Solution of Algebraic Quadratic Equations Taking into Account Transitional Processes in Oscillation Systems

Alexander A. Antonov

Research Center of Information Technologies "TELAN Electronics", Kiev, Ukraine
E-mail: telan@bk.ru

Abstract

The manuscript demonstrates that the solution of algebraic quadratic equations on the set of real numbers sometimes gives the results which contradict the transitional processes in oscillation systems. The solution of algebraic quadratic equations on the set of complex numbers, on the contrary, always agrees with the transitional processes in oscillation systems.

1. Introduction

In the 15th century the solution of algebraic equations used to be the favorite topic of parlor talk. These conversations were often joined by the Spanish inquisitor Tomas de Torquemada (1420 – 1498), who sentenced over 10 thousand people to be burnt at the stake. He considered himself to be an expert in mathematics and argued that solution of algebraic equations of high degree had been made impossible for the people by the will of God. However, during one of these discussions, his friend, mathematician Paolo Valmes (m. 1486) dared to object and said that he knew how to solve these equations [9]. That same night professor Valmes was arrested, and in three weeks sentenced to death without bloodshed, i.e. to burning at the stake, for "resisting the will of God". The way professor

Vlames was going to solve algebraic equations, and whether he intended to use imaginary numbers, remained unknown. Later, in 1545, Girolamo Cardano (1501-1576) discovered imaginary numbers, and in 1777 Leonhard Euler (1707 – 1783) suggested the symbol $i = \sqrt{-1}$. The term "complex numbers" was introduced in 1831 by Carl Friedrich Gauss (1777 - 1856) [10].

500 years have passed. However, as will be seen from the manuscript, the centuries-old history of searching for correct solutions of algebraic equations is not over yet.

2. The existing algorithms of solving quadratic equations

The course of algebra usually teaches that the algebraic quadratic equation $ax^2 + bx + c = 0$ can be solved two different ways.

The first algorithm of solving this equation is as follows. If only real numbers are used, the quadratic equation can have either two, or one, or none solutions, depending on the correlation of values of its parameters a, b, c. This statement is often illustrated by finding the intersection points (see Fig. 1a) of plotted parabola $y = ax^2 + bx + c$ and straight line $y = 0$, which corresponds to graphic solution of the set of equations

$$\begin{cases} y = ax^2 + bx + c \\ y = 0 \end{cases} \tag{1}$$

Fig. 1a clearly shows that the parabola curve corresponding to the initial quadratic equation, depending on its location, can actually intersect the x axis in two points, or touch the x axis in one point, or may not intersect the x axis at all.

The second solution algorithm is different. According to this algorithm, the solution of the quadratic equation $ax^2 + bx + c = 0$ is found using the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

according to which, at any parameters of a, b, c the equation always has two solutions. However, depending on the value of the discriminant $b^2 - 4ac$ these solutions can be different:

- if $b^2 - 4ac > 0$ both solutions are real and different;
- if $b^2 - 4ac = 0$ both solutions are real and equal;
- if $b^2 - 4ac < 0$ both solutions are different and complex conjugate numbers.
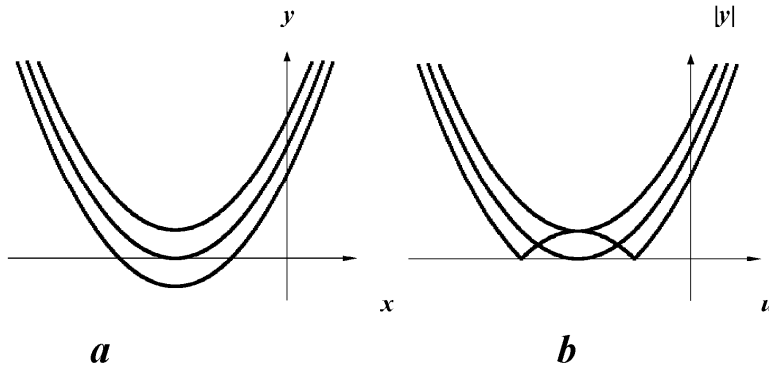
Fig. 1 (Graphic solution of a quadratic equation with the use of real numbers)

It is easy to notice that at $b^2 - 4ac > 0$ and $b^2 - 4ac = 0$ both algorithms of solving the quadratic equation discussed above actually give the same results. However, at $b^2 - 4ac < 0$ the above given algorithms give the results which are mutually exclusive. In fact, in accordance with the first algorithm, it is argued that at $b^2 - 4ac < 0$ the solution does not exist, whereas, according to the second algorithm, it is considered that at $b^2 - 4ac < 0$ the solution does exist.

So, where is the truth? It is obvious that only one of the two solutions can be true.

3. Characteristic algebraic equations

In order to answer the question above, let us recall that during the analysis of transitional processes in engineering sciences, the so called algebraic characteristic equations are solved, which correspond to differential equations of processes in dynamic systems [6] – [8].

Transitional processes are frequently dealt with in engineering, and they correspond to the transition of any dynamic (in particular, oscillation) system from one energy state into another. For instance, rocking of a pendulum after it was pushed by a man is a transitional process.

Characteristic equations are solved in order to determine the nature and parameters of transitional processes, with different solutions of a characteristic equation corresponding to different types of transitional processes. Thus, the solution of the quadratic characteristic equation at $b^2 - 4ac > 0$ corresponds to an aperiodic transitional process. The solution of the quadratic characteristic equation at $b^2 - 4ac = 0$ corresponds to a critical transitional process. And the solution of the quadratic characteristic equation at $b^2 - 4ac < 0$ corresponds to an oscillation

(in the form of damped oscillations – remember the motion of a pendulum after it was pushed) transitional process.

Thus, transitional processes corresponding to any solution of the characteristic quadratic equations using complex numbers actually exist in nature. Therefore, we have to make the conclusion that these solutions are the only true solutions.

Similar conclusions can be made for the solutions of algebraic characteristic equations of a higher degree.

If engineers believed that the only true solutions of algebraic characteristic equations are those using only real numbers, than they would have to claim that oscillation transitional processes should not exist in nature. But they do exist.

Therefore, the conclusion that the only true solution of quadratic equations is that on the set of complex numbers is indisputable.

4. Graphic illustrations of the solutions of characteristic algebraic equations

Certainly, the solution of quadratic equations using complex numbers needs graphical explanation, at least, in order to reconsider the charts on Fig. 1a. This is obviously a complicated task, since, if x and y are complex numbers, then the function $y = f(x)$ needs to be plotted in the four-dimensional space. However, psychologists found out that a human thinks with visual (i.e. not more that three-dimensional) images. To make sure, try to imagine a cube in the four-dimensional space. Therefore, we cannot either plot or imagine the graph of function of a complex variable $y = f(x)$.

Nevertheless, the above mentioned obstacle can be overcome if we remember that, apart from the algebraic form of presenting complex numbers, there are also the trigonometric and exponential forms of representing them, which allow clearly seeing that a complex number takes zero value only at zero value of its module, irrespective of its phase value. Therefore, the task can be simplified, and, instead of the four-dimensional function of the complex variable $y = f(x)$, the three-dimensional graph of the function $|y| = |f(u + iv)|$ can be plotted. Therefore, the solution of the quadratic equation using complex numbers can be explained with the help of graphic solution of the system of equations

$$\begin{cases} |y| = |ax^2 + bx + c| = |a(u+iv)^2 + b(u+iv) + c| \\ |y| = 0 \end{cases} \tag{3}$$

i.e. by finding intersection points of the plane $|y| = 0$ and surface $|y| = |ax^2 + bx + c = a(u \pm iv)^2 + b(u \pm iv) + c|$, where $|y|$ is the module of the complex variable y.

Graphic solutions of this system of equations for different values of parameters a,b,c are given on Fig. 2a, 2b and 2c. Fig. 2a corresponds to the solution of the quadratic equation in case $b^2 - 4ac > 0$. Fig. 2b corresponds to the solution of the quadratic equation in case $b^2 - 4ac = 0$. Fig. 2c corresponds to the solution of the quadratic equation if $b^2 - 4ac < 0$.

Then how should we understand the above given solution with the use of real numbers, and the corresponding graphs (Fig. 1a)? Why was it that earlier only the solutions in the form of real numbers were obtained?

It is simple. The point is that, when looking for the solution of the quadratic equation in accordance with the first algorithm, it was supposed that the solution should be found only in the form of real numbers. This is why only this solution was found.

This is easy to confirm if we plot the graphs (Fig. 1b) of the real variable $|y| = |au^2 + bu + c|$, where $u = \mathrm{Re}(x)$ is the real component of the complex argument x. They are plotted in bold lines on Fig. 2a, 2b and 2c, and, as can be seen, lie in the plane of real numbers $v = 0$.

These graphs, contrary to graphs on Fig. 1a, allow interpreting the solutions of the quadratic equation using real numbers, similar to graphs on Fig. 2, as points of contact of the curve $|y| = |au^2 + bu + c|$ and the u axis, i.e. the real component of the argument x.
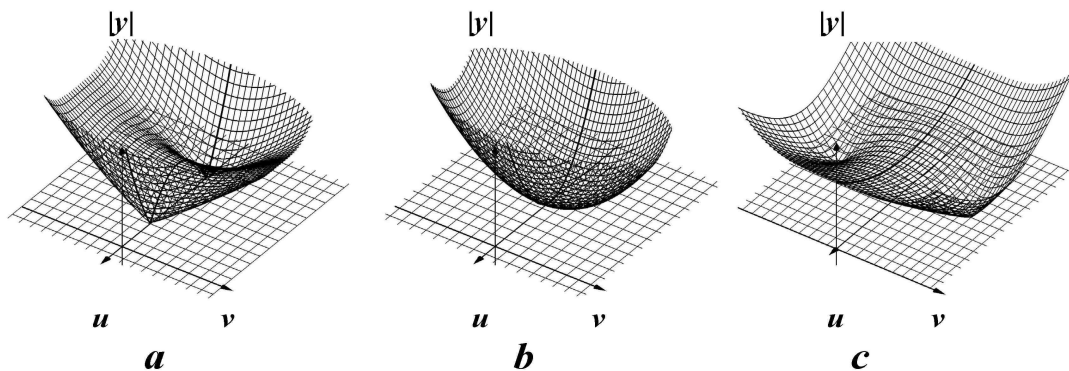

Fig. 2 (Graphic solution of a quadratic equation with the use of complex numbers)

For convenient comparison, these graphs on Fig. 1b are plotted along with the graphs given in Fig. 1a.

5. Summary

Quadratic characteristic equations used when studying transitional processes in

oscillation systems, always have the solution in the form of a pair of complex-conjugate numbers. These solutions correspond to actual real oscillation transitional processes.

There are also other proofs of a physical reality of complex numbers [1] – [5].

But these physically real values are complex only for the time of damped oscillations. In other words, only for this time the described above additional dimensions appear in our physical world.

Therefore, the solutions obtained using the formula (2) should be considered the only true solutions of quadratic algebraic equations, which correspond to tasks of any nature.

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