*Gen. Math. Notes, Vol. 7, No.2, December 2011, pp. 15-23*
*ISSN 2219-7184; Copyright © ICSRS Publication, 2010*
*www.i-csrs.org*
*Available free online at http://www.geman.in*

# Products of Menger Probabilistic Normed Spaces

**K.P.R. Sastry[1], G.A. Naidu[2], V. Madhavi Latha[3], S.S.A. Sastri[4] and I. Laxmi Gayatri[5]**

[1]8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam-530017, India
E-mail: kprsastry@hotmail.com
[2]Department of Mathematics, Andhra University, Visakhapatnam-530003, India
E-mail: drgolivean@yahoo.com
[3]Department of Mathematics, Andhra University, Visakhapatnam-530003, India
E-mail: vml_gsm@yahoo.co.in
[4]Department of Mathematics, G.V.P. College of Engineering, Madhuravada, Visakhapatnam-530048, India
E-mail: sambharasas@yahoo.co.in
[5]Department of Engg. Mathematics, GIT, GITAM University, Visakhapatnam- 530045, India
E-mail: ilg.gitam@gmail.com

(Received: 26-5-11/ Accepted: 3-8-11)

**Abstract**

*In this paper, we study Menger Probabilistic Normed (PN) spaces in a detailed way; we introduce the notation of a finite product of Menger Probabilistic Normed spaces, show that a finite product of complete Menger probabilistic normed spaces is itself complete and every Cauchy sequence in a Menger probabilistic normed space is norm totally bounded under certain conditions. We also introduce the notion of L-Menger PN spaces and show that in a L-Menger PN space* $(X,\tilde{F}, *)$*, the distribution map* $\tilde{F}$ *determines and is determined by a single distribution function on* $\mathbb{R}$.

**Keywords**: *Probabilistic metric space, Menger probabilistic normed space, L-Menger probabilistic normed space and Menger spaces.*

# 1 Introduction

It is well known that the theory of probabilistic normed spaces is a new frontier branch between probabilistic theory and functional analysis and has an important background which contains the common metric space as a special case. One can study the completeness theorems in probabilistic normed spaces. This study has important applications, for example on the fixed point theory etc.
Now, we begin with the following definitions

**Definition 1.1.** *[2] A function $F: \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing, left continuous, $inf_{x \in \mathbb{R}} F(x) = 0$ and $sup_{x \in \mathbb{R}} F(x) = 1$.*
*The set of distribution functions $F$ such that $F(0) = 0$ is denoted by $\mathfrak{D}^+$.*
*Also denote b $H$, the Heaviside distribution function $H(t) = \begin{cases} 1 & if\ t > 0 \\ 0 & if\ t \leq 0 \end{cases}$*

**Definition 1.2.** *[2] Let $S$ be a non-empty set and let $F: SxS \to \mathfrak{D}^+$. For any $p, q \in S$. We denote the image of the pair $(p, q)$ by $F_{p,q}$ which is a distribution function so that $F_{p,q}(t) \in [0,1]$, for every real $t$. Suppose $F$ satisfies:*

*a)* $F_{p,q}(t) = 1$ *for all* $t > 0$ *if and only if* $p = q$
*b)* $F_{p,q}(t) = F_{q,p}(t)$
*c)* *If* $F_{p,q}(t_1) = 1$ *and* $F_{q,r}(t_2) = 1$ *then* $F_{p,r}(t_1 + t_2) = 1$ *where* $p, q, r \in S$.

*Then $(S, F)$ is called a Probabilistic metric space (PM space).*

**Definition 1.3.** *[2] A triangular norm $*: [0,1]\ x\ [0,1] \to [0,1]$ is a function satisfying the following conditions*

*(i)* $a * 1 = a \quad \forall\, a \in [0,1]$
*(ii)* $a * b = b * a \quad \forall\, a, b \in [0,1]$
*(iii)* $c * d \geq a * b\ \forall\, a, b, c, d \in [0,1]$ *with* $c \geq a$ *and* $d \geq b$
*(iv)* $(a * b) * c = a * (b * c)\ \ \forall\, a, b, c \in [0,1]$

*A triangular norm is also denoted by t-norm.*

**Definition 1.4.** *[2] A Menger Probabilistic metric space (Menger PM space) is a triplet $(S, F, *)$ where $(S, F)$ is a Probabilistic metric space, $*$ is a t-norm satisfying the condition*

$F_{p,r}(t_1 + t_2) \geq F_{p,q}(t_1) * F_{q,r}(t_2)$ *for all* $t_1, t_2 \geq 0$ *and* $p, q, r \in S$.

*We observe that a Menger PM space is a probabilistic metric space.*

**Definition 1.5.** *[2] A triplet $(S, F, *)$ is called a Menger Probabilistic normed space (Menger PN space) if $S$ is a real vector space, $F: S \to \mathfrak{D}^+$(for $x \in S$, the distridution function $F(x)$ is denoted by $F_x$ and $F_x(t)$ is the value of $F_x$ at $t \in \mathbb{R}$) and $*$ is a t-norm, satisfying the following conditions:*

*(i)* $F_x(0) = 0$
*(ii)* $F_x(t) = H(t)$ *for every* $t > 0$ *iff* $x = 0$
*(iii)* $F_{\alpha x}(t) = F_x\left(\frac{t}{|\alpha|}\right)\ \forall \alpha \in \mathbb{R}, \alpha \neq 0$
(iv) $F_{x+y}(t_1 + t_2) \geq F_x(t_1) * F_y(t_2)\ \forall\ x, y \in S$ *and* $t_1, t_2 \in \mathbb{R}^+$

**Remark 1.6.** [1] Let $(S, F, *)$ be a Menger PN space and $S$ be a real vector space. Then $(S, \tilde{F}, *)$ is a Menger PM space where $\tilde{F}_{x,y}(t) = F_{x-y}(t)$. $(S, \tilde{F}, *)$ is called the induced Menger PM space of the Menger PN space $(S, F, *)$.
Schweizer, Sklar and Thorp [3] proved that if $(S, F, *)$ is a Menger PM space with $sup_{0<t<1}(t * t) = 1$, then $(S, F, *)$ is a Hausdorff topological space in the topology $\mathcal{T}$ induced by the family of $(\epsilon, \lambda)$ neighbourhoods $\{U_p(\epsilon, \lambda): p \in S, \epsilon > 0, \lambda > 0\}$ where $U_p(\epsilon, \lambda) = \{u \in S: F_{u,p}(\epsilon) > 1 - \lambda\}$.

**Definition 1.7.** *[2] Let $(S, F, *)$ be a Menger PM space with $sup_{0<t<1}(t * t) = 1$*

*(i)* *A sequence $\{u_n\}$ in $S$ is $\mathcal{T}$-convergent to $u \in S(u_n \overset{\mathcal{T}}{\to} u)$ if for any given $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{u_n,u}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.*
*(ii)* *A sequence $\{u_n\}$ in $S$ is a $\mathcal{T}$-Cauchy sequence if for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{u_n,u_m}(\epsilon) > 1 - \lambda$ whenever $m, n \geq N$.*
*(iii)* *A Menger PM space $(S, F, *)$ is said to be $\mathcal{T}$-complete if each $\mathcal{T}$-Cauchy sequence in $S$ is $\mathcal{T}$-convergent to some point in $S$.*

**Example 1.8.** (i) Let $(E, \|\ \ \|_E)$ be a normed real vector space. Define $\widetilde{F}: E \to \mathfrak{D}^+$ by

$$\tilde{F}_x(t) = \begin{cases} \frac{t}{t+\|x\|_E} & if\ t > 0 \\ 0 & if\ t \leq 0 \end{cases}$$

Then $(E, \widetilde{F}, *)$ is a Menger PN space where $*$ is any t-norm.
(ii) For $x \in E$, define $\tilde{F}_x(t) = H(t - \|x\|)\ \forall\ t \in \mathbb{R}$.
Then $(E, \widetilde{F}, *)$ is also a Menger PN space.

**Lemma 1.9 ([1], Lemma 2.6).** *If $(\mathbb{R}, F, *)$ is a Menger PN space then $|x| \leq |y| \Rightarrow F_x(t) \leq F_y(t)$ for all $x, y \in \mathbb{R}$ and $t \geq 0$.*

**Definition 1.10.** *[1] Let* $(\mathbb{R}, F,*)$ *be a Menger PN space and* $(E, \|\ \ \|)$ *be a normed real vector space. We define the mapping* $\tilde{F}: E \to \mathfrak{D}^+$ *by* $\tilde{F}_x(t) = F_{\|x\|}(t)$.
*(We observe that* $\tilde{F}_x(t) = \tilde{F}_y(t)\ \ if\ \|x\| = \|y\|$*)*
*Then the following proposition says that* $(E, \tilde{F}\ ,*)$ *is a Menger PN space.*

**Proposition 1.11 ([1], Proposition 2.8).** *Let* $(\mathbb{R}, F,*)$ *be a Menger PN space then* $(E, \tilde{F},*)$ *is also a Menger PN space.*
*The following theorem also is proved in [1].*

**Theorem 1.12 ([1], Theorem 3.1).** *Let* $(S, F,*)$ *be a Menger PM space with a continuous t- norm* $*$. *Suppose* $\{x_n\}$ *is a Cauchy sequence which has a convergent subsequence. Then* $\{x_n\}$ *is convergent.*
*The following theorem also is proved in [1], but it is not well formed*

**Theorem 1.13 ([1], Theorem 3.2).** *Let* $(E, \tilde{F},*)$ *be a complete Menger PN space, where E is a real vector space and* $\tilde{F}: \mathbb{R} \to \mathfrak{D}^+$ *is defined by* $\tilde{F}_x(t) = F_{\|x\|}(t)$.
*Then* $(\mathbb{R}, F,*)$ *is complete.*
*In this theorem, F is not well defined. In fact when* $x \neq y$ *and* $\|x\| = \|y\|$, $\tilde{F}_x(t)$ *may be different from* $\tilde{F}_y(t)$ *as seen in the following Example.*

**Example 1.14:** Let $X = \mathbb{R} \times \mathbb{R}, for\ any\ x = (\alpha_1, \alpha_2) \in X, \|x\| = |\alpha_1| + |\alpha_2|$.
Define $F_x(t) = \frac{t}{t+|\alpha_1|+2|\alpha_2|}\ \ \forall\, x \in X$ and $* = \ min$.
Then $(X, F,*)$ is a Menger PN space.
Let $e_1 = \ (1, 0)$ and $e_2 = \ (0, 1)$ so that $\|e_1\| = 1 = \|e_2\|$
But $F_{\|e_1\|}(t) \neq F_{\|e_2\|}(t)$
since $F_{\|e_1\|}(t) = F_{e_1}(t) = \frac{t}{t+1}\ where\ as\ \ F_{\|e_2\|}(t) = \frac{t}{t+2}$
The following theorem says that a Menger PN space $(E, \tilde{F},*)$ may induce a number of probabilistic distribution maps $F$ on $\mathbb{R}$. This theorem can be easily established.

**Theorem 1.15.** *Let* $(E, \tilde{F},*)$ *be a Menger PN space. For any* $x \in X, x \neq 0$, *define* $F_1(t) = \tilde{F}_{\frac{x}{\|x\|}}(t)$ *and* $F_a(t) = F_1\left(\frac{t}{|a|}\right)\ \ for\ a \neq 0$.
*Then* $(\mathbb{R}, F,*)$ *is a Menger PN space. If* $(E, \tilde{F},*)$ *is complete, then so is* $(\mathbb{R}, F,*)$.
*The following theorem is a kind of converse to Theorem 1.15. This theorem says that every distribution function induces a probability distribution map on a normed linear space under any t-norm* $*$.

**Theorem 1.16.** *Let X be a normed linear space,* $*$ *be a t-norm and F be a distribution function such that* $F(t) < 1$ *for at least one* $t > 0$. *For* $x \in X$, *define* $\tilde{F}_x: E \to \mathfrak{D}^+$ *by*

$$\tilde{F}_x(t) = \begin{cases} F(\frac{t}{\|x\|})\ \ if\ x \neq 0 \\ H(t)\ if\ x = 0 \end{cases}$$

*Then* $(X, \widetilde{F}, *)$ *is a Menger PN space.*

**Proof.** The result can be easily established.

**Definition 1.17.** *A Menger PN space* $(X, F, *)$ *is said to be a L- Menger PN space if* $\|x\| = \|y\| \Rightarrow F_x = F_y$

*We observe that* $(X, \widetilde{F}, *)$ *of Theorem 1.16 is a L-Menger PN space.*

*It may be noted that the Menger PN space* $(X, F, *)$ *of Example 1.14 is not a L-Menger PN space.*

*The following theorem shows that a L-Menger PN space determines and is determined by a distribution function F with* $0 < F(t) < 1$ *for at least one* $t > 0$.

**Theorem 1.18.** *Let* $(\mathbb{R}, F, *)$ *be a Menger PN space and X be a vector space. Define* $\tilde{F}: X \to \mathfrak{D}^+$ *by* $\tilde{F}_x(t) = F_1(\frac{t}{\|x\|})$. *Then* $(X, \tilde{F}, *)$ *is a L-Menger PN space. Conversely if* $(X, \widetilde{F}, *)$ *is a L-Menger space, then* $(\mathbb{R}, F, *)$ *is a Menger PN space where* $F_1(t) = \tilde{F}_x(t)$ *for any* $x$ *( and hence every* $x$*) with* $\|x\| = 1$ *and* $0 < F_1(t) < 1$ *for at least one* $t > 0$.

*The following theorem also may be taken as a modification of Theorem 1.13.*

**Theorem 1.19.** *Let* $(X, \tilde{F}, *)$ *be a L-Menger PN space. Define* $F: \mathbb{R} \to \mathfrak{D}^+$ *by* $F_\alpha(t) = \tilde{F}_{\alpha x}(t)$ *for any* $x \in E$ *with* $\|x\| = 1$. *Then* $(\mathbb{R}, F, *)$ *is a Menger PN space.*

**Proof.** The proof of this theorem can be easily established.

**Definition 1.20.** *Let* $\{(\mathbb{R}, F_i, *): i = 1,2, \;.., k\}$ *be Menger PN spaces.*

*Write* $e_i = (0,0, \ldots, 1,0, \ldots, 0)$ *where 1 is in the* $i^{th}$ *place. Then* $\{e_1, e_2, \ldots, e_k\}$ *is a basis to the real vector space* $\mathbb{R}^k$. *Define* $\tilde{F}: \mathbb{R}^k \to \mathfrak{D}^+$ *is defined by*

$\tilde{F}_x(t) = F_{1x_1}(t) * F_{2x_2}(t) * \ldots . * F_{kx_k}(t)$ *where* $x = x_1e_1 + x_2e_2 + \cdots + x_ke_k$.

*Then it can be easily verified that* $(\mathbb{R}^k, \tilde{F}, *)$ *is a Menger PN space.*

$(\mathbb{R}^k, \tilde{F}, *)$ *is called the product Menger PN space of* $\{(\mathbb{R}, F_i, *): i = 1,2, \;.., k\}$.

$F_i$ *is called the* $i^{th}$ *component of* $\tilde{F}$.

*The following theorem can be easily established.*

**Theorem 1.21.** *Let* $(\mathbb{R}^k, \tilde{F}, *)$ *be a Menger PN space. Suppose*

$\tilde{F}_x(t) = F_{x_1e_1}(t) * F_{x_2e_2}(t) * \ldots . * F_{x_ke_k}(t)$ *whenever* $x = (x_1, x_2, \ldots, x_k)$.

*Then* $(\mathbb{R}^k, \tilde{F}, *)$ *is the product Menger PN space of* $\{(\mathbb{R}, F_i, *): i = 1,2, \;.., k\}$.

*The following theorem is proved in [1].*

**Theorem 1.22 ([1], Theorem 3.5).** *Let* $(\mathbb{R}, F, *)$ *be a Menger PN space where* $F_x(\bullet)$ *is continuous at zero and* $*$ *is a continuous t-norm. Then* $(\mathbb{R}, F, *)$ *is a complete Menger PN space.*

*In the next section, we prove this theorem without the restriction that "*$F_x(\bullet)$ *is continuous at zero and* $*$ *is a continuous t-norm".*

# 2 Main Results

In this section, we show that a Menger PN space $(\mathbb{R}, \mathrm{F}, *)$ is complete ($*$ is not assumed to be continuous), $(\mathbb{R}^k, \tilde{F}, *)$ is complete for $k = 1,2, \ldots.$ if $*$ is continuous.
Incidentally, we also show that a Cauchy sequence in $(\mathbb{R}, F, *)$ is bounded.

**Theorem 2.1**. *If* $(\mathbb{R}, F, *)$ *is a Menger PN space, then*

*(i)* *Every Cauchy sequence in* $(\mathbb{R}, F, *)$ *is bounded and*
*(ii)* $(\mathbb{R}, F, *)$ *is complete.*

**Proof.** Suppose $(\mathbb{R}, \mathrm{F}, *)$ is a Menger PN space

(i) Let $\{x_n\}$ be a Cauchy sequence in $(\mathbb{R}, \mathrm{F}, *)$.

We show that $\{x_n\}$ is bounded in $(\mathbb{R}, \mathrm{F}, *)$.
Otherwise, there exists a sub sequence $\{x_{n_k}\}$ such that $|x_{n_k}| \to \infty$
Since $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R}, F, *)$, for a given $\lambda \in (0,1)$ there exists N such that $F_{x_n - x_m}(1) > 1 - \lambda \ \forall \ m, n \geq N$
Hence $F_1\left(\frac{1}{|x_{n_k} - x_N|}\right) > 1 - \lambda \ \ \forall \ n_k > N$ ……… (1)

Since $F_1$ is increasing, $F_1(0^+)$ exists. Write $F_1(0^+) = \alpha$, then $\alpha \in [0,1]$.
$\alpha = 1 \Rightarrow F_1(t) = 1 \Rightarrow 1 = 0$, a contradiction
Hence $\alpha < 1$
Choose $\lambda \in (0,1)$ $such\ that$ $1 - \lambda > \alpha$
From (1), letting $k \to \infty$
$\alpha = F_1(0^+) \geq 1 - \lambda > \alpha$, a contradiction.
Therefore $\alpha = 1$, so that $F_1(0^+) = 1$, again a contradiction.
Hence $\{x_n\}$ is a bounded sequence in $\mathbb{R}$.

(ii) Let $\{x_n\}$ be a Cauchy sequence in $(\mathbb{R}, F, *)$. Then by (i), $\{x_n\}$ is bounded.

Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to β (say).
Then $F_{x_{n_k} - \beta}(t) = F_1(\frac{t}{x_{n_k} - \beta}) \to 1$ as $k \to \infty$.
$\therefore x_{n_k} \to \beta$ in $(\mathbb{R}, F, *)$
Hence $\{x_n\}$ is convergent by Theorem 1.12.
Therefore $(\mathbb{R}, F, *)$is complete.
Now we state a Lemma, whose proof is a consequence of the continuity of $*$ at 1.

**Lemma 2.2.** *Suppose* $*: [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm and* $k$ *is a positive integer.*

*Then for a given* $\lambda \in (0,1)$ *there exists* $\lambda' \in (0,1)$ *such that*

$$\underbrace{(1-\lambda') * (1-\lambda') * \ldots * (1-\lambda')}_{k\text{-times}} > 1-\lambda.$$

*Making use of the above Lemma, in the following theorem, we establish that* $(\mathbb{R}^k, \tilde{F}, *)$ *is complete if* $*$ *is continuous.*
*This theorem may be taken as a modification to Theorem 1.13.*

**Theorem 2.3.** *If* $(\mathbb{R}, F, *)$ *is a Menger PN space with continuous t-norm, then* $(\mathbb{R}^k, \tilde{F}, *)$ *is complete where* $\tilde{F}$ *is defined by* $\tilde{F}_x(t) = F_{\alpha_1}(t) * F_{\alpha_2}(t) * \ldots.* F_{\alpha_k}(t)$ *where* $x = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^K$.

**Proof.** By Definition 1.20, $(\mathbb{R}^k, \tilde{F}, *)$ is a Menger PN space.
Now we show that it is complete.
Let $\{x_n\}$ be a Cauchy sequence in $(\mathbb{R}^k, \tilde{F}, *)$.
Then for a given $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $N_0 \in N$ such that for any $m, n \geq N_0$
$\tilde{F}_{x_n - x_m}(\epsilon) > 1 - \lambda$
Let $x_n = (\alpha_{1n}, \alpha_{2n}, \ldots., \alpha_{kn}), n = 1,2, \ldots$

Now $1 - \lambda < \tilde{F}_{x_n - x_m}(\epsilon) = F_{1\alpha_n - 1\alpha_m}(\epsilon) * F_{(\alpha_{2n} - \alpha_{2m})}(\epsilon) * \ldots.* F_{(\alpha_{kn} - \alpha_{km})}(\epsilon)$

$\Rightarrow F_{1\alpha_n - 1\alpha_m}(\epsilon) > 1 - \lambda$, ......, $F_{\alpha_{kn} - \alpha_{km}}(\epsilon) > 1 - \lambda$

Therefore $\{\alpha_{in}\}$ is a Cauchy sequence in $(\mathbb{R}, F, *)$ $for\ 1 \leq i \leq k$.
Hence $\{\alpha_{in}\}$ converges, say, to $\alpha_i$, $i = 1,2, \ldots k$.
Write $x = (\alpha_1, \alpha_2, \ldots, \alpha_k)$
By Lemma 2.2, choose $\lambda' \in (0,1)$ such that

$$\underbrace{(1-\lambda') * (1-\lambda') * \ldots * (1-\lambda')}_{\text{k-times}} > 1-\lambda.$$

For $\lambda' \in (0,1)$ there exists $N_i$ such that for $n \geq N_i$
$F_{\alpha_{in} - \alpha_i}(\epsilon) > 1 - \lambda'\ for\ i = 1,2, \ldots. k$
Put $N = \max\{N_1, N_2, \ldots, N_k\}$, so that for $n \geq N$

$$\begin{aligned} \tilde{F}_{x_n - x}(\epsilon) &= F_{\alpha_{1n} - \alpha_1}(\epsilon) * F_{\alpha_{2n} - \alpha_2}(\epsilon) * \ldots.* F_{\alpha_{kn} - \alpha_k}(\epsilon) \\ &> (1-\lambda') * (1-\lambda') * \ldots * (1-\lambda') \\ &> 1 - \lambda \end{aligned}$$

Therefore $\{x_n\}$ converges to $x$.
Hence $(\mathbb{R}^k, \tilde{F}, *)$ is complete**.**

**Definition 2.4.** *If $X$ is a normed space and $(X, F,*)$ is a Menger PN space, then we say that $(X, F,*)$ is a normed Menger PN space.*

**Definition 2.5.** *If $X$ is a finite dimensional vector space and $(X, F,*)$ is a Menger PN space, then we say that $(X, F,*)$ is a finite dimensional Menger PN space.*

**Theorem 2.6.** *Let $X$ be a finite dimensional vector space with $(e_1, e_2, \dots, e_n)$ as a basis for $X$. Then $\left(X, \tilde{F},*\right)$, where $*$ is continuous, is a complete Menger PN space whenever $(\mathbb{R}, F_i,*)$, $i = 1,2, \dots$ are Menger PN spaces and $\tilde{F}$ satisfies*
$\tilde{F}_x(t) = F_{1\alpha_1}(t) * F_{2\alpha_2}(t) * \dots * F_{k\alpha_k}(t)$ *where* $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_k e_k$

**Proof.** Same as in Theorem 2.3.

In Theorem 2.1, we have shown that every Cauchy sequence in the Menger PN space $(\mathbb{R}, F,*)$ is bounded and hence totally bounded in $\mathbb{R}$. In the following Theorem we extend it to a general Menger PN space, under certain conditions.

**Theorem 2.7.** *Let $(X, F,*)$ be a normed Menger PN space, where $\{F_x(\bullet): \|x\| = 1\}$ is equicontinuous at zero and $*$ is a continuous t-norm. Then every Cauchy sequence in $(X, F,*)$ is norm totally bounded.*

**Proof.** Suppose $\{x_n\}$ is a Cauchy sequence in$(X, F,*)$.
We have to prove $\{x_n\}$ is totally bounded in norm.
Suppose $\{x_n\}$ is not totally bounded in norm.
Then there exists $\epsilon > 0$ such that what ever $N$ be there exists $n_N > N$ such that $x_{n_N} \notin \bigcup_{i=1}^{N} S_\epsilon(x_i)$
i.e. $\left\|x_N - x_{n_N}\right\| > \epsilon \ for\ i = 1,2, \dots\dots N$ … (1)
Since $\{F_x(\bullet): \|x\| = 1\}$ is equicontinuous at zero, for the given $(1-\lambda) > 0 \ \exists\ \delta > 0$ such that $F_x(t) < 1 - \lambda \ \ for\ every\ t < \delta$ and $\|x\| = 1$ …… (2)
Since $\{x_n\}$ is a Cauchy sequence in $(X, F,*)$,
for the given $\delta\epsilon > 0$ and $\lambda \in (0,1)$ there exists $M \in \boldsymbol{N}$ such that
$F_{x_n - x_m}(\delta\epsilon) > 1 - \lambda \ for\ all\ n, m \geq M.$
Now, by (1), $\frac{\epsilon}{\|x_{n_M} - x_M\|} < 1$, so that $\frac{\delta\epsilon}{\|x_{n_M} - x_M\|} < \delta$.
Consequently $F_{\frac{x_{n_M} - x_M}{\|x_{n_M} - x_M\|}}\left(\frac{\delta\epsilon}{\|x_{n_M} - x_M\|}\right) > 1 - \lambda$ $(\because n_M \geq M)$
Therefore by (2), we have
$1 - \lambda < F_{\frac{x_{n_M} - x_M}{\|x_{n_M} - x_M\|}}\left(\frac{\delta\epsilon}{\|x_{n_M} - x_M\|}\right) < 1 - \lambda$, a contradiction.
Hence $\{x_n\}$ is totally bounded.
We conclude the paper with an open problem.

**Open problem 2.8.** Is Theorem 2.3 valid if “continuity of t-norm” is dropped?

# References

[1] A. Aghajani and A. Razani, Some completeness theorems in the Menger probabilistic metric spaces, *Applied Sciences*, 10(2008), 1-8.

[2] S.S. Chang, B.S. Lee, Y.J. Cho, Y.Q. Chen, S.M. Kang and J.S. Jung, Generalized contraction mapping principle and differential equations in probabilistic metric spaces, *Proc. Amer. Math. Soc.*, 124(1996), 2367-2376.

[3] B. Schweizer, A. Sklar and E. Throp, The metrization of statistical metric spaces, *Pacific J. Math.* 10(1960), 673-675.

[4] B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific J. Math.*, 10(1960), 313-334.