alpha_4\rho_1-2\,a_2a_4\rho_2-2\,a_3a_4\rho_3+\mu_4\rho_4-2\,a_4a_5\rho_5-2\,a_4a_6\rho_6-2\,a_4a_7\rho_7 \\ -2\,\alpha_0\alpha_5\rho_0-2\,\alpha_1\alpha_5\rho_1-2\,a_2a_5\rho_2-2\,a_3a_5\rho_3-2\,a_4a_5\rho_4+\mu_5\rho_5-2\,\alpha_5\alpha_6\rho_6-2\,a_5a_7\rho_7 \\ -2\,\alpha_0\alpha_6\rho_0-2\,\alpha_1\alpha_6\rho_1-2\,a_2a_6\rho_2-2\,a_3a_6\rho_3-2\,a_4a_6\rho_4-2\,\alpha_5\alpha_6\rho_5+\mu_6\rho_6-2\,a_6a_7\rho_7 \\ -2\,\alpha_0\alpha_7\rho_0-2\,\alpha_1\alpha_7\rho_1-2\,a_2a_7\rho_2-2\,a_3a_7\rho_3-2\,a_4a_7\rho_4-2\,a_5a_7\rho_5-2\,a_6a_7\rho_6+\mu_7\rho_7 \end{bmatrix}.$$

Therefore, the solution x of the equation (1) is

$$\begin{aligned} x \;=\; & \varpi\,[(\mu_0\rho_0+2\alpha_0\;(\alpha_1\rho_1+\alpha_2\rho_2+\alpha_3\rho_3+\alpha_4\rho_4+\alpha_5\rho_5+\alpha_6\rho_6+\alpha_7\rho_7))\,e_0 \\ & +(-2\,\alpha_0\alpha_1\rho_0+\mu_1\rho_1-2\,\alpha_1\alpha_2\rho_2-2\,\alpha_1\alpha_3\rho_3-2\,\alpha_1\alpha_4\rho_4-2\,\alpha_1\alpha_5\rho_5-2\,\alpha_1\alpha_6\rho_6-2\alpha_1\alpha_7\rho_7)\,e_1 \\ & +(-2\,\alpha_0\alpha_2\rho_0-2\,\alpha_1\alpha_2\rho_1+\mu_2\rho_2-2\,\alpha_2\alpha_3\rho_3-2\,\alpha_2\alpha_4\rho_4-2\,\alpha_2\alpha_5\rho_5-2\,\alpha_2\alpha_6\rho_6-2\,\alpha_2\alpha_7\rho_7)\,e_2 \\ & +(-2\,\alpha_0\alpha_3\rho_0-2\,\alpha_1\alpha_3\rho_1-2\,a_2a_3\rho_2+\mu_3\rho_3-2\,a_3a_4\rho_4-2\,a_3a_5\rho_5-2\,a_3a_6\rho_6-2\,a_3a_7\rho_7)\,e_3 \\ & +(-2\,\alpha_0\alpha_4\rho_0-2\,\alpha_1\alpha_4\rho_1-2\,a_2a_4\rho_2-2\,a_3a_4\rho_3+\mu_4\rho_4-2\,a_4a_5\rho_5-2\,a_4a_6\rho_6-2\,a_4a_7\rho_7)\,e_4 \\ & +(-2\,\alpha_0\alpha_5\rho_0-2\,\alpha_1\alpha_5\rho_1-2\,a_2a_5\rho_2-2\,a_3a_5\rho_3-2\,a_4a_5\rho_4+\mu_5\rho_5-2\,\alpha_5\alpha_6\rho_6-2\,a_5a_7\rho_7)\,e_5 \\ & +(-2\,\alpha_0\alpha_6\rho_0-2\,\alpha_1\alpha_6\rho_1-2\,a_2a_6\rho_2-2\,a_3a_6\rho_3-2\,a_4a_6\rho_4-2\,\alpha_5\alpha_6\rho_5+\mu_6\rho_6-2\,a_6a_7\rho_7)\,e_6 \\ & +(-2\,\alpha_0\alpha_7\rho_0-2\,\alpha_1\alpha_7\rho_1-2\,a_2a_7\rho_2-2\,a_3a_7\rho_3-2\,a_4a_7\rho_4-2\,a_5a_7\rho_5-2\,a_6a_7\rho_6+\mu_7\rho_7)\,e_7]\,. \end{aligned}$$

Next we will consider the following example to illustrate the result of Theorem 3.

Consider the following equation:

$$(e_0+4e_1-e_2+3e_4+e_5-7e_6+5e_7)\,x\,(e_0+4e_1-e_2+3e_4+e_5-7e_6+5e_7) = -e_0+2e_1+3e_2+5e_3-3e_4-e_6$$

in $\mathbf{O}$. For $\alpha = e_0 + 4e_1 - e_2 + 3e_4 + e_5 - 7e_6 + 5e_7$ and $\rho = -e_0 + 2e_1 + 3e_2 + 5e_3 - 3e_4 - e_6$, this equation is of the form

$$\alpha(x\alpha) = (\alpha x)\alpha = \alpha x \alpha = \rho.$$

For $\alpha_0 = 1,\ \alpha_1 = 4,\ \alpha_2 = -1,\ \alpha_3 = 0,\ \alpha_4 = 3,\ \alpha_5 = 1,\ \alpha_6 = -7,\ \alpha_7 = 5$ and $\rho_0 = -1,\ \rho_1 = 2,\ \rho_2 = 3,\ \rho_3 = 5,\ \rho_4 = -3,\ \rho_5 = 0,\ \rho_6 = -1,\ \rho_7 = 0$, from the formula (10) given for x in Theorem 3, we obtain the solution x as

$$x = 53/5202e_0+47/2601e_1+155/5202e_2+5/102e_3-53/1734e_4-1/2601e_5-37/5202e_6-5/2601e_7.$$

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References

[1] Y.H. Au-Yeung, An explicit solution for the quaternionic equation $x^2+bx+xc+d=0$, *Southeast Asian Bulletin of Mathematics*, 26(2002), 717–724.

[2] C. Bolat and A. Ipek, On the solutions of linear matrix quaternionic equations and their systems, Submitted.

[3] C. Flaut, Some equation in algebras obtained by Cayley–Dickson process, *An. St. Univ. Ovidius Constanta*, 9(2) (2001), 45–68.

[4] J. Helmstetter, The quaternionic equation $ax+xb=c$, *Adv. Appl. Clifford Algebras*, DOI 10.1007/s00006-012-0322-z.

[5] R.E. Johnson, On the equation $x\alpha = x+\beta$ over an algebraic division ring, *Bulletin of the American Mathematical Society*, 50(1944), 202-207.

[6] R.M. Porter, Quaternionic linear and quadratic equations, *J. Nat. Geom.*, 11(2) (1997), 101–106.

[7] S.V. Shpakivskyi, Linear quaternionic equations and their systems, *Adv. Appl. Clifford Algebras*, 21(2011), 637-645.

[8] Y. Tian, Similarity and consimilarity of elements in the real Cayley–Dickson algebras, *Adv. Appl. Clifford Alg.*, 9(1) (1999), 61–76.