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Remez Algorithm Applied to the Best Uniform Polynomial Approximations

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Abstract

In this paper, we show the uniqueness of polynomial of best approximation of a function $f \in E = \mathcal{C}^0([a, b])$ by a polynomial of degree $\leq n$ and to characterize it. Then introduce the algorithm of Remez and prove its convergence. Illustrations and numerical simulations are given to prove the efficiency of our work.

Keywords: *Uniform approximation, Remez's algorithm, numerical simulation.*

1 Historical Introduction

Much of the work in approximation theory concerns the approximation of a given function f on some compact set I in the real or complex plane by polynomials. Classical results in this area deal with the best approximation problem

$$\min_{p \in \mathcal{P}_n} \|f - p\|_I, \tag{1}$$

where $\|g\|_I = \max_{x \in I} |g(x)|$, $\mathcal{P}_n$ denotes the set of polynomials of degree at most n. (Note that since in (1) we seek an approximation from a finite dimensional subspace, the minimum is indeed attained by some polynomial $p_* \in \mathcal{P}_n$).

Scalar approximation problems of the form (1) have been studied since the mid 1850s. Accordingly, numerous results on existence and uniqueness of the solution as well as estimates for the value of (1) are known.

Consider the following best polynomial approximation problem: Given a continuous function f defined on an interval $I = [a, b]$, find a function p_* in the space $\mathcal{P}_n$ of polynomials of degree less or equal to n such that

$$\|f - p_*\| \leq \|f - p\| \text{ for all } p \in \mathcal{P}_n,$$

where $\|.\|$ is the supremum norm $\|g\|= \max_{x\in I} |g(x)|$. The approximation p_* exists and is unique, and is known as the best, uniform, Chebyshev or minimax approximation to f.

Discussions of this problem can be found in almost every book on approximation theory Cheney ([2]), Davis ([5]), Laurent ([7]), Lorentz ([8]), Meinardus ([9]), Mhaskar and Pai ([10]), Powell ([13]), Remez ([15]) and Rice ([19]).

Starting with Chebyshev himself, the best approximation problem was studied from the second half of the 19^{th} century to the early 20^{th} century, and by 1915 the main results had been established (see Steffens [20]). A second wave of interest came in the 1950s and 1960s when computational aspects were investigated. The focus of much of this work was the algorithm introduced by Evgeny Yakovlevich Remez in a series of three papers published in 1934 ([16], [17] and [18]), and in this period were developed a deep understanding of its theoretical properties as well as numerous variations for its practical implementation.

A good introduction and another vision on this subject is in R. Pachón and L.N. Trefethen ([11])

In the 1970s the Remez algorithm also became a fundamental tool of digital signal processing, where it was introduced by Parks and McClellan ([12]) in the context of filter design.

Theorem 1.1 *(Equioscillation Property): A polynomial $p \in \mathcal{P}_n$ is the best approximation to f (that is, $p = p_*$) if and only if there exists a set of $n+2$ distinct points $\{x\}_{i=0}^{n+1}$ such that*

$$f(x_i) - p(x_i) = \lambda\sigma_i \|f - p_*\|, i = 0, ..., n+1, \tag{2}$$

where $\sigma_i := (-1)^i$ and $\lambda = 1$ or $\lambda = -1$ is fixed.

A set of points $E^* := \{x\}_{i=0}^{n+1}$ that satisfies (2) is called a reference. Analogous properties hold for other types of approximations such as best rational, CF and Padé (see Trefethen [21]).

Theorem 1.1 can be generalized for approximations that satisfy the Haar condition (see Laurent [7] and Powel [13], p. 77), of which polynomials are a special case. This allows us to look for best approximations in other sets of functions, for example trigonometric polynomials, which are the ones used for the Parks-McClellan algorithm. This paper works only with polynomials, but we believe that our methods can be carried over to the trigonometric case.

The second theorem, proved by de la Vallee Poussin in 1910 (de la Vallée Poussin [6]), establishes an inequality between the alternating error of a trial polynomial and the error of the best approximation (see Cheney [2], p. 77 and Powell [13], Thm. 7.7).

Theorem 1.2 *(de la Vallée Poussin) Let $p \in \mathcal{P}_n$ and $\{y_i\}_{i=0}^{n+1}$ be a set of $n+2$ distinct points in I such that*

$$sign(f(y_i) - p(y_i)) = \lambda\sigma_i, i = 0, ..., n+1,$$

with σ_i and λ defined as in Theorem 1.1. Then, for every $q \in \mathcal{P}_n$,

$$\min_i |f(y_i) - p(y_i)| \leq \max_i |f(y_i) - q(y_i)| \tag{3}$$

and in particular,

$$\min_i |f(y_i) - p(y_i)| \leq \|f - p_*\| \leq \|f - p\| \tag{4}$$

Theorem 1.2 asserts that a polynomial $p \in \mathcal{P}_n$ whose error oscillates $n+2$ times is near-best in the sense that

$$\|f - p\| \leq C\,\|f - p_*\|\,, \qquad C = \frac{\|f - p\|}{\min_i |f(y_i) - p(y_i)|} \geq 1.$$

The Remez algorithm constructs a sequence of trial references $\{E_k\}$ and trial polynomials $\{p_k\}$ that satisfy this alternation condition in such a way that $C \longrightarrow 1$ as $k \longrightarrow 1$. At the k^{th} step the algorithm starts with a trial reference E_k and then computes a polynomial p_k such that

$$f(x_i) - p_k(x_i) = \sigma_i h_k, \quad x_i \in E_k \tag{5}$$

where h_k is the levelled error (positive or negative), defined as

$$h_k := f(x_i) - p_k(x_i) \quad \text{for all} \quad x_i \in E_k.$$

Then, a new trial reference E_{k+1} is computed from the extrema of $f-\ p_k$ in such a way that $|h_{k+1}| \geq |h_k|$ is guaranteed. This monotonic increase of the levelled error is the key observation in showing that the algorithm converges to p_* (Powell [13], Thm 9.3).

In Section 2.1 we explain how to compute a trial polynomial and levelled error from a given trial reference, and in Section 2.2 we show how to adjust the trial reference from the error of the trial polynomial.

2 Main Results

Denote $\mathcal{P}_n$ the set of functions or one variable polynomials, with real coefficients, of degree at most n.

Consider a closed and bounded interval $[a,\ b]$ of $\mathbb{R}$.

Denote $\mathcal{C}^m([a,\ b])$ the space of the real functions m time continuously differentiables on $[a,\ b]$.

We are going to show the uniqueness of polynomial of the best approximation of a function $f \in E = \mathcal{C}^0([a, b])$ by a polynomial of degree $\leq n$ and characterize it. Then introduce Remez algorithm and prove its convergence. Illustrations and numerical simulations are given to prove the effectiveness of our work.

2.1 The Chebyshev Best Uniform Polynomial Approximation

Let us to place in the case where $E = \mathcal{C}^0([a, b])$ provided with the uniform convergence norm,

$$\|f\|_\infty = \max_{x\in[a,\ b]} |f(x)| .$$

We are going to show the uniqueness of polynomial of the best approximation of a function $f \in E$ by a polynomial of degree $\leq n$ and and characterize it.

First let us show the following lemma :

Lemma 2.1 *Let us given a function $f \in E$ and $(n+2)$ reals $x_0 < x_1 < ... < x_{n+1}$ belonging to $[a, b]$, there exists a unique polynomial $p \in \mathcal{P}_n$ such that*

$$\forall i = 1,\ 2,\ ...,\ n+1,\ \ f(x_i) - p(x_i) = (-1)^i(f(x_0) - p(x_0))\ ; \tag{6}$$

it is the polynomial $p \in \mathcal{P}_n$ such that

$$\forall q \in \mathcal{P}_n \ \ \text{with}\ \ q \neq p,\ \max_{0\leq i\leq n+1} |f(x_i) - p(x_i)| < \max_{0\leq i\leq n+1} |f(x_i) - q(x_i)| . \tag{7}$$

***Proof*.** We have

- equations (6) form a linear system of $(n+1)$ equations with $(n+1)$ unknowns, (the coefficients of p).
- To prove the existence, it is enough to prove the uniqueness, therefore it is enough to show that (6) implies (7). Let $q \in \mathcal{P}_n$ and suppose p verifies (6) and that

$$\max_{i=0}^{n+1} |f(x_i) - q(x_i)| \leq \max_{i=0}^{n+1} |f(x_i) - p(x_i)|\ \ ;$$

even if it means multiplying at the need f, p and q by (-1), we can suppose

$$f(x_0) - p(x_0) \geq 0.$$

Then we have

$$(-1)^i(q(x_i)-p(x_i)) \geq |f(x_i) - p(x_i)| - |f(x_i) - q(x_i)| \geq 0, \ (0 \leq i \leq n+1)$$

hence

$$(-1)^i \int_{x_i}^{x_{i+1}} (q^{'}(s) - p^{'}(s))ds \leq 0, \quad (0 \leq i \leq n).$$

We deduct from it that :

a) $q^{'} - p^{'} \equiv 0$, hence $q - p = c$ (constant) and, since $q - p$ alters at least $(n+1)$ times of sign, $q - p \equiv 0$;

b) or $\exists\ \xi_i \in]x_i,\ x_{i+1}[$ with $(-1)^i(q^{'}(\xi_i) - p^{'}(\xi_i)) < 0$, (for $0 \leq i \leq n$); according to the intermediate value theorem, the polynomial $q^{'} - p^{'}$ has a root on every interval $]\xi_i,\ \xi_{i+1}[$, therefore $q^{'} - p^{'}$ has at least n roots what is incompatible with the fact that

$$deg(q^{'} - p^{'}) \leq n - 1.$$

Definition 2.2 *We call a function f is equioscillated on $(k+1)$ points of $[a,\ b]$ if there exist $(k+1)$ reals $x_0 < x_1 < x_2 < ... < x_k$ belonging to $[a,\ b]$ such that*

$$\forall i = 0,\ 1,\ ...,\ k, \quad |f(x_i)| = \|f\|_\infty \ \ \textit{and} \ \forall i = 0,\ 1,\ ...,\ k-1,\ f(x_i) = -f(x_{i+1}).$$

Proposition 2.3 *Let $p \in \mathcal{P}_n$ be a polynomial such that $f - p$ equioscillates in $(n+2)$ points of $[a,\ b]$. Then we get, according to (7),*

$$q \in \mathcal{P}_n, \ \ \textit{with} \ \ q \neq p, \ \ \textit{implies} \ \ \|f - p\|_\infty < \|f - q\|_\infty .$$

Proof. Indeed,

$$\|f - p\|_\infty = \max_{i=0}^{n+1} |f(x_i) - p(x_i)| < \max_{i=0}^{n+1} |f(x_i) - q(x_i)| \leq \|f - q\|_\infty .$$

The polynomial p is then the best approximation of f in the Chebyshev meaning and this best approximation is then unique.

We are going to show that really there is a polynomial $p \in \mathcal{P}_n$ such that $f-p$ equioscillates in $(n+2)$ points of $[a,\ b]$; this polynomial will be obtained as a limit of sequence of polynomials constructed by means of the second Remez's algorithm.

Since the best approximation is unique, we can define the operator that assigns to each continuous function its best polynomial approximation of fixed degree. It is well known that this operator, although continuous, is nonlinear (for an example see Lorentz [8], p. 33), and so we need iterative methods to compute p_*. The Remez algorithm is one such method. Other important algorithms are the differential correction algorithms, which rely on ideas of linear programming (see Rabinowitz [14]).

A good discussion of the Remez's algorithm is doing by recalling two theorems that are essential to it. The first was first proved by Borel in 1905 (Borel [1], Cheney [2], p. 75 and Powell [13], p. 77).

2.2 Remez's Algorithm

Step 0 : Initialization.

Let us given $(n+2)$ distinct points $x_0^0 < x_1^0 < ... < x_{n+1}^0$.

Step k : (of the algorithm)

Assume known $(n+2)$ points $x_0^k < x_1^k < ... < x_{n+1}^k$. Associate to them the polynomial $p_k \in \mathcal{P}_n$ such that

$$\forall i = 1, 2, ..., n+1, \quad f(x_i^k) - p_k(x_i^k) = (-1)^i (f(x_0^k) - p_k(x_0^k))\,;$$

such polynomial exists according to the Lemma 2.1. Then there are two possible cases :

Case 1 :

$$\|f - p_k\|_\infty = \left| f(x_i^k) - p_k(x_i^k) \right|\,;$$

in this case $f - p_k$ equioscillates on the $(n+2)$ points x_i^k.

p_k achieves then the best approximation in the Chebyshev meaning; we stop the algorithm.

Case 2 : There exists $y \in [a,\ b]$ such that

$$\|f - p_k\|_\infty = |f(y) - p_k(y)| > \left| f(x_i^k) - p_k(x_i^k) \right|, \text{ for } i = 0, 1, ..., n+1.$$

Construct a new sequence of points $x_0^{k+1} < x_1^{k+1} < ... < x_{n+1}^{k+1}$ by exchanging in the previous sequence one of the points x_i^k with y in such a way that

$$\forall j = 1, 2, ..., n+1, \quad (f(x_j^{k+1}) - p_k(x_j^{k+1}))(f(x_{j-1}^{k+1}) - p_k(x_{j-1}^{k+1})) \leq 0.$$

Having said there are six possibilities :

1) $y \in \left[a,\ x_0^k\right[$ and

$$(f(x_0^k) - p_k(x_0^k))(f(y) - p_k(y)) \geq 0.$$

One takes

$$x_0^{k+1} = y \quad \text{and} \quad \forall j = 1,\, 2,\, ...,\, n+1, \quad x_j^{k+1} = x_j^k.$$

2) $y \in \left[a,\ x_0^k\right[$ and

$$(f(x_0^k) - p_k(x_0^k))(f(y) - p_k(y)) < 0.$$

One takes

$$x_0^{k+1} = y \quad \text{and} \quad \forall j = 1,\, 2,\, ...,\, n+1, \quad x_j^{k+1} = x_{j-1}^k.$$

3) $y \in \left]x_i^k,\ x_{i+1}^k\right[$ and

$$(f(x_i^k) - p_k(x_i^k))(f(y) - p_k(y)) \geq 0.$$

One takes

$$x_i^{k+1} = y \quad \text{and} \quad \forall j \neq i, \quad x_j^{k+1} = x_j^k.$$

4) $y \in \left]x_i^k,\ x_{i+1}^k\right[$ and

$$(f(x_i^k) - p_k(x_i^k))(f(y) - p_k(y)) < 0.$$

One takes

$$x_{i+1}^{k+1} = y \quad \text{and} \quad \forall j \neq i+1, \quad x_j^{k+1} = x_j^k.$$

5) $y \in \left]x_{n+1}^k,\ b\right]$ and

$$(f(x_{n+1}^k) - p_k(x_{n+1}^k))(f(y) - p_k(y)) \geq 0.$$

One takes

$$x_{n+1}^{k+1} = y \quad \text{and} \quad \forall j \leq n, \quad x_j^{k+1} = x_j^k.$$

6) $y \in \left]x_{n+1}^{k+1},\ b\right]$ and

$$(f(x_{n+1}^k) - p_k(x_{n+1}^k))(f(y) - p_k(y)) < 0.$$

We take

$$x_{n+1}^{k+1} = y \quad \text{and} \quad \forall j \leq n, \quad x_j^{k+1} = x_{j+1}^k.$$

When the Remez algorithm stops at the end of a finite number of steps, we acquire directly the best approximation of f.

We are now going to be situated in the case where the algorithm continues indefinitely and study its convergence.

Lemma 2.4 *Suppose that Remez algorithm does not stop at the end of a finite number of steps ; then we get that*

$$\forall k \geq 0, \quad \varepsilon_k < \varepsilon_{k+1} \leq \inf_{q \in \mathcal{P}_n} \|f - q\|_\infty \,,$$

where

$$\varepsilon_k = \max_{i=0}^{n+1} \left| f(x_i^k) - p_k(x_i^k) \right| .$$

Proof. The inequality

$$\varepsilon_{k+1} \leq \inf_{q \in \mathcal{P}_n} \|f - q\|_\infty$$

is an immediate consequence of the Lemma 2.1. It is thus enough to show that $\varepsilon_k < \varepsilon_{k+1}$.

Let i_0 be an index such that $x_{i_0}^{k+1}$ be the new point of the sequence x_i^{k+1} in comparison with the sequence x_i^k. Even if it means multiplying at the need f, p_k and p_{k+1} by (-1), we can suppose that

$$f(x_{i_0}^{k+1}) - p_k(x_{i_0}^{k+1}) = \|f - p_k\|_\infty \,.$$

Then we have

$$\forall i \neq i_0, \quad f(x_i^{k+1}) - p_k(x_i^{k+1}) = (-1)^{i-i_0}\, \varepsilon_k,$$

(according to the property of alternation of the signs).

Also, we have

$$\forall i, \quad f(x_i^{k+1}) - p_{k+1}(x_i^{k+1}) = (-1)^{i-i_0}\, \widetilde{\varepsilon}_{k+1},$$

by putting

$$\widetilde{\varepsilon}_{k+1} = \, f(x_{i_0}^{k+1}) - p_{k+1}(x_{i_0}^{k+1}), \;\; (= \pm\varepsilon_{k+1}),$$

we deduct from it that

$$p_{k+1}(x_{i_0}^{k+1}) - p_k(x_{i_0}^{k+1}) = \|f - p_k\|_\infty - \widetilde{\varepsilon}_{k+1}, \tag{8}$$

and

$$\forall i \neq i_0, \quad p_{k+1}(x_i^{k+1}) - p_k(x_i^{k+1}) = (-1)^{i-i_0}\, (\varepsilon_k - \widetilde{\varepsilon}_{k+1}).$$

The polynomial $p_{k+1} - p_k$ thus spells under the form

$$p_{k+1} - p_k = \, (\widetilde{\varepsilon}_{k+1} - \varepsilon_k)\, q_k,$$

where q_k is the unique polynomial of $\mathcal{P}_n$ such that

$$\forall i \neq i_0, \quad q_k(x_i^{k+1}) = (-1)^{i-i_0+1}. \tag{9}$$

Remark that

$$q_k(x_{i_0}^{k+1}) > 0,$$

else, or q_k would change $(n+1)$ time of sign points x_i^{k+1}, or q_k would change $(n-1)$ time of sign and would have a double zero at $x_{i_0}^{k+1}$, what is incompatible with the fact that $q_k \in \mathcal{P}_n$.

By returning to (6), we obtain,

$$(\widetilde{\varepsilon}_{k+1} - \varepsilon_k)\, q_k(x_{i_0}^{k+1}) = \|f - p_k\|_\infty - \widetilde{\varepsilon}_{k+1} \geq \inf_{q \in \mathfrak{p}_n} \|f - q\|_\infty - \varepsilon_{k+1} \geq 0,$$

we deduct that

$$\widetilde{\varepsilon}_{k+1} - \varepsilon_k \geq 0,$$

hence

$$\varepsilon_{k+1} = \widetilde{\varepsilon}_{k+1} \quad \text{and} \quad \varepsilon_{k+1} \geq \varepsilon_k.$$

Equality $\varepsilon_{k+1} = \varepsilon_k$ implies

$$\|f - p_k\|_\infty = \varepsilon_{k+1} = \varepsilon_k,$$

the algorithm would stop then to the k^{th} step.

Remark 2.1 *We proved that*

$$0 \leq \|f - p_k\|_\infty - \varepsilon_{k+1} \leq \|q_k\|_\infty (\varepsilon_{k+1} - \varepsilon_k). \tag{10}$$

Lemma 2.5 *there exists $\eta > 0$ such that*

$$\forall k \geq 0, \quad \forall i \in \{0,\, 1,\, ...,\, n\}, \ \left|x_{i+1}^k - x_i^k\right| \geq \eta.$$

Proof. By the absurd. If it was not true, according to the compactness of $[a,\ b]$, we could find an increasing sequence of integers $(k_s)_{s\in\mathbb{N}}$, such that

$$\forall i = 0,\, 1,\, ...,\, n+1, \ \lim_{(s \longrightarrow +\infty)} x_i^{k_s} = x_i \in [a,\ b]\,,$$

and such that $x_i,\ i = 0,\, 1,\, ...,\, n+1$, are not all distinct. In this case, there exists a polynomial $p \in \mathcal{P}_n$ such that

$$\forall i = 0,\, 1,\, ...,\, n+1,\ p(x_i) = f(x_i).$$

We have then, for all $s \geq 1$, according to (7)

$$0 \leq \varepsilon_0 < \varepsilon_1 \leq \varepsilon_{k_s} = \max_{i=0}^{n+1} \left|f(x_i^{k_s}) - p_{k_s}(x_i^{k_s})\right| \leq \max_{i=0}^{n+1} \left|f(x_i^{k_s}) - p(x_i^{k_s})\right|$$

which is incompatible with the fact that

$$\lim_{(s \longrightarrow +\infty)} \max_{i=0}^{n+1} \left|f(x_i^{k_s}) - p_{k_s}(x_i^{k_s})\right| = \max_{i=0}^{n+1} |f(x_i) - p(x_i)| = 0.$$

We have the following theorem :

Theorem 2.6 *There exists an unique polynomial p_n of degree $\leq n$ such that*

$$\forall i = 0,\, 1,\, ...,\, n, \quad p_n(x_i) = f(x_i).$$

This polynomial spells

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, \ell_i(x),$$

where we put

$$\ell_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{x\, -\, x_j}{x_i\, -\, x_j}.$$

Now we are liable to prove the following theorem :

Theorem 2.7 *Let us given a function $f \in \mathcal{C}^0([a,\ b])$, there exists one and only one polynomial of best approximation of f by a polynomial of degree $\leq n$ in the sense of the norm of the uniform convergence in $[a,\ b]$.*

This polynomial of best approximation p is characterized by the fact that $f - p$ equioscillates in $(n+2)$ points of $[a,\ b]$.

Moreover, Remez algorithm provides a sequence of polynomials p_k which converge uniformly to p on $[a,\ b]$.

Proof. According to Lemma 2.1, the sequence (ε_k) is increasing and bounded, thus it is convergent. Let

$$\varepsilon = \lim_{(k \longrightarrow +\infty)} \varepsilon_k.$$

According to expression (7) and Theorem 2.6, we get that

$$q_k(x) = \sum_{i\neq i_0} (-1)^{i-i_0+1} \prod_{j\neq i,\, j\neq i_0} \frac{x\, -\, x_j^{k+1}}{x_i^{k+1}\, -\, x_j^{k+1}}\,;$$

using a Lemma 2.4, we deduct that

$$\|q_k\|_\infty \leq \frac{(n+1)(b-a)^n}{\eta^n} = c_n \ \text{(independently of } k)\,,$$

which gives us by putting back in (8),

$$\lim_{(k \longrightarrow +\infty)} \|f - p_k\|_\infty = \varepsilon \tag{11}$$

The sequence of polynomials (p_k) is thus bounded in the finite dimensional space $\mathcal{P}_n$.

By using the compactness of $[a,\ b]$, we deduct that there exists an increasing sequence of integers $(k_s)_{s\in\mathbb{N}}$, and a polynomial $p \in \mathcal{P}_n$ such that

$$\lim_{(s\longrightarrow+\infty)} p_{k_s} = p$$

and

$$\forall i = 0,\ 1,\ ...,n+1,\ \lim_{(s\longrightarrow+\infty)} x_i^{k_s} = x_i \in [a,\ b]\,.$$

According to Lemma 2.1,

$$x_0 < x_1 < ... < x_{n+1}.$$

Moreover, passing to the limit in the relations, (with $k = k_s$),

$$\varepsilon_k = \left|f(x_i^k) - p_k(x_i^k)\right| \quad \text{and} \quad f(x_{i+1}^k) - p_k(x_{i+1}^k) = -(f(x_i^k) - p_k(x_i^k)),$$

we obtain

$$\varepsilon = |f(x_i) - p(x_i)| \ \text{ and } \ (f-p)(x_{i+1}) = -(f-p)(x_i).$$

Moreover, according to (9),

$$\varepsilon = \|f - p\|_\infty\,.$$

The function $f - p$ equioscillates then on $(n+2)$ points x_i. Therefore p is the unique polynomial which achieves the best approximation of f.

Moreover, according to expression (9), all limit point of the sequence (p_k) achieves the best approximation of f. The sequence (p_k) has thus an unique limit point. As it belongs to a compact (the sequence is bounded in a finite dimensional space), it is convergent.

Example 2.8 *We know that the $(n+2)^{th}$ Chebyshev polynomial is*

$$p_{n+1}(x) = 2^{-n}\,t_{n+1}(x),$$

with

$$t_n(x) = \cos(narc\ \cos x).$$

The polynomial p_{n+1} is then equioscillating at $(n+2)$ points,

$$x_k = \cos\frac{k\,\pi}{n\,+\,1},\ k = 0,\ 1,\ ...,\ n+1,$$

of $[-1,\ +1]$.

We deduct from the previous theorem that the polynomial q_n of degree n,

$$q_n(x) = x^{n+1} - p_{n+1}(x),$$

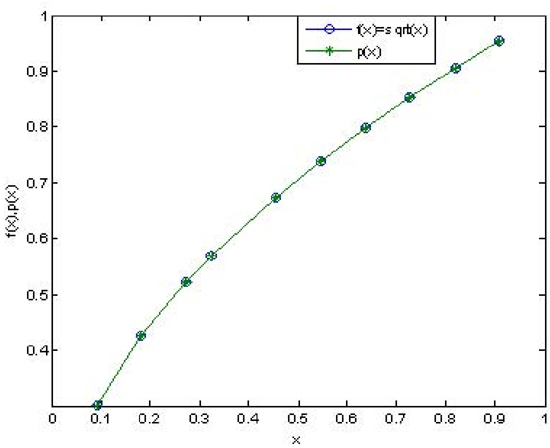
achieves the best approximation of the function x^{n+1} on $[-1,\ +1]$ in $\mathcal{P}_n$ in the uniform convergence meaning.

2.3 Numerical Experiments

In this section, we apply the above algorithm to several test examples. The proposed algorithm is programmed in Fortran 95 for working on the windows XP system. Numerical result prove that the method is efficient.

The computations of best approximations of $\sqrt{x}$ in $[0,\ 1]$ and $|x|$ in $[-1,\ 1]$ are equivalent.

Remez himself used his algorithm and this equivalence to compute the best approximations to $|x|$ by polynomials of odd degrees up to 11 with an accuracy of 10^{-5}.



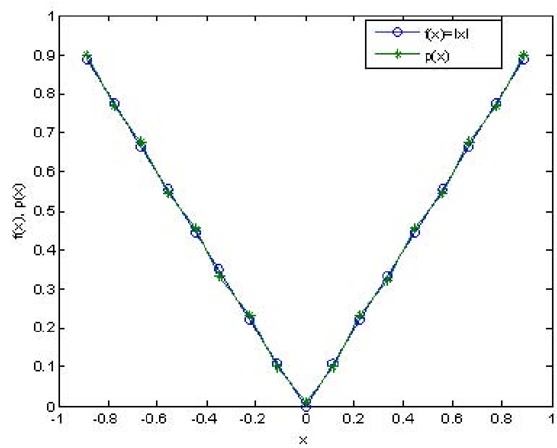
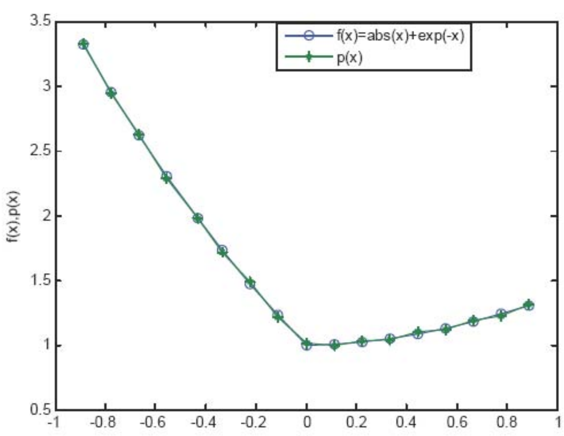


Figure (1.Left): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = \sqrt{x}$ *on* $[0, 1]$ *with an accuracy of* 10^{-5} *by the Remez algorithm.*

Figure (2.Right): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = |x|$ *with an accuracy of* 10^{-5} *on* $[-1, 1]$ *by the Remez algorithm.*



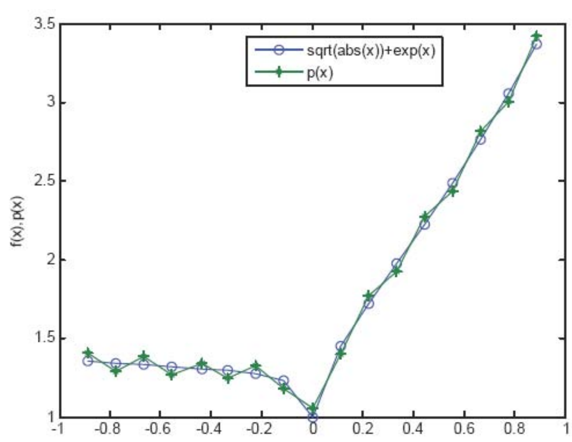
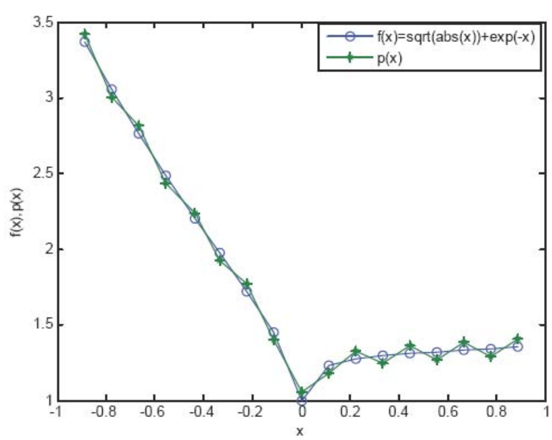


Figure (3.Left): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = |x| + exp(-x)$ *with an accuracy of* 10^{-10} *on* $[-1, 1]$ *by the Remez algorithm.*

Figure (4.Right): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = \sqrt{|x|} + exp(x)$ *with an accuracy of* 10^{-10} *on* $[-1, 1]$ *by the Remez algorithm.*



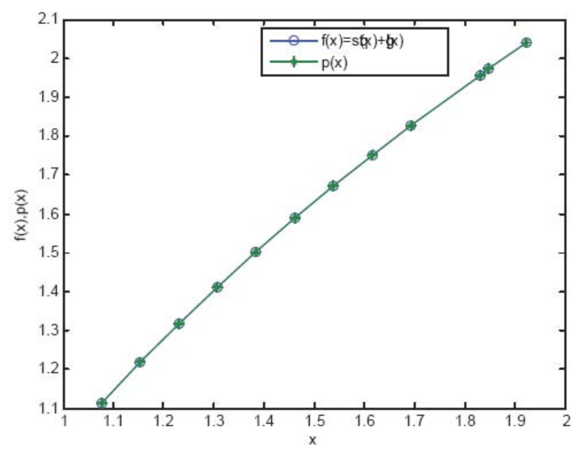
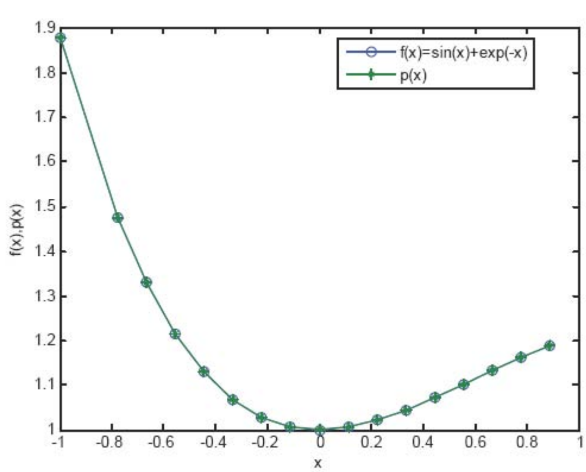


Figure (5.Left): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = \sqrt{|x|}+ exp(-x)$ *with an accuracy of* 10^{-10} *on* $[-1, 1]$ *by the Remez algorithm.*

Figure (6.Right): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = \sqrt{x} + log(x)$ *with an accuracy of* 10^{-5} *on* $[1, 2]$ *by the Remez algorithm.*



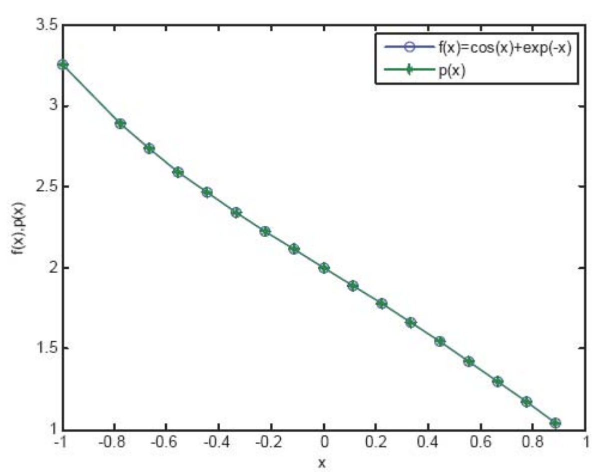
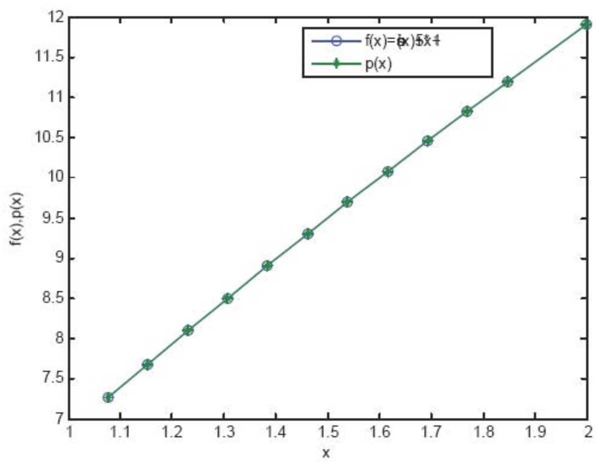


Figure (7.Left): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = sin(x)+exp(-x)$ *with an accuracy of* 10^{-10} *on* $[-1, 1]$ *by the Remez algorithm.*

Figure (8.Right): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = cos(x) + exp(-x)$ *with an accuracy of* 10^{-10} *on* $[-1, 1]$ *by the Remez algorithm.*



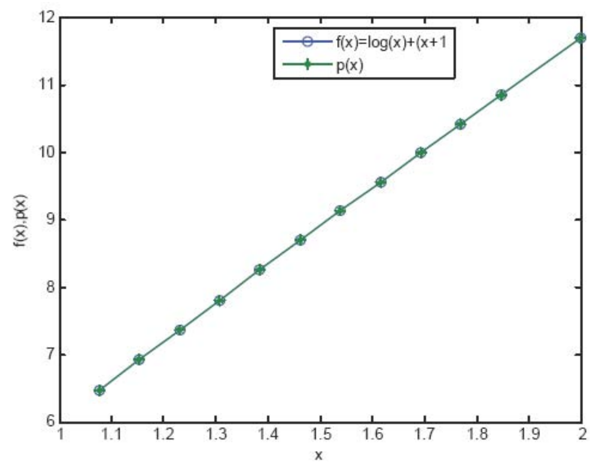


Figure (9.Left): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = sin(x) + exp(-x)$ *with an accuracy of* 10^{-10} *on* $[-1, 1]$ *by the Remez algorithm.*

Figure (10.Right): *Polynomial approximation* $p_n(x)$ *of the function* $f(x) = log(x) + 5x + 1$ *with an accuracy of* 10^{-4} *on* $[1, 2]$ *by the Remez algorithm.*

3 Conclusion

The computations of best approximations of our functions in these examples are best and give best results.

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