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Optical Fresnel-Wavelet Transforms for Certain Space of Generalized Functions

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Abstract

A theory of the diffraction Fresnel transform is extended to certain spaces of Schwartz distributions. The diffraction Fresnel transform is obtained as a continuous function in the space of Boehmians. Convergence with respect to δ *and* Δ *convergences is shown to be well defined .*

Keywords: *Fresnel Transform; Wavelet transform; Distribution space; Boehmian space.*

1 Introduction

Integral transforms play an important role in various fields of science. In optics, several integral transforms are of great importance. Some of these transforms are: the Fresnel transform [10, 12, 25, 26]; the fractional Fourier transform [5, 6, 11, 13, 18]; the linear canonical transform [22, 23]; the wavelet transform [20, 21]; the diffraction Fresnel transform [27,28] and, many others. The wavelet transform is described in [20, 21] as

$$\Omega_f(\mu,\lambda) = \frac{1}{\sqrt{\mu}}{}_R f(x)\,\psi^*\left(\frac{x-\lambda}{\mu}\right)dx \tag{1.1}$$

where $\psi(x)$ is named as mother wavelet satisfying ${}_R dx\psi(x) = 0$. The parameters λ and μ are, respectively, the translate and dilate of w, whereas, w^* is the conjugate of w. The optical diffraction transform is described by the Fresnel integration [27, 28]

$$F(x_2) = \frac{1}{\sqrt{2\pi i\gamma_1}} {}_R f(x_1) K_{x_2}(x_1)\, dx_1 \tag{1.2}$$

where $K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; x_1, x_2) = \exp\left(\frac{i}{2\gamma_1}\left(\alpha_1 x_1^2 - 2x_1x_2 + \alpha_2 x_2^2\right)\right)$ is the transform kernel whose parameters $\alpha_1, \gamma_1, \gamma_2, \alpha_2$ represent a ray transfer Matrix M, in an optical system, with $\alpha_1\alpha_2 - \gamma_1\gamma_2 = 1$.

We consider the combined optical transform obtained jointly from (1.1) and (1.2), named as the Fresnel-wavelet transform [10, Equ. (36)]

$$F_w(\lambda, \mu, x_2) = \frac{1}{\sqrt{2\pi i\gamma_1}} {}_R K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; x_1, x_2, \lambda, \mu) f(x_1)\, dx. \tag{1.3}$$

where
$K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; x_1, x_2, \lambda, \mu) = \exp\left(\frac{i}{2\gamma_1}\left(\frac{\alpha_1(x_1-\lambda)^2}{\mu^2} - \frac{2x_2(x_1-\lambda)}{\mu} + \alpha_2 x_2^2\right)\right)$
is the transform kernel.

Parameters: $\alpha_1, \gamma_1, \gamma_2$, and α_2 appearing in the above expression are elements of a 2×2 matrix with unit determinant. Since the general single-mode squeezing operator F in the generalized Fresnel transform is in wave optics, applications of F is a faithful representation in the Fresnel-wavelet transform [10]. Hence, the combined Fresnel-wavelet transform can be more conveniently studied by the general single-more squeezed operation.

In the literature, it has not yet been reported that the Fresnel-wavelet transform is extended to a space of generalized functions . Thus, we, in this article, aim at extending the Fresnel-wavelet transform to certain generalized function space (Boehmian space). Such extension is mainly related to the fact that the optical Fresnel-wavelet transform of a good function is certainly a C^∞ function.

We spread the article into five sections: In Section 2, we introduce the notion of Boehmian spaces. In Section 3, we consider the Boehmian space $\mathfrak{B}_\star$ from [4]. Section 4 is devoted for a general construction of the space $\mathfrak{B}_{Fw}$, where images of the extended Fresnel-wavelet transform lie. In the last section, we establish that the optical Fresnel-wavelet transform of an arbitrary Boehmian in $\mathfrak{B}_\star$ is another Boehmian in $\mathfrak{B}_{Fw}$. Moreover, we discuss linearity and continuity conditions with respect to certain types of convergence.

Let $\varepsilon(R_+)$ be the test function space of all C^∞ functions of arbitrary supports and $\varepsilon'(R_+)$ be its strong duals of distributions of compact supports. The kernel function $K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; x_1, x_2, \lambda, \mu)$ of the Fresnel-wavelet transform is clearly in $\varepsilon(R_+)$. This leads to define the distributional transform on the dual of distributions of compact support by the relation $F_w(\lambda, \mu, x_2) = \frac{1}{\sqrt{2\pi i\gamma_1}} \langle f(x_1), K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; x_1, x_2, \lambda, \mu)\rangle$, for every $f \in \varepsilon'(R_+)$.

2 General Boehmian Spaces

Let G be a linear space and H be a subspace of G. Assume to each pair of elements $f, g \in G$ and $\psi, \phi \in H$, is assigned the product $f \bullet g$ such that: $\phi \bullet \psi \in H$ and $\phi \bullet \psi = \psi \bullet \phi, \forall \phi, \psi \in H$, $(f \bullet \phi) \bullet \psi = f \bullet (\phi \bullet \psi)$,and $(f+g) \bullet \phi = f \bullet \phi + g \bullet \phi$, $k(f \bullet \phi) = (kf) \bullet \phi, k \in \mathbf{C}$. A family of sequences Δ from H, is said to be delta sequence if for each $f, g \in G, (\psi_n), (\delta_n) \in \Delta$, the following should satisfy: $f \bullet \delta_n = g \bullet \delta_n \, (n = 1, 2, ...)$, implies $f = g$, and $(\phi_n \bullet \psi_n) \in \Delta$. Let $\mathcal{O}$ be a class of pair of sequences

$$\mathcal{O} = \left\{ ((f_n), (\phi_n)) : (f_n) \subseteq G^{\mathbf{N}}, (\phi_n) \in \Delta \right\},$$

for each $n \in \mathbb{N}$. An element $((f_n), (\phi_n)) \in \mathcal{O}$ is said to be a quotient of sequences, denoted by $\frac{f_n}{\phi_n}$ if $f_i \bullet \phi_j = f_j \bullet \phi_i, \forall i, j \in \mathbb{N}$. Two quotients of sequences $\frac{f_n}{\phi_n}$ and $\frac{g_n}{\psi_n}$ are *equivalent,* $\frac{f_n}{\phi_n} \sim \frac{g_n}{\psi_n}$, if $f_i \bullet \psi_j = g_j \bullet \phi_i, \forall i, j \in \mathbb{N}$. The relation $\sim$ is an equivalent relation on $\mathcal{O}$ and hence, splits $\mathcal{O}$ into equivalence classes. The equivalence class containing $\frac{f_n}{\phi_n}$ is denoted by $\left[\frac{f_n}{\phi_n}\right]$ .These equivalence classes are called *Boehmians* and the *space of all Boehmians* is denoted by $\mathfrak{B}_\bullet$. The sum of two Boehmians and multiplication by a scalar is defined in a natural way $\left[\frac{f_n}{\phi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{(f_n \bullet \psi_n) + (g_n \bullet \phi_n)}{\phi_n \bullet \psi_n}\right]$ and $\alpha \left[\frac{f_n}{\phi_n}\right] = \left[\alpha \frac{f_n}{\phi_n}\right], \alpha \in \mathbb{C}$. The operation $\bullet$ and the differentiation are defined by $\left[\frac{f_n}{\phi_n}\right] \bullet \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n \bullet g_n}{\phi_n \bullet \psi_n}\right]$ and $D^\alpha \left[\frac{f_n}{\phi_n}\right] = \left[\frac{D^\alpha f_n}{\phi_n}\right]$. The relationship between the notion of convergence and the product $\bullet$ are given by:

$1-$*If* $f_n \to f$ *as* $n \to \infty$ *in* G *and,* $\phi \in H$ *is any fixed element, then*$f_n \bullet \phi \to f \bullet \phi$, as $n \to \infty$ in G.

$2-$*If* $f_n \to f$ *as* $n \to \infty$ *in* G *and* $(\delta_n) \in \Delta$, *then* $f_n \bullet \delta_n \to f$ as $n \to \infty$ in G. In $\mathfrak{B}_\bullet$ two types of convergence:

$\delta-$**convergence** : *Let* $(\beta_n) \in \mathfrak{B}_\bullet$ then $\beta_n \xrightarrow{\delta} \beta$, *if there is* $(\delta_n) \in \Delta$, $(\beta_n \bullet \delta_n), (\beta \bullet \delta_n) \in G, \forall k, n \in \mathbb{N}$,*and* $(\beta_n \bullet \delta_k) \to (\beta \bullet \delta_k)$ *as* $n \to \infty$,*in* G, $\forall k \in \mathbb{N}$.

$\Delta-$**convergence :** (β_n) *in* $\mathfrak{B}_\bullet$ *is* $\Delta-$*convergent to* β *in* $\mathfrak{B}_\bullet$, $\beta_n \xrightarrow{\Delta} \beta$, *if there is* $(\delta_n) \in \Delta$ *such that* $(\beta_n - \beta) \bullet \delta_n \in G, \forall n \in \mathbb{N}$, *and* $(\beta_n - \beta) \bullet \delta_n \to 0$ *as* $n \to \infty$ *in* G. For further analysis, see [1-4, 8, 14, 15, 17].

3 The Boehmian Space $\mathfrak{B}_\star$

Let f and g be C^∞ functions , over R_+. Then the convolution between f and g is defined by [4, Equ.3.2]

$$(f \triangleright g)(x) =_{R_+} f\left(xy^{-1}\right) \phi(y) \, y^{-1} dy, \qquad (3.1)$$

where x is a non-negative real number.

In the rest of investigations, it is more convenient to use the noation $\star$ instead of the used one, $\triangleright$. Further, we retain likewise notations and the results established in [4].

Let $\mathcal{D} = \mathcal{D}(R_+)$, be the Schwartz' space of all C^∞ complex-valued functions which are compactly supported in R_+. Then, we recall the following definition [4]

Definition 3.1. *Let* $\mathcal{S} = \left\{\phi \in \mathcal{D}(R_+) : \phi \geq 0 \text{ and } {}_{R_+}\phi = 1\right\}$ *and* Δ *be the set of all delta sequences* $\phi_n, n = 0, 1, 2, ...,$ *from* $\mathcal{S}$, *such that supp* $\phi_n \to 0$ *as* $n \to \infty$. *Then,* $(\phi_n) \in \Delta$ *if and only if* $(\phi_n) \in \mathcal{D}(R_+)$ *, and*

Δ_1 ${}_{\mathbf{R}_+}\phi_n = 1, \forall n \in \mathbb{N}$;

Δ_2 $\phi_n \geq 0, \forall n \in \mathbb{N}$;

Δ_3 $\inf\{\epsilon > 0 : supp\phi_n \subseteq (0, \epsilon)\} \to 0, as\ n \to \infty$.

The following are proved in [4]

Lemma 3.2. *Let* $f \in C^\infty(R_+)$ *and* $\phi \in \mathcal{S}$, *then* $f \star \phi \in C^\infty(R_+)$.

Lemma 3.3. *Let* $f, g \in C^\infty(R_+), \phi, \psi \in \mathcal{S}$ *and,* $\alpha \in \mathbb{C}$(The set of complex numbers) . *Then, the following are true*

(1) $(f + g) \star \phi = f \star \phi + g \star \phi$.

(2) $(\alpha f \star \phi) = \alpha (f \star \phi)$.

(3) $\phi \star \psi = \psi \star \phi$.

(4) $f \star (\phi \star \psi) = (f \star \phi) \star \psi$.

Theorem 3.4. *If* $\lim_{n\to\infty} f_n = f$ *, in* $C^\infty(R_+)$, *and* $\phi \in \mathcal{S}$ *, then*

$$\lim_{n\to\infty} f_n \star \phi = f \star \phi \text{ in } C^\infty(R_+).$$

Lemma 3.5. *Let* $f_n \to f$ *, in* $C^\infty(R_+)$, *and* $(\delta_n) \in \Delta$. *Then,*$f_n \star \delta_n \to f$ *in* $C^\infty(R_+)$.

Theorem 3.6 . *Given* $(\phi_n), (\psi_n) \in \Delta$. *Then,* $(\phi_n \star \psi_n) \in \Delta$..

After this sequence of results, the desired Boehmian space $\mathfrak{B}_\star$ was constructed in [4].

In $\mathfrak{B}_\star$, it is needful to have the following definition:

Definition 3.7. *Let* $\left[\frac{f_n}{\delta_n}\right], \left[\frac{g_n}{\phi_n}\right] \in \mathfrak{B}_\star$. *Then, the convolution of two Boehmians is defined as*

$$\left[\frac{f_n}{\delta_n}\right] \star \left[\frac{g_n}{\phi_n}\right] = \left[\frac{f_n \star g_n}{\delta_n \star \phi_n}\right], \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Equ.(3.2) is well-defined by Theorem 3.6 and Lemma 3.2.

Differentiation is defined by

$$D^{\alpha}\left[\frac{f_n}{\phi_n}\right]=\left[\frac{D^{\alpha}f_n}{\phi_n}\right].$$

Addition and scalar multiplication is defined in $\mathfrak{B}_{\star}$ as

$$\left[\frac{f_n}{\phi_n}\right]+\left[\frac{g_n}{\psi_n}\right]=\left[\frac{(f_n\star\psi_n)+(g_n\star\phi_n)}{\phi_n\star\psi_n}\right] and \alpha\left[\frac{f_n}{\phi_n}\right]=\left[\alpha\frac{f_n}{\phi_n}\right], \alpha\in\mathbb{C}.$$

4 The Boehmian Space $\mathfrak{B}_{F_w}$

Let $S\left(R_+^3\right)$, be the space of rapidly decreasing functions on $R_+^3=R_+\times R_+\times R_+$ [19, 7]. Then the Fresnel-wavelet transform of $f\in S\left(R_+^3\right)$ is indeed a $C^{\infty}\left(R_+\right)$ function. Let $f\in S\left(R_+^3\right)$ and $\psi\in C^{\infty}\left(R_+\right)$.

We define a mapping $\otimes: S\left(R_+^3\right)\rightarrow C^{\infty}\left(R_+\right)$ by

$$\left(f\otimes\psi\right)\left(\lambda,\mu,x_2\right)=\int_{R_+}f\left(\lambda t^{-1},\mu t^{-1},x_2\right)\psi\left(t\right)dt. \tag{4.1}$$

Following theorem is very needful

Lemma 4.1. *Let $f\in S\left(R_+^3\right)$ and $\psi\in C^{\infty}\left(R_+\right)$ then*

$$f\otimes\psi\in S\left(R_+^3\right).$$

Proof. To show $f\otimes\psi\in S$, we establish the following three relations

$$D_{\lambda}\left(f\otimes\psi\right)\left(\lambda,\mu,x_2\right)=\left(D_{\lambda}f\otimes\psi\right)\left(\lambda,\mu,x_2\right); \tag{4.2}$$

$$D_{\mu}\left(f\otimes\psi\right)\left(\lambda,\mu,x_2\right)=\left(D_{\lambda}f\otimes\psi\right)\left(\lambda,\mu,x_2\right); \tag{4.3}$$

and

$$D_{x_2}\left(f\otimes\psi\right)\left(\lambda,\mu,x_2\right)=\left(D_{x_2}f\otimes\psi\right)\left(\lambda,\mu,x_2\right). \tag{4.4}$$

To establish (4.2), let $\mu_0, x_{20}>0$ be fixed and, λ_0 vary over R_+ then

$D_{\lambda}\left(f\otimes\psi\right)\left(\lambda_0,\mu_0,x_{20}\right)=$

$$\lim_{\lambda\rightarrow\lambda_0 R_+}\frac{f\left(\lambda t^{-1},\mu_0 t^{-1},x_{20}\right)-f\left(\lambda_0 t^{-1},\mu_0 t^{-1},x_{20}\right)}{\lambda-\lambda_0}\psi\left(t\right)dt$$
$$=_{R_+}D_{\lambda}f\left(\lambda_0 t^{-1},\mu_0 t^{-1},x_{20}\right)\psi\left(t\right)dt$$
$$=\left(D_{\lambda}f\otimes\psi\right)\left(\lambda_0,\mu_0,x_{20}\right).$$

Thus,

$$D_{\lambda}\left(f\otimes\psi\right)\left(\lambda_0,\mu_0,x_{20}\right)=\left(D_{\lambda}f\otimes\psi\right)\left(\lambda_0,\mu_0,x_{20}\right).$$

Proof of (4.3) and (4.4) is analogous. Induction on the partial differention with respect to λ,μ and x_2 yields

$$D_\lambda^k (f \otimes \psi) = D_\lambda^k f \otimes \psi, D_\mu^k (f \otimes \psi) = D_\mu^k \otimes \psi \text{ and } D_{x_2}^k (f \otimes \psi) = D_{x_2}^k f \otimes \psi. \tag{5.5}$$

Hence, using the topology of S we have

$$\|f * \psi\|_S \leq \|\psi\|_{L^1} \|f\|_S \tag{1}$$

Lemma 4.2 *$f \otimes \psi_n \to f$ for every $f \in S\left(R_+^3\right)$ and $(\psi_n) \in \Delta$.*

Proof. Using $(4.2) - (4.4)$, mean value theorem and Δ_3 we write

$$\left|\lambda^i D_\lambda^k (f \otimes \psi_n - f)(\lambda, \mu, x_2)\right| = \left|\lambda^i \left(D_\lambda^k f \otimes \psi_n - D_\lambda^k f\right)(\lambda, \mu, x_2)\right|.$$

Hence, using (4.1), we get

$$\left|\lambda^i D_\lambda^k (f \otimes \psi_n - f)(\lambda, \mu, x_2)\right| \leq$$
$$_{R_+} \left|\lambda^i D_\lambda^k \left(f\left(\lambda t^{-1}, \mu t^{-1}, \varkappa_2\right) - f(\lambda, \mu, x_2)\right) \psi(t)\right| dt.$$

Hence the above expression approaches 0 as $n \to \infty$.

It can be similarly proved that

$\left|\mu^i D_\mu^k (f \otimes \psi_n - f)(\lambda, \mu, x_2)\right|$ and $\left|x_2^i D_{x_2}^k (f \otimes \psi_n - f)(\lambda, \mu, x_2)\right|$ approach 0 as $n \to \infty$.

This completes the proof of the lemma.

Lemma 4.3 *$f_n \otimes \psi \to f \otimes \psi$ for every $f_n, f \in S\left(R_+^3\right)$ and $\psi \in C^\infty(R_+)$.*

Proof. Employing (4.1)-(4.4) the lemma can easily be established in a manner similar to that of above Lemma . The Boehmian space $\mathfrak{B}_{Fw}(S, \otimes, \Delta)$ is therefore established. Operations such as addition, scalar multiplication, Differentiation and the operation $\otimes$ between two Boehmians in $\mathfrak{B}_{Fw}$ can be defined similarly as done in the previous section.

5 Fresnel-Wavelet Transform of Boehmians

Following is lemma suggesting a new definition for the Fresnel-wavelet transform of a Boehmian in the space $\mathfrak{B}_\star$.

Lemma 5.1 *Given $f \in S\left(R_+^3\right)$ and $\psi \in C^\infty(R_+)$ then*

$$F_w (f \star \psi)(\lambda, \mu, x_2) = f \otimes F_w \psi,$$

Proof. The Fresnel-Wavelet transform is written in the form

$$F_w (f(x_1))(\lambda, \mu, x_2) = \int_{R_+} f(x_1) K'_{\lambda,\mu,x_2}(x_1)\, dx_1 \tag{5.1}$$

where $K'_{\lambda,\mu,x_2}(x_1) = K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; x_1, x_2, \lambda, \mu)$ and

$$K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; x_1, x_2, \lambda, \mu) = \exp\left(\alpha_1 \tfrac{(x_1-\lambda)^2}{\mu} - 2x_2 \tfrac{(x_1-\lambda)^2}{\mu} + \alpha_2 x_2^2\right).$$

Hence

$$\begin{aligned} F_w(f \star \psi)(\lambda, \mu, x_2) &= \int_{R_+} (f \star \psi)(x_1) K'_{\lambda,\mu,x_2}(x_1)(x_1)\, dx_1 \\ &= \int_{R_+} \left(\int_{R_+} f\left(x_1 y^{-1}\right) \psi(y)\, y^{-1} dy \right) K'_{\lambda,\mu,x_2}(x_1)\, dx_1. \end{aligned}$$

The substitution $x_1 = yt$ implies

$$\begin{aligned} F_w(f \star \psi)(\lambda, \mu, x_2) &= \int_{R_+} f(t) \left(\int_{R_+} \psi(y) K'_{\lambda t^{-1}, \mu t^{-1}, x_2}(y)\, dy \right) dt \\ &= (F_w \psi \otimes f)(\lambda, \mu, x_2). \end{aligned}$$

This completes the proof. Hence, we define the Fresnel-wavelet transform of a Boehmian in $\mathfrak{B}_\star$ as

$$\mathfrak{S}\left[\tfrac{f_n}{\partial_n}\right] = \left[\tfrac{F_w f_n}{\partial_n}\right]. \tag{5.2}$$

in the space $\mathfrak{B}_{Fw}\left(S\left(R_+^3\right), \otimes, \Delta\right)$.

The definition, in (5.2), is well defined. For, if $\frac{f_n}{\delta_n} \sim \frac{f_n}{\delta_n}$ in $\mathfrak{B}_\star$ then $f_n \star \delta_m = g_m \star \delta_m$. Applying the Fresnel-wavelet transform and Theorem 5.1 imply $F_w f_n \otimes \delta_m = F_w g_m \otimes \delta_n,$. Hence $\frac{F_w f_n}{\delta_n} \sim \frac{F_w g_n}{\delta_n}$. Therefore $\left[\frac{F_w f_n}{\delta_n}\right] = \left[\frac{F_w g_n}{\delta_n}\right]$ in $\mathfrak{B}_{Fw}$.

Theorem 5.2. *The* $\mathfrak{S} : \mathfrak{B}_\star \to \mathfrak{B}_{Fw}$ *is linear.*

Proof. is obvious.

Theorem 5.3: *The* $\mathfrak{S} : \mathfrak{B}_\star \to \mathfrak{B}_{Fw}$ *is continuous with respect to* Δ *convergence.*

Proof. If $\beta_v \xrightarrow{\Delta} \beta$ in $\mathfrak{B}_\star$ then $(\beta_v \to \beta) \star \delta_v = \left[\frac{f_v \star \delta_i}{\delta_i}\right]$ for some $\delta_i \in \Delta, f_n \in C^\infty(R_+)$ and $f_v \to 0$ as $v \to \infty$. Thus $F_w f_v \to 0$ in $S\left(R_+^3\right)$ since $f_v \to 0$ as $v \to \infty$. Hence we conclude $F_w \beta_v \xrightarrow{\Delta} F_w \beta$ as $v \to \infty$. This completes the proof of te theorem.

Theorem 5.4. $\mathfrak{S} : \mathfrak{B}_\star \to \mathfrak{B}_{Fw}$ *is continuous with respect to the* δ *convergence.*

Proof. Let $\beta_v \overset{\delta}{\to} \beta$ as $v \to \infty$ in $\mathfrak{B}_\star$ then using [15] there can be found $f_{v,,j}, f_j$ such that

$$f_{v,j} \to f_j \ , \text{as } v \to \infty, \tag{5.3}$$

where

$$\left[\frac{f_{v,j}}{\delta_j}\right] = \beta_v \text{ and } \left[\frac{f_j}{\delta_j}\right] = \beta$$

Applying the Fresnel-wavelet transform on (5.3) we get

$$F_w f_{v,j} \to F_w f_j \ \text{ as } v \to \infty.$$

Thus

$$\left[\frac{F_w f_{v,j}}{\delta_j}\right] \to \left[\frac{F_w f_j}{\delta_j}\right].$$

Hence the theorem.

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