Gen. Math. Notes, Vol. 21, No. 1, March 2014, pp.10-26
ISSN 2219-7184; Copyright ©ICSRS Publication, 2014
www.i-csrs.org
Available free online at http://www.geman.in

Generalized c-Distance and a Common Fixed Point Theorem in Cone Metric Spaces

Sushanta Kumar Mohanta[1] and Rima Maitra[2]

[1,2]Department of Mathematics, West Bengal State University
Barasat, 24 Parganas (North), West Bengal
Kolkata 700126, India
[1]E-mail: smwbes@yahoo.in
[2]E-mail: rima.maitra.barik@gmail.com

(Received: 3-1-14 / Accepted: 12-2-14)

Abstract

The main purpose of this work is to obtain a common fixed point theorem for a pair of self mappings in cone metric spaces by using the concept of generalized c-distance. Our results will improve and supplement some results in the existing literature. Finally, an example is provided to show that the generalized c-distances form a bigger category than that of c-distances.

Keywords: *c-Distance, Generalized c-distance, Point of coincidence, Weakly compatible mappings, Common fixed point.*

1 Introduction

Over the past two decades a considerable amount of research work for the development of fixed point theory have executed by several authors. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a cone metric space initiated by Huang and Zhang[8]. After that a series of articles have been dedicated to the improvement of fixed point theory. In most of those articles, the authors used normality property of cones in their results. The idea of common fixed point was initially given by Junck[11]. In fact, they considered commuting mappings to obtain common fixed point. Afterwards, many generalizations of this common fixed point result were obtained by several mathematicians viz., Hadzic[9], Singh and Meade[20], Pathak[16], Yeh[23] etc.. Recently, Wang et. al.[22] introduced the concept of c-distance on a cone metric space, which is a cone version of the w-distance of Kada et.al.[12] and proved a common fixed point theorem. In this paper, we introduce the concept of generalized c-distance on a cone metric space and obtain sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings satisfying generalized contractive conditions. The cone under consideration is not assumed to be normal.

2 Preliminaries

In this section we need to recall some basic notations, definitions, and necessary results from existing literature. Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

$$\begin{array}{ll} (i) & P \text{ is closed, nonempty and } P \neq \{\theta\}; \\ (ii) & a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \ \Rightarrow\ ax + by \in P; \\ (iii) & P \cap (-P) = \{\theta\}. \end{array}$$

For any cone $P \subseteq E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq k\,\|y\|. \tag{1}$$

The least positive number satisfying the above inequality is called the normal constant of P.

Definition 2.1 *[8] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

(i) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.2 *[8] Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n). We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.*

Definition 2.3 *[8] Let (X, d) be a cone metric space, (x_n) be a sequence in X. If for any $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X.*

Definition 2.4 *[8] Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.*

Here we point to some elementary results of [8].
Let (X, d) be a cone metric space, P a normal cone, $x \in X$ and (x_n) a sequence in X. Then

(i) (x_n) converges to x if and only if $d(x_n, x) \to \theta$(Lemma 1).

(ii) Limit point of every sequence is unique (Lemma 2).

(iii) Every convergent sequence is Cauchy (Lemma 3).

(iv) (x_n) is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$ (Lemma 4).

(v) If $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$ (Lemma 5).

Lemma 2.5 *[19] Let E be a real Banach space with a cone P. Then*
(i) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(ii) If $a \preceq b$ and $b \ll c$, then $a \ll c$.

Lemma 2.6 *[8] Let E be a real Banach space with cone P. Then one has the following.*
(i) If $\theta \ll c$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.
(ii) If $a_n,\ b_n$ are sequences in E such that $a_n \to a,\ b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.

Proposition 2.7 *[10] If E is a real Banach space with cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$ then $a = \theta$.*

Definition 2.8 *[2] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.*

Definition 2.9 *[11] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:*

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.10 *[2] Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

Definition 2.11 *[22] Let (X, d) be a cone metric space. Then a mapping $q : X \times X \to E$ is called a c-distance on X if the following are satisfied :*

(i) *$\theta \preceq q(x, y)$ for all x, $y \in X$;*

(ii) *$q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;*

(iii) *for all $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$ and all $n \geq 1$, then $q(x, y) \preceq u$ whenever (y_n) is a sequence in X converging to a point $y \in X$;*

(iv) *for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.*

Example 2.12 *[22] Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.*

Example 2.13 *[22] Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(u, y)$ for all $x, y \in X$, where $u \in X$ is a fixed point. Then q is a c-distance.*

Example 2.14 *[22] Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) =\mid x - y \mid$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance.*

Remark 2.15 *[22]*(1)*$q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.*
(2) *$q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.*

Definition 2.16 *Let (X, d) be a cone metric space and $j \in \mathbb{N}$. A function $q : X \times X \to E$ is called a generalized c-distance of order j on X if the following conditions are satisfied:*

(q1) *$\theta \preceq q(x, y)$, for all $x, y \in X$;*

(q2) $q(x,z) \preceq \sum_{i=0}^{j} q(x_i, x_{i+1})$, *for all* $x, z \in X$ *and for all distinct points* $x_i \in X$, $i \in \{1,2,3,\cdots,j\}$ *each of them different from* $x(=x_0)$ *and* $z(=x_{j+1})$;

(q3) *for all* $x \in X$, *if* $q(x, y_n) \preceq u$ *for some* $u = u_x \in P$ *and all* $n \geq 1$, *then* $q(x,y) \preceq u$ *whenever* (y_n) *is a sequence in* X *converging to a point* $y \in X$;

(q4) *for all* $c \in E$ *with* $\theta \ll c$, *there exists* $e \in E$ *with* $\theta \ll e$ *such that* $q(z,x) \ll e$ *and* $q(z,y) \ll e$ *imply* $d(x,y) \ll c$.

We see that every c-distance is a generalized c-distance of order 1. In fact, every c-distance may also be considered as a generalized c-distance of any order $j \in \mathbb{N}$. But the converse does not hold (see Example 3.8).

3 Main Result

In this section we always suppose that E is a real Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P. We begin with the following Lemma that will play a crucial role in the proof of the main theorem.

Lemma 3.1 *Let* (X,d) *be a cone metric space and* q *is a generalized c-distance of order* j *on* X. *Let* (x_n) *and* (y_n) *be sequences in* X. *Suppose that* (α_n) *and* (β_n) *are sequences in* P *converging to* θ, *and let* $x,y,z \in X$. *Then the following hold :*
(i) If $q(x_n, y_n) \preceq \alpha_n$ *and* $q(x_n, z) \preceq \beta_n$ *for any* $n \in \mathbb{N}$, *then* (y_n) *converges to* z;
(ii) If $q(x_n, y) \preceq \alpha_n$ *and* $q(x_n, z) \preceq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $q(x,y) = \theta$ *and* $q(x,z) = \theta$, *then* $y = z$;
(iii) If $q(x_n, x_m) \preceq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* (x_n) *is a Cauchy sequence.*

(i) Let $c \in E$ with $\theta \ll c$. Then there exists $\delta > 0$ such that $\| x \| < \delta$ implies $c - x \in int\,(P)$. Since (α_n) and (β_n) are converging to θ, there exists $n_0 \in \mathbb{N}$ such that $\| \alpha_n \| < \delta$ and $\| \beta_n \| < \delta$ for all $n > n_0$. Thus $c - \alpha_n \in int\,(P)$ and $c - \beta_n \in int\,(P)$ for all $n > n_0$ and so $\alpha_n \ll c$ and $\beta_n \ll c$ for all $n > n_0$. By hypothesis, $q(x_n, y_n) \preceq \alpha_n \ll c$ and $q(x_n, z) \preceq \beta_n \ll c$ for all $n > n_0$. Now from $(q4)$ with $e = c$ it follows that $d(y_n, z) \ll c$ for all $n > n_0$. Therefore (y_n) converges to z.
Clearly, (ii) is immediate from (i).
(iii) Let $c \in E$ with $\theta \ll c$. Then by the arguments similar to that used above,

there exists a positive integer n_0 such that $q(x_n, x_m) \preceq \alpha_n \ll c$ for all $m > n$ with $n > n_0$. This implies that $q(x_n, x_{n+1}) \preceq \alpha_n \ll c$ and $q(x_n, x_{m+1}) \preceq \alpha_n \ll c$ for all $m > n$ with $n > n_0$. From $(q4)$ with $e = c$ it follows that $d(x_{n+1}, x_{m+1}) \ll c$ for all $m > n$ with $n > n_0$. This shows that (x_n) is a Cauchy sequence in X.

Theorem 3.2 *Let (X, d) be a cone metric space and q is a generalized c-distance of order j on X. Suppose the mappings $f,\ g : X \to X$ satisfy*

$$q(fx, fy) \preceq a_1\, q(gx, gy) + a_2\, q(gx, fx) + a_3\, q(gy, fy) \tag{2}$$

for all $x, y \in X$, and $a_1,\ a_2,\ a_3 \in [0, 1)$ with $a_1 + a_2 + a_3 < 1$ and that

$$inf\, \{q(gx, y) + q(fx, y) + q(gx, fx) : x \in X\} \succ \theta \tag{3}$$

for all $y \in X$ with y is not a point of coincidence of f and g. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq g(X)$, there exists an $x_1 \in X$ such that $fx_0 = gx_1$. Proceeding in this way, a sequence (x_n) can be chosen such that $fx_n = gx_{n+1},\ n = 0,\ 1,\ 2,\ \cdots$.

For any natural number n, we have by using condition (2) that

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq a_1\, q(gx_{n-1}, gx_n) + a_2\, q(gx_{n-1}, fx_{n-1}) + a_3\, q(gx_n, fx_n) \\ &= a_1\, q(gx_{n-1}, gx_n) + a_2\, q(gx_{n-1}, gx_n) + a_3\, q(gx_n, gx_{n+1}). \end{aligned}$$

So, it must be the case that

$$q(gx_n, gx_{n+1}) \preceq r\, q(gx_{n-1}, gx_n) \tag{4}$$

where $r = \frac{a_1 + a_2}{1 - a_3} \in [0, 1)$.
By repeated application of (4), we obtain

$$q(gx_n, gx_{n+1}) \preceq r^n\, q(gx_0, gx_1). \tag{5}$$

We can suppose that $gx_n \neq gx_m$ for all distinct $n, m \in \{0,\ 1,\ 2,\ \cdots\}$. In fact, if $gx_n = gx_m$ for some $n, m \in \{0,\ 1,\ 2,\ \cdots\}$, $m \neq n$ then assuming $m > n$, we may write

$$gx_n = gx_{n+k},\ \ where\ k = m - n \geq 1. \tag{6}$$

Put $y = gx_n$. Then

$$\begin{aligned} q(y, gx_{n+1}) &= q(gx_n, gx_{n+1}) \\ &= q(gx_{n+k}, gx_{n+1}) \\ &= q(fx_{n+k-1}, fx_n) \\ &\preceq a_1\, q(gx_{n+k-1}, gx_n) + a_2\, q(gx_{n+k-1}, fx_{n+k-1}) + a_3\, q(gx_n, fx_n) \\ &= a_1\, q(gx_{n+k-1}, gx_{n+k}) + a_2\, q(gx_{n+k-1}, gx_{n+k}) + a_3\, q(y, gx_{n+1}) \end{aligned}$$

which gives that

$$q(y, gx_{n+1}) \preceq r\, q(gx_{n+k-1}, gx_{n+k}). \tag{7}$$

By repeated use of (4), we obtain from (7)

$$\begin{aligned} q(y, gx_{n+1}) &\preceq r\, q(gx_{n+k-1}, gx_{n+k}) \\ &\preceq r^2\, q(gx_{n+k-2}, gx_{n+k-1}) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\preceq r^k\, q(gx_n, gx_{n+1}) \\ &= r^k\, q(y, gx_{n+1}). \end{aligned}$$

Since $0 \leq r < 1$, by using Proposition 2.7 it follows that $q(y, gx_{n+1}) = \theta$ and so

$$q(gx_n, gx_{n+1}) = \theta. \tag{8}$$

Now using conditions (2), (4) and (8), we obtain

$$\begin{aligned} q(y, y) &= q(gx_n, gx_n) \\ &= q(gx_{n+k}, gx_{n+k}) \\ &= q(fx_{n+k-1}, fx_{n+k-1}) \\ &\preceq a_1 q(gx_{n+k-1}, gx_{n+k-1}) + (a_2 + a_3) q(gx_{n+k-1}, fx_{n+k-1}) \\ &= a_1 q(gx_{n+k-1}, gx_{n+k-1}) + (a_2 + a_3) q(gx_{n+k-1}, gx_{n+k}) \\ &\preceq a_1 q(gx_{n+k-1}, gx_{n+k-1}) + (a_2 + a_3) r^{k-1} q(gx_n, gx_{n+1}) \\ &= a_1 q(gx_{n+k-1}, gx_{n+k-1}). \end{aligned}$$

Similarly,

$$\begin{aligned} q(gx_{n+k-1}, gx_{n+k-1}) &\preceq a_1 q(gx_{n+k-2}, gx_{n+k-2}) + (a_2 + a_3) r^{k-2} q(gx_n, gx_{n+1}) \\ &= a_1 q(gx_{n+k-2}, gx_{n+k-2}). \end{aligned}$$

Proceeding in this way, we obtain at the k-th step that

$$q(gx_{n+1}, gx_{n+1}) \preceq a_1 q(gx_n, gx_n) + (a_2 + a_3) q(gx_n, gx_{n+1}) = a_1 q(gx_n, gx_n). \tag{9}$$

Thus,

$$q(y,y) \preceq a_1 q(gx_{n+k-1}, gx_{n+k-1}) \preceq \cdots \preceq a_1^k q(gx_n, gx_n) = a_1^k q(y,y). \tag{10}$$

Again by using Proposition 2.7, $q(y,y) = \theta$. Since $q(y, gx_{n+1}) = \theta$ and $q(y,y) = \theta$, by using Lemma 3.1(ii), we have $gx_{n+1} = y$. Therefore, $fx_n = y = gx_n$ which implies that y is a point of coincidence of f and g.
Thus in the sequel of the proof we can assume that $gx_n \neq gx_m$ for all distinct $n, m \in \{0,\ 1,\ 2,\ \cdots\}$.

Let $m, n \in \mathbb{N}$ with $m > n$. Taking $m = n + p$ where $p = 1,\ 2,\ 3,\ \cdots$ and using (2) and (5), we have

$$\begin{aligned}
q(gx_n, gx_m) &= q(fx_{n-1}, fx_{m-1}) \\
&\preceq a_1 q(gx_{n-1}, gx_{m-1}) + a_2 q(gx_{n-1}, fx_{n-1}) + a_3 q(gx_{m-1}, fx_{m-1}) \\
&= a_1 q(gx_{n-1}, gx_{m-1}) + a_2 q(gx_{n-1}, gx_n) + a_3 q(gx_{m-1}, gx_m) \\
&\preceq a_1 q(gx_{n-1}, gx_{m-1}) + a_2 r^{n-1} q(gx_0, gx_1) + a_3 r^{m-1} q(gx_0, gx_1) \\
&\preceq a_1 q(gx_{n-1}, gx_{m-1}) + (a_2 + a_3) r^{n-1} q(gx_0, gx_1),
\end{aligned}$$

since $r^{m-1} \leq r^{n-1}$.
Continuing in this way, we obtain at the n-th step that

$$\begin{aligned}
q(gx_n, gx_m) &\preceq a_1^n q(gx_0, gx_p) + (a_2 + a_3)[r^{n-1} + a_1 r^{n-2} + \cdots + a_1^{n-1}] q(gx_0, gx_1) \\
&= a_1^n q(gx_0, gx_p) + \beta_n q(gx_0, gx_1), \qquad (11)
\end{aligned}$$

where $\beta_n = (a_2 + a_3)[r^{n-1} + a_1 r^{n-2} + \cdots + a_1^{n-1}]$.

We now show that

$$q(gx_0, gx_p) \preceq \frac{1}{1 - a_1^j} \left(\frac{1}{1-r} + \beta_j \right) M, \tag{12}$$

where $M = q(gx_0, gx_1) + q(gx_0, gx_2) + \cdots + q(gx_0, gx_j) \in P$.

If $p \leq j$, then

$$\begin{aligned}
q(gx_0, gx_p) &\preceq (1 + \beta_j)\, q(gx_0, gx_p) \\
&\preceq [(1 + r + r^2 + \cdots) + \beta_j]\, q(gx_0, gx_p) \\
&= \left(\frac{1}{1-r} + \beta_j \right) q(gx_0, gx_p) \\
&\preceq (1 + a_1^j + (a_1^j)^2 + \cdots) \left(\frac{1}{1-r} + \beta_j \right) q(gx_0, gx_p) \\
&\preceq \frac{1}{1 - a_1^j} \left(\frac{1}{1-r} + \beta_j \right) M.
\end{aligned}$$

If $p > j$, then there exists $s \in \mathbb{N}$ such that $p = sj + t$, where $0 \le t < j$. If $t = 0$, then by using conditions (5) and (11)

$$\begin{aligned} q(gx_0, gx_p) &\preceq q(gx_0, gx_1) + q(gx_1, gx_2) + \cdots + q(gx_{j-1}, gx_j) + q(gx_j, gx_p) \\ &\preceq q(gx_0, gx_1) + rq(gx_0, gx_1) + \cdots + r^{j-1} q(gx_0, gx_1) \\ &\quad + a_1^j q(gx_0, gx_{p-j}) + \beta_j q(gx_0, gx_1) \\ &= \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j \right) q(gx_0, gx_1) + a_1^j q(gx_0, gx_{p-j}). \end{aligned} \tag{13}$$

By repeated application of (13), we obtain at $(s-1)$-th step that

$$\begin{aligned} q(gx_0, gx_p) &\preceq \left[1 + a_1^j + (a_1^j)^2 + \cdots + (a_1^j)^{s-2}\right] \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j \right) q(gx_0, gx_1) \\ &\quad + (a_1^j)^{(s-1)} q(gx_0, gx_j) \\ &\preceq \left[1 + a_1^j + (a_1^j)^2 + \cdots + (a_1^j)^{s-2}\right] \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j \right) q(gx_0, gx_1) \\ &\quad + (a_1^j)^{(s-1)} \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j \right) q(gx_0, gx_j) \\ &\preceq \left[1 + a_1^j + (a_1^j)^2 + \cdots + (a_1^j)^{s-1}\right] \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j \right) M \\ &\preceq \frac{1}{1 - a_1^j} \left(\frac{1}{1-r} + \beta_j \right) M. \end{aligned}$$

If $t \neq 0$, then

$$\begin{aligned} q(gx_0, gx_p) &\preceq q(gx_0, gx_1) + q(gx_1, gx_2) + \cdots + q(gx_{j-1}, gx_j) + q(gx_j, gx_p) \\ &\preceq \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j \right) q(gx_0, gx_1) + a_1^j q(gx_0, gx_{p-j}). \end{aligned} \tag{14}$$

By repeated application of (14), we obtain at s-th step that

$$\begin{aligned} q(gx_0, gx_p) &\preceq \left[1 + a_1^j + (a_1^j)^2 + \cdots + (a_1^j)^{s-1}\right] \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j \right) q(gx_0, gx_1) \\ &\quad + (a_1^j)^s q(gx_0, gx_t) \\ &\preceq \left[1 + a_1^j + (a_1^j)^2 + \cdots + (a_1^j)^{s-1}\right] \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j \right) q(gx_0, gx_1) \end{aligned}$$

$$
\begin{aligned}
&+(a_1^j)^s \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j\right) q(gx_0, gx_t) \\
\preceq\ & \left[1 + a_1^j + (a_1^j)^2 + \cdots + (a_1^j)^s\right] \left(\sum_{\nu=0}^{j-1} r^\nu + \beta_j\right) M \\
\preceq\ & \frac{1}{1-a_1^j}\left(\frac{1}{1-r} + \beta_j\right) M.
\end{aligned}
$$

Thus, for the case $p > j$, we have

$$q(gx_0, gx_p) \preceq \frac{1}{1-a_1^j}\left(\frac{1}{1-r} + \beta_j\right) M. \tag{15}$$

It now follows from (11) that for all $m, n \in \mathbb{N}$ with $m > n$,

$$
\begin{aligned}
q(gx_n, gx_m) &\preceq \frac{a_1^n}{1-a_1^j}\left(\frac{1}{1-r} + \beta_j\right) M + \beta_n\, q(gx_0, gx_1) \\
&\preceq b_n\, M,
\end{aligned} \tag{16}
$$

where $b_n = \frac{a_1^n}{1-a_1^j}\left(\frac{1}{1-r} + \beta_j\right) + \beta_n \to 0$ as $n \to \infty$. By using Lemma 3.1(iii), we conclude that (gx_n) is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists an element $u \in g(X)$ such that $gx_n \to u$ as $n \to \infty$.
By (16) and $(q3)$, we have

$$q(gx_n, u) \preceq b_n\, M. \tag{17}$$

Suppose that u is not a point of coincidence of f and g. Then by hypothesis, (5) and (17), we have

$$
\begin{aligned}
\theta &\prec inf\{q(gx, u) + q(fx, u) + q(gx, fx) : x \in X\} \\
&\preceq inf\{q(gx_n, u) + q(fx_n, u) + q(gx_n, fx_n) : n \in \mathbb{N}\} \\
&= inf\{q(gx_n, u) + q(gx_{n+1}, u) + q(gx_n, gx_{n+1}) : n \in \mathbb{N}\} \\
&\preceq inf\{b_n\, M + b_{n+1}\, M + r^n\, q(gx_0, gx_1) : n \in \mathbb{N}\} \\
&= \theta,
\end{aligned}
$$

which is a contradiction. Therefore, u is a point of coincidence of f and g. So there exists $z \in X$ such that $fz = gz = u$.
For uniqueness, let there exists $w(\neq u) \in X$ such that $fx = gx = w$ for some $x \in X$. Then

$$
\begin{aligned}
q(u, u) = q(fz, fz) &\preceq a_1\, q(gz, gz) + a_2\, q(gz, fz) + a_3\, q(gz, fz) \\
&= (a_1 + a_2 + a_3)\, q(u, u).
\end{aligned}
$$

Using Proposition 2.7, it follows that $q(u,u)=\theta$.
By the arguments similar to that used above, we have $q(w,w)=\theta$.
Now,

$$\begin{aligned} q(u,w)=q(fz,fx) &\preceq a_1\,q(gz,gx)+a_2\,q(gz,fz)+a_3\,q(gx,fx) \\ &= a_1\,q(u,w)+a_2\,q(u,u)+a_3\,q(w,w) \\ &= a_1\,q(u,w). \end{aligned}$$

Again, by Proposition 2.7, we have $q(u,w)=\theta$.

Thus, $q(u,w)=\theta$ and $q(u,u)=\theta$ imply that $u=w$. Therefore, f and g have a unique point of coincidence in X.
If f and g are weakly compatible, then by Proposition 2.10 , f and g have a unique common fixed point in X.

Remark 3.3 *We see that if u is a point of coincidence of f and g, then $q(u,u)=\theta$.*

Corollary 3.4 *Let (X,d) be a complete cone metric space and q is a generalized c-distance of order j on X. Suppose the mapping $f:X\to X$ satisfies*

$$q(fx,fy)\preceq a_1\,q(x,y)+a_2\,q(x,fx)+a_3\,q(y,fy) \tag{18}$$

for all $x,y\in X$, and a_1, a_2, $a_3\in[0,1)$ with $a_1+a_2+a_3<1$ and that

$$inf\,\{q(x,y)+q(fx,y)+q(x,fx):x\in X\}\succ\theta \tag{19}$$

for all $y\in X$ with $y\neq fy$. Then f has a unique fixed point in X.

The proof follows from Theorem 3.2 by taking $g=I$, the identity mapping.
As an application of Corollary 3.4, we have the following results.

Theorem 3.5 *[8] Let (X,d) be a complete cone metric space, P a normal cone. Suppose the mapping $T:X\to X$ satisfies the contractive condition*

$$d(Tx,Ty)\preceq\alpha\,[d(x,Tx)+d(y,Ty)] \tag{20}$$

for all $x,y\in X$, where $\alpha\in[0,\frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Since P is normal, it follows that d is a c-distance on X. So, we may consider d as a generalized c-distance of any order j. The condition (20) can be restated as

$$d(Tx,Ty)\preceq 0\cdot d(x,y)+\alpha\,d(x,Tx)+\alpha\,d(y,Ty) \tag{21}$$

for every $x, y \in X$, where $2\alpha \in [0, 1)$ is a constant. Thus, condition (18) of Corollary 3.4 is satisfied.
Assume that there exists $y \in X$ with $y \neq Ty$ and

$$inf\,\{d(x, y) + d(Tx, y) + d(x, Tx) : x \in X\} = \theta. \tag{22}$$

Then, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} \{d(x_n, y) + d(Tx_n, y) + d(x_n, Tx_n)\} = \theta, \tag{23}$$

which implies that $d(x_n, y) \to \theta,\ d(Tx_n, y) \to \theta,\ d(x_n, Tx_n) \to \theta.$
By condition (20), we have

$$d(Tx_n, Ty) \preceq \alpha\,[d(x_n, Tx_n) + d(y, Ty)] \tag{24}$$

for any $n \in \mathbb{N}$. Since P is normal, it follows that

$$d(y, Ty) \preceq \alpha\, d(y, Ty) \tag{25}$$

which gives that $d(y, Ty) = \theta$ i.e., $y = Ty$, a contradiction.
Hence, if $y \neq Ty$, then

$$inf\,\{d(x, y) + d(Tx, y) + d(x, Tx) : x \in X\} \succ \theta. \tag{26}$$

So, using Corollary 3.4, we have the desired result.

Theorem 3.6 *[8]Let (X, d) be a complete cone metric space, P a normal cone. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \preceq \alpha\, d(x, y) \tag{27}$$

for all $x, y \in X$, where $\alpha \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

As in Theorem 3.5, we treat d as a generalized c-distance of any order j. The condition (27) can be restated as

$$d(Tx, Ty) \preceq \alpha\, d(x, y) + 0 \cdot\ d(x, Tx) + 0 \cdot\ d(y, Ty) \tag{28}$$

for every $x, y \in X$, where $\alpha \in [0, 1)$ is a constant. Thus, condition (18) of Corollary 3.4 is satisfied.
Assume that there exists $y \in X$ with $y \neq Ty$ and

$$inf\,\{d(x, y) + d(Tx, y) + d(x, Tx) : x \in X\} = \theta. \tag{29}$$

Then, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} \{d(x_n, y) + d(Tx_n, y) + d(x_n, Tx_n)\} = \theta, \tag{30}$$

which implies that $d(x_n, y) \to \theta,\ d(Tx_n, y) \to \theta,\ d(x_n, Tx_n) \to \theta$.
By using condition (27), we have

$$d(Tx_n, Ty) \preceq \alpha\, d(x_n, y) \tag{31}$$

for any $n \in \mathbb{N}$. Since P is normal, it follows that

$$d(y, Ty) \preceq \theta \tag{32}$$

which gives that $d(y, Ty) = \theta$ i.e., $y = Ty$, a contradiction.
Hence, if $y \neq Ty$, then

$$inf\,\{d(x, y) + d(Tx, y) + d(x, Tx) : x \in X\} \succ \theta. \tag{33}$$

So, by Corollary 3.4, T has a unique fixed point in X.

Theorem 3.7 *[7] Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose that the mapping $f : X \to X$ is continuous and satisfies the contractive condition:*

$$q(fx, fy) \preceq a_1\, q(x, y) + a_2\, q(x, fx) + a_3\, q(y, fy) \tag{34}$$

for all $x, y \in X$, where $a_1,\ a_2,\ a_3 \in [0, 1)$ are nonnegative real numbers such that $a_1 + a_2 + a_3 < 1$. Then f has a unique fixed point in X.

We treat q as a generalized c-distance of order 1. Assume that there exists $y \in X$ with $y \neq fy$ and

$$inf\,\{q(x, y) + q(fx, y) + q(x, fx) : x \in X\} = \theta. \tag{35}$$

Then, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} \{q(x_n, y) + q(fx_n, y) + q(x_n, fx_n)\} = \theta, \tag{36}$$

which implies that $q(x_n, y) \to \theta,\ q(fx_n, y) \to \theta,\ q(x_n, fx_n) \to \theta$.
Since $q(x_n, y) \to \theta$ and $q(x_n, fx_n) \to \theta$, by Lemma 3.1, we have (fx_n) converges to y.
By using Condition (34), we obtain

$$q(fx_n, f^2x_n) \preceq a_1\, q(x_n, fx_n) + a_2\, q(x_n, fx_n) + a_3\, q(fx_n, f^2x_n). \tag{37}$$

So, it must be the case that

$$q(fx_n, f^2x_n) \preceq \frac{a_1 + a_2}{1 - a_3}\, q(x_n, fx_n). \tag{38}$$

Now, by using condition (38)

$$\begin{aligned} q(x_n, f^2x_n) &\preceq q(x_n, fx_n) + q(fx_n, f^2x_n) \\ &\preceq q(x_n, fx_n) + \frac{a_1 + a_2}{1 - a_3} q(x_n, fx_n) \\ &\longrightarrow \theta. \end{aligned}$$

Again, by Lemma 3.1, (f^2x_n) converges to y.
Since f is continuous, we have

$$fy = f(\lim_{n\to\infty} fx_n) = \lim_{n\to\infty} f^2x_n = y, \tag{39}$$

which is a contradiction.
Hence, if $y \neq fy$, then

$$inf\,\{q(x, y) + q(fx, y) + q(x, fx) : x \in X\} \succ \theta. \tag{40}$$

So, by Corollary 3.4, f has a unique fixed point in X.
We conclude with an example.

Example 3.8 *Let $E = \mathbb{R}^2$, the Euclidean plane and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ a cone in E. Let $X = \{\alpha,\ \beta,\ \gamma,\ \delta\} \subseteq \mathbb{R}$ and define $d : X \times X \to E$ by*

$$d(x, y) = (a \mid x - y \mid, b \mid x - y \mid) \tag{41}$$

for all $x, y \in X$, where a, b are positive constants. Then (X, d) is a cone metric space. Let $q : X \times X \to E$ be defined by

$$q(\alpha, \beta) = q(\beta, \alpha) = (9, 9),\ q(\alpha, \gamma) = q(\gamma, \alpha) = q(\beta, \gamma) = q(\gamma, \beta) = (3, 3), \tag{42}$$

$$q(\alpha, \delta) = q(\delta, \alpha) = q(\beta, \delta) = q(\delta, \beta) = q(\gamma, \delta) = q(\delta, \gamma) = (5, 5) \tag{43}$$

$$and\ q(x, x) = (0, 0)\ for\ every\ x \in X. \tag{44}$$

Then q satisfies condition $(q2)$ of Definition 2.16 for $j = 2$. The conditions $(q1)$ and $(q3)$ are immediate. To show $(q4)$, for any $c \in E$ with $\theta \ll c$, put $e = (\frac{1}{2}, \frac{1}{2})$. Then

$$q(z, x) \ll e\ and\ q(z, y) \ll e\ imply\ d(x, y) \ll c. \tag{45}$$

Thus q is a generalized c-distance of order 2 on X but it is not a c-distance on X since it lacks the triangular property:

$$q(\alpha, \beta) = (9, 9) \npreceq q(\alpha, \gamma) + q(\gamma, \beta) = (3, 3) + (3, 3) = (6, 6). \tag{46}$$

We define $f, g : X \to X$ by

$$fx = \gamma,\ for\ all\ x \in X \tag{47}$$

and

$$\begin{aligned} gx &= \gamma, \; for \; x \in \{\alpha, \gamma, \delta\} \\ &= \delta, \; for \; x = \beta. \end{aligned}$$

We see that γ is the unique point of coincidence of f and g.
It is easy to verify that

$$inf\,\{q(gx, y) + q(fx, y) + q(gx, fx) : x \in X\} \succ \theta \tag{48}$$

for all $y \in X$ with $y \neq \gamma$. Also, for every $x, y \in X$ one has

$$q(fx, fy) \preceq a_1\, q(gx, gy) + a_2\, q(gx, fx) + a_3\, q(gy, fy) \tag{49}$$

where a_1, a_2, $a_3 \in [0, 1)$ with $a_1 + a_2 + a_3 < 1$. Thus, we have all the conditions of Theorem 3.2 and γ is the unique common fixed point of f and g in X.

Conflict of Interests: The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] T. Abdeljawad and E. Karapinar, Quasi cone metric spaces and generalizations of Caristi Kirk's theorem, *Fixed Point Theory and Applications*, Article ID 574387(2009), 9 pages.

[2] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.*, 341(2008), 416-420.

[3] H. Aydi, E. Karapinar and S. Moradi, Coincidence points for expansive mappings under c-distance in cone metric spaces, *International Journal of Mathematics and Mathematical Sciences*, Article ID 308921(2012), 11 pages.

[4] I. Beg and M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, *Fixed Point Theory and Applications*, Article ID 74503(2006), 7 pages.

[5] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, 57(2000), 31-37.

[6] Z.M. Fadail and A.G.B. Ahmad, Common coupled fixed point theorems of single-valued mapping for c-distance in cone metric spaces, *Abstract and Applied Analysis*, Article ID 901792(2012), 24 pages.

[7] Z.M. Fadail, A.G.B. Ahmad and Z. Golubovic, Fixed point theorems of single-valued mapping for c-distance in cone metric spaces, *Abstract and Applied Analysis*, Article ID 826815(2012), 11 pages.

[8] L.G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 332(2007), 1468-1476.

[9] O. Hadzic, Common fixed point theorem for family of mappings in complete metric spaces, *Math. Japon*, 29(1984), 127-134.

[10] D. Ilić and V. Rakočević, Common fixed points for maps on cone metric space, *J. Math. Anal. Appl.*, 341(2008), 876-882.

[11] G. Jungck, Common fixed points for noncontinuous nonself maps on non-metric spaces, *Far East J. Math. Sci.*, 4(1996), 199-215.

[12] O. Kada, T. Suzuki and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japonica*, 44(1996), 381-391.

[13] R. Kannan, Some results on fixed points-II, *Amer. Math. Monthly*, 76(1969), 405-408.

[14] S.K. Mohanta, A fixed point theorem via generalized w-distance, *Bulletin of Mathematical Analysis and Applications*, 3(2011), 134-139.

[15] J.O. Olaleru, Some generalizations of fixed point theorems in cone metric spaces, *Fixed Point Theory and Applications*, Article ID 657914(2009), 10 pages.

[16] H.K. Pathak, Weak commuting mappings and fixed points, *Ind. J. Pure Appl. Math.*, 17(1986), 201-211.

[17] B.E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Analysis: Theory, Methods and Applications*, 47(2001), 2683-2693.

[18] S. Rezapour and R. Hamlbarani, Some notes on the paper cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 345(2008), 719-724.

[19] S. Rezapour, M. Derafshpour and R. Hamlbarani, A review on topological properties of cone metric spaces, *In Proceedings of the Conference on Analysis, Topology and Applications (ATA'08)*, Vrnjacka Banja, Serbia, May-June (2008).

[20] S.P. Singh and B.A. Meade, On common fixed point theorem, *Bull. Aus. Math. Soc.*, 16(1977), 49-53.

[21] P. Vetro, Common fixed points in cone metric spaces, *Rend. Circ. Mat. Palermo*, 56(2007), 464-468.

[22] S. Wang and B. Guo, Distance in cone metric spaces and common fixed point theorems, *Applied Mathematics Letters*, 24(2011), 1735-1739.

[23] C.C. Yeh, On common fixed points of continuous mappings, *Ind. J. Appl. Math.*, 10(1979), 415-420.