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The Category of Q-P Quantale Modules

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Abstract

In this paper, we introduce the concept of Q-P quantale modules. A series of categorical properties of Q-P quantale modules are studied, we prove that the category of Q-P quantale modules is not only pointed and connected, but also completed.

Keywords: *Q-P quantale modules; Morphisms; Category.*

1 Introduction

The first lattice analogy of a ring module was introduced in[1]by A.Joyal and M.Tierney. The idea of quantale module appeared in work[2] of S.Abransky and S.Vickers. With the development of the theory of quantale, many people have stuied this structure. The paper[3]investigate the relations of quantale module with quantale matrix. Every prime give wise to a strong module, which be generalized for prime matrix. Every quantale module can be viewed as a matrix.Pedre Resende [4] defined a sup-lattice bimorphism which are equivalent to Galois connections, and study their relation to quantale modules. Jan paska [5] introduced concept of Girard bimodules and studied of properties of Girard bimodules. In the paper [6][7]discussed a series of properties of Hilbert modules, and gave some important resultes on Hilbert modules. So, the quantale theory has aroused great interests of many scholar and experts, a great deal of new ideas and applications of quantale have been proposed in twenty years($[6-17]$).

In this paper, we introduced the concept of Q-P quantale modules, and study deeply and systemly the categorical properties of Q-P quantale modules, some interesting categorical properties of Q-P quantale modules are obtained.

For facts concerning category in general we refer to [18].

The paper is organized as follows. In section 1, we recall the notions of quantale modules and introduce the definition of Q-P quantale modules. In section 2, we prove that the category of the Q-P quantale modules is pointed and connected.The equalizer, the coequlizer, the product, the coproduct, the mutiplipullback in the category of Q-P quantale modules are studied.we prove that the each projection of the category of Q-P quantale modules is retract, and the category of Q-P quantale modules has kernel and cokernel.

2 Preliminaries

Definition 2.1 (10) *A quantale is a complete lattice Q with an associative binary operation&satisfying: a &(sup*$_\alpha b_\alpha$*)=sup*$_\alpha$*(a & b*$_\alpha$*)and (sup*$_\alpha b_\alpha$*) & a=sup*$_\alpha$*(b*$_\alpha$ *& a)for all a∈ Q and b*$_\alpha$ *⊆Q.*

Definition 2.2 (6) *Let Q be a quantale, a left module over Q(briefly, a left Q-module)is a sup-lattice M, together with a module action $\cdot$:Q×M⟶M satisfying*

(1) $(\bigvee_{i\in I} a_i)\cdot m = \bigvee_{i\in I}(a_i\cdot m)$;

(2) $a\cdot(\bigvee_{j\in J} m_j) = \bigvee_{j\in J}(a\cdot m_j)$;

(3) $(a\&b)\cdot m = a\cdot(b\cdot m)$. *for all a,b,*$a_i$ *∈Q, m,*m_j *∈M.*

The right modules are defined analogously.

If Q is untial and e·m=m for every m∈M, we say that M is unital.

Definition 2.3 (10) *Let M and N are Q-quantales. A mapping* $f: M\longrightarrow N$ *is said to be module homomorphism if* $f(\bigvee_{i\in I} m_i) = \bigvee_{i\in I} f(m_i)$*, and* $f(a\cdot m) = a\cdot f(m)$*for all a ∈Q, m,* m_i *∈M.*

Definition 2.4 *Let* Q, P *be a quantale, a Q-P quantale module over* Q, P *(briefly, a Q-P-module) is a complete lattice* M*, together with a mapping* $T: Q\times M\times P\longrightarrow M$ *satisfies the following conditions:*

(1) $T(\bigvee_{i\in I} a_i, m, \bigvee_{j\in J} b_j) = \bigvee_{i\in I}\bigvee_{j\in J} T(a_i, m, b_j)$;

(2) $T(a, (\bigvee_{k\in K} m_k), b) = \bigvee_{k\in K} T(a, m_k, b)$;

(3) $T(a\&b, m, c\&d) = T(a, T(b, m, c), d)$.

for all $a_i, a, b\in Q$*,*$b_j, c, d\in P$*,* $m_k, m\in M$*.*

We shall denote the Q-P quantale module M *over* Q, P *by* $(M,\ T)$*.*

Definition 2.5 *Let (M_1, T_1) and $(M_2, T_2)$are Q-P quantale modules. A mapping $f : M_1 \longrightarrow M_2$is saied to be Q-P quantale module homomorphism if satisfying*

(1) $f(\bigvee_{i\in I} m_i) = \bigvee_{i\in I} f(m_i)$;

(2) $f(T_1(a, m, b)) = T_2(a, f(m), b)$*for all* $a\in Q, b\in P$, $m_i \in M$.

Definition 2.6 *Let (M, T_M) be Q-P quantale module over Q and P, N is the subset of M, N is said to be submodule of M if N is closed under arbitrary join and T_M(a, n, b)∈N for all a∈ Q,b∈ P, n∈N.*

3 Equalizer, Intersection, Product and Pull Back

Definition 3.1 *Let* ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *be the category whose objects are the Q-P quantale modules, and morphisms are* $f : M \longrightarrow N$ *which is the Q-P quantale module homomorphism,i.e.,*

$\mathcal{O}b({}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}})=\{M : M$ *is Q-P quantale modules*$\}$,

$\mathcal{M}or({}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}})=\{f : M \longrightarrow N$ *is the Q-P quantale modules homorphism*$\}$

Hence, the category ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *is a concrete category.*

Theorem 3.2 *Every constant morphism of the category* ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *is exactly a zero morphism.*

Proof: Let Q,P are quantales, M and N are double quantale modules, the mapping f : M $\longrightarrow$ N is a morphism of Q-P quantale modules. Suppose id_M : M $\longrightarrow$N is a identity morphism, 0_M : M$\longrightarrow$M is a zero morphism. Since f∘id_M=f∘0_M,then f∘$\mathrm{id}_M(m)$=f∘$0_M(m)$ for all m∈M. Thus f(m)=0_N for all m∈M.

Conversely, If f(m)=0_N for all m∈M, then f∘r=f∘s for all r,s∈ Hom(M, N).

Theorem 3.3 *Every coconstant morphism of the category* ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *is exactly a zero morphism.*

Theorem 3.4 *The category* ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *is a pointed.*

Theorem 3.5 *(1) The category* ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *has terminal objects.*

(2) The category ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *has initial objects.*

(3) The category ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *is connected.*

Proof: (1) Let Q,P are quantales, (M, T_M) is a Q-P quantale module. It is easy to prove that $(\{0\}, T_{\{0\}})$ is a Q-P quantale module,define mapping f : M $\longrightarrow$ $\{0\}$such that f(m)=0 for all m∈M, then

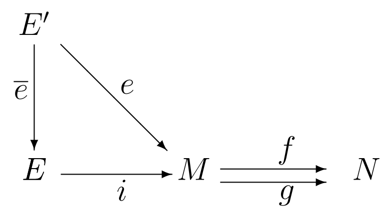
$f(\bigvee_{i\in I} m_i) = 0 = \bigvee_{i\in I} 0 = \bigvee_{i\in I} f(m_i)$,

$f(T_M(a,m,b) = 0 = T_{\{0\}}(a,0,b) = T_{\{0\}}(a,f(m),b)$ for all a∈Q,b∈P,m,m_i ∈M, therefore the mapping f is a Q-P quantale module morphism.

(2) Let M is a Q-P quantale module, f : $\{0\} \longrightarrow M$ is a Q-P quantale module morphism, then f(0)=0_M. We can see that f is only morphism in Hom({0}, M), therefore the category $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ has initial objects.

(3)It is clearly.

Theorem 3.6 *The category* $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$*has equalizers.*



Proof: Let Q,P are quantales, (M, T_M) and (N, T_N) are Q-P quantale modules, f and g : M⟶ N are Q-P quantale module morphisms. Suppose E=$\{m \in M \mid f(m) = g(m)\}$, then f($0_M$)=$0_N$=g($0_M$), implies $0_M \in E \neq \emptyset$.

For all $\{m_i \mid i \in I\} \subseteq E, a \in Q, b \in P$, m∈ E,

$f(\bigvee_{i\in I} m_i) = \bigvee_{i\in I} f(m_i) = \bigvee_{i\in I} g(m_i) = g(\bigvee_{i\in I} m_i), i.e., \bigvee_{i\in I} m_i \in E;$

$f(T_M(a,m,b)) = T_N(a,f(m),b) = T_N(a,g(m),b) = g(T_M(a,m,b)), i.e.,$ $T_M(a,m,b) \in E$, then E is a submodule of M, therefore the inclusion mapping i : E↪M is a Q-P quantale module morphism. We will show (E, i)is equalizer of f and g,

(1) It is clear know that f∘i=g∘i;

(2) Let E' is a Q-P quantale module, mapping e : $E' \longrightarrow M$ is a Q-P quantale module morphism, and satisfy f∘ e=g∘e. Define mapping $\overline{e} : E' \longrightarrow E$ such that $\overline{e}(x) = e(x)$for all $x \in E'$. Since$f(e(x)) = g(e(x))$ for all $x \in E'$, then $\bar{e}$ is well defined.

Let$\{x_i \mid i \in I\} \subseteq E', a \in Q, b \in P, x \in E'$, then $\overline{e}(\bigvee_{i\in I} x_i) = e(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} e(x_i) = \bigvee_{i\in I} \overline{e}(x_i)$;

$\overline{e}(T_M(a,x,b)) = e(T_M(a,x,b)) = T_M(a,e(x),b) = T_M(a,\overline{e}(x),b)$,thus $\overline{e}$is a Q-P quantale module morphism. For all $x \in E'$, we have that$(i \circ \overline{e})(x) = i(\overline{e}(x)) = i(e(x)) = e(x)$,then$e = i \circ \overline{e}$.

It's easy to prove that there is a only one Q-P quantale module morphism from $E'$to E with $e(x) = i \circ \overline{e}(x)$for all $x \in E'$,therefore(E, i)is the equalizer of f and g.

Theorem 3.7 *The category* $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$*has multiple equalizers.*

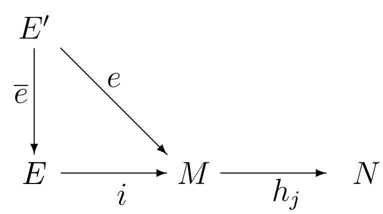
Proof: Let Q,P are quantales, (M, T_M) and (N, T_N) are Q-P quantale modules, $\{h_j \mid M \longrightarrow N\}_{j\in J}$ are Q-P quantale module morphisms. Suppose $E = \{m \in M \mid \forall j_1, j_2 \in J, h_{j_1}(m) = h_{j_2}(m)\}$. Since $h_{j_1}(0_M) = 0_N = h_{j_2}(0_M)$for all $j_1, j_2 \in J$, then$0_M \in E \neq \emptyset$.

Let$\{m_i \mid i \in I\} \subseteq E, a \in Q, b \in P, m \in E, j_1, j_2 \in J$, we have

$h_{j_1}(\bigvee_{i\in I} m_i) = \bigvee_{i\in I} h_{j_1}(m_i) = \bigvee_{i\in I} h_{j_2}(m_i) = h_{j_2}(\bigvee_{i\in I} m_i), i.e., \bigvee_{i\in I} m_i \in E$;

$h_{j_1}(T_M(a, m, b) = T_N(a, h_{j_1}(m), b) = T_N(a, h_{j_2}(m), b) = h_{j_2}(T_M(a, m, b)), i.e., T_M(a, m, b) \in E$,

thus the set E is a submodule of M, therefore the mapping i : E $\hookrightarrow M$ is a Q-P quantale module morphism,



We will prove that (E, i) is the multiple equalizer of $\{h_j\}_{j\in J}$.

(1) It' is clearly that $h_{j_1} \circ i = h_{j_2} \circ i$for all$j_1, j_2 \in J$;

(2) Suppose $(E', T_{E'}$ is a Q-P quantale module, mapping $e : E' \longrightarrow M$ is a Q-P quantale module morphism,and satisfy $h_{j_1} \circ e = h_{j_2} \circ e$ for all $j_1, j_2 \in J$. Define $\overline{e} : E' \longrightarrow E$, $\overline{e}(x) = e(x) forallx \in E'$. Because $h_{j_1}(e(x)) = h_{j_2}(e(x))$ for all $x \in E', j_1, j_2 \in J$,thus $\overline{e}(x) \in E$ for all $x \in E'$, therefore $\overline{e}$ is well defined.

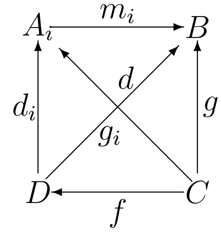
Let $\{x_i \mid i \in I\} \subseteq E', a \in Q, b \in P, x \in E^{'}$, then

$\overline{e}(\bigvee_{i\in I} x_i) = e(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} e(x_i) = \bigvee_{i\in I} \overline{e}(x_i)$;

$\overline{e}(T_{E'}(a, x, b)) = e(T_{E'}(a, x, b)) = T_M(a, e(x), b) = T_M(a, \overline{e}(x), b)$,

thus the mapping $\overline{e}$ is Q-P quantale module morphism. Since $(i \circ \overline{e})(x) = i(\overline{e}(x)) = i(e(x)) = e(x)$,then $e = i \circ \overline{e} forallx \in E'$.It's easy to prove that there is a only one Q-P quantale module morphism from E'to E with $e(x) = i \circ \overline{e}(x)$ for all $x \in E'$, therefore (E, i) is the equalizer of $\{h_j\}_{j\in J}$.

Theorem 3.8 *The category* ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *has intersection.*



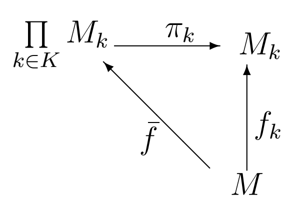
Proof: Let $(A_i, m_i)_{i\in I}$ is a family submodules of B,i.e.,there is a morphism $m_i : A_i \longrightarrow B$ for all $i \in I$. It's easy to prove that m_i is a homomorphism for all $i \in I$, then $m_i(A_i)$ is a submodule of B, and $m_i(A_i)$ is isomorphic to A_i.

Let mapping m_i^o is the corestrict of m_i on $m_i(A)$, $(m_i^o)^{-1}$ is the inverse mapping of m_i^o, $D = \bigcap\limits_{i\in I} m_i(A_i)$, It's evident that D is the submodule of B,thus D is the submodule of A_i for all $i \in I$. Suppose $d : D \longrightarrow B$is a inclusion map. We will prove that (D, d) is the intersection of $(A_i, m_i)_{i\in I}$in the category. In fact, we have that

(1) Let $d_i = (m_i^\circ)^{-1}|_D : D \longrightarrow A_i$ is the restrict of $(m_i^o)^{-1}$ on D for all $i \in I$, then d_i is the Q-P quantale module, and $d = m_i \circ d_i$ for all $i \in I$.

(2) Let $g : C \longrightarrow B$ and $g_i : C \longrightarrow A_i$ are the Q-P quantale modlue morphisms such that $g = m_i \circ g_i$ for all $i \in I$, then $g_i(C)$ is the submodule of D for all $i \in I$, thus $g(C) = m_i(g_i(C))$ is the submodule of $m_i(A_i)$, we know that g(C)is the submodule of D. Suppose f is the restrict of g on D,then f is a Q-P quantale module morphism, and $d \circ f = g$. It's easy to prove that there is a only one morphism such that $d \circ f = g$, therefore (D,d) is the intersection of $(A_i, m_i)_{i\in I}$ in the category.

Theorem 3.9 *The category* $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *has products.*



Proof: Let $\{(M_k, T_k) \mid k \in K\}$is a family Q-P quantale modules, define $T : Q \times \prod\limits_{k\in K} M_k \times Q \longrightarrow \prod\limits_{k\in K} M_k$ such that $T(a, m, b) = (T_k(a, m_k, b))_{k\in K}$ for all $a \in Q, b \in P, m = (m_k)_{k\in K}$, then

(1) $\prod\limits_{k\in K} M_k$ is a complete lattice with pointwise.

(2) $\prod\limits_{k\in K} M_k$ is a Q-P quantale module. In fact, for all $\{a_i \mid i \in I\} \subseteq Q$, $\{b_h \mid h \in H\} \subseteq P$, $\{m^{(j)} = (m_k^{(j)})_{k\in K} \mid j \in J\} \subseteq \prod\limits_{k\in K} M_k, a, b \in Q, c, d \in P, m = (m_k)_{k\in K} \in \prod\limits_{k\in K} M_k$, $k \in K$, we have that

$$(T(\bigvee_{i\in I} a_i, m, \bigvee_{h\in H} b_h))_k = T_k(\bigvee_{i\in I} a_i, m_k, \bigvee_{h\in H} b_h) = \bigvee_{i\in I}\bigvee_{h\in H} T_k(a_i, m_k, b_h)$$
$$= \bigvee_{i\in I}\bigvee_{h\in H} T(a_i, m, b_h)_k$$
$$= (\bigvee_{i\in I}\bigvee_{h\in H} T(a_i, m, b_h))_k;$$

$(T(a,\bigvee\limits_{j\in J}m^{(j)},c))_k = T_k(a,(\bigvee\limits_{j\in J}m^{(j)})_k,c) = T_k(a,\bigvee\limits_{j\in J}m_k^{(j)},c) = \bigvee\limits_{j\in J}T_k(a,m_k^{(j)},c) = \bigvee\limits_{j\in J}(T(a,m^{(j)},c))_k$;

$(T(a\&b,m,c\&d))_k = T_k(a\&b,m_k,c\&d) = T_k(a,T_k(b,m_k,c),d) = T_k(a,(T(b,m,c)_k,d)) = (T(a,T(b,m,c),d)_k$.

(3) Let $k\in K$, define $\pi_k:\prod\limits_{k\in K}M_k\longrightarrow M_k$ is a project,i.e.,$\pi_k(m)=m_k$ for all $m=(m_k)_{k\in K}\in\prod\limits_{k\in K}M_k$. Suppose $\{m^{(i)}=(m_k^{(i)})_{k\in K}\mid i\in I\}\subseteq\prod\limits_{k\in K}M_k$, $a\in Q, b\in P, m=(m_k)_{k\in K}\in\prod\limits_{k\in K}M_k$, then

$\pi_k(\bigvee\limits_{i\in I}m^{(i)}) = (\bigvee\limits_{i\in I}m^{(i)})_k = \bigvee\limits_{i\in I}m_k^{(i)} = \bigvee\limits_{i\in I}\pi_k(m^{(i)})$;

$\pi_k(T(a,m,b)) = (T(a,m,b))_k = T_k(a,m_k,b) = T_k(a,\pi_k(m),b)$,

therefore $\pi_k:\prod\limits_{k\in K}M_k\longrightarrow M_k$ is a Q-P quantale module morphism for all $k\in K$.

(4) we will prove that $(\prod\limits_{k\in K}M_k,\{\pi_k\}_{k\in K})$is the products of $\{M_k\mid k\in K\}$.

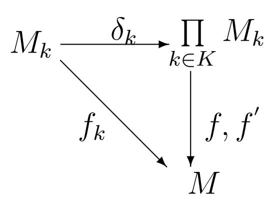
Let (M,T_M) is the a Q-P quantale module, $f_k:M\longrightarrow M_k$ for all $k\in K$, define $\overline{f}:M\longrightarrow M_k$ such that $(\overline{f}(m))_k=f_k(m)$ for all $m\in M, k\in K$. For all $a\in Q, b\in Q, m\in M, \{m_i\mid i\in I\}\subseteq M$,$k\in K$, we have

$(\overline{f}(\bigvee\limits_{i\in I}m_i))_k = f_k(\bigvee\limits_{i\in I}m_i) = \bigvee\limits_{i\in I}f_k(m_i) = \bigvee\limits_{i\in I}(\overline{f}(m_i))_k = (\bigvee\limits_{i\in I}\overline{f}(m_i))_k$,

$\overline{f}(T_M(a,m,b))_k = f_k(T_M(a,m,b)) = T_k(a,f_k(m),b) = T_k(a,(\overline{f}(m))_k,b) = (T_M(a,\overline{f}(m),b))_k$,

Therefore$\overline{f}$ is a Q-P quantale module morphism,It's clear that $\pi_k\circ\overline{f}=f_k$for all $k\in K$. It's easy to prove that there is a only one morphism satisfy the condition.Hence $(\prod\limits_{k\in K}M_k,\{\pi_k\}_{k\in K})$is the products of $\{M_k\mid k\in K\}$.

Theorem 3.10 *The category* $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *has coproducts.*



Proof: Let $\{(M_k,T_k)\mid k\in K\}$is a family Q-P quantale modules. By the theorem 2.7, we can see that $(\prod\limits_{k\in K}M_k,T)$ is a Q-P quantale modules.

For all $k\in K$, we have that

(1) For all $\{m_i\mid i\in I\}\subseteq M_k$, then $(\delta_k(\bigvee\limits_{i\in I}m_i))_k = \bigvee\limits_{i\in I}m_i = \bigvee\limits_{i\in I}(\delta_k(m_i))_k = (\bigvee\limits_{i\in I}\delta_k(m_i))_k$,

For all $l \in K$, and $l \neq k$, $(\delta_k(\bigvee\limits_{i\in I} m_i))_l = 0_{M_l} = \bigvee\limits_{i\in I} 0_{M_l} = \bigvee\limits_{i\in I}(\delta_k(m_i))_l = (\bigvee\limits_{i\in I} \delta_k(m_i))_l$,
i.e.,$\delta_k(\bigvee\limits_{i\in I} m_i) = \bigvee\limits_{i\in I} \delta_k(m_i)$;

(2)For all $a, b \in Q, b \in P, m \in M_k$, we have

$(\delta_k(T_k(a, m, b))_k = T_k(a, m, b) = T_k(a, (\delta_k(m))_k, b) = (T(a, \delta_k(m), b))_k$,

For all $l \in K$,and$l \neq k$,we have $(\delta_k(T_k(a, m, b)))_l = 0_{M_l} = T_l(a, 0_{M_l}, b) = T_l(a, (\delta_k(m))_l, b) = (T(a, \delta_k(m), b))_l$, i.e.,$\delta_k(T_k(a, m, b)) = T(a, \delta_k(m), b)$.

Therefore δ_k is a Q-P quantale module morphism for all $k \in K$.

Let M is a Q-P quantale module, mapping $f_k : M_k \longrightarrow M$is a Q-P quantale module morphism for all $k \in K$. Define $f : \prod\limits_{k\in K} M_k \longrightarrow M$such that $f(x) = \bigvee\limits_{k\in K} f_k(x_k)$with $x \in \prod\limits_{k\in K} M_k$, then for all $\{x^{(i)} \mid i \in I\} \subseteq \prod\limits_{k\in K} M_k, a \in Q, b \in P, x \in \prod\limits_{k\in K} M_k$,

$f(\bigvee\limits_{i\in I} x^{(i)}) = \bigvee\limits_{k\in K} f_k((\bigvee\limits_{i\in I} x^{(i)})_k) = \bigvee\limits_{k\in K} f_k(\bigvee\limits_{i\in I} x_k^{(i)}) = \bigvee\limits_{k\in K}(\bigvee\limits_{i\in I} f_k(x_k^{(i)}))$
$= \bigvee\limits_{i\in I}\bigvee\limits_{k\in K} f_k(x_k^{(i)}) = \bigvee\limits_{i\in I} f(x^{(i)})$;

$f(T(a, x, b)) = \bigvee\limits_{k\in K} f_k(T(a, x, b)_k) = \bigvee\limits_{k\in K} f_k(T_k(a, x_k, b)) = \bigvee\limits_{k\in K}(T_M(a, f_k(x_k), b))$
$= T_M(a, \bigvee\limits_{k\in K} f_k(x_k), b) = T_M(a, f(x), b)$,

thus f is a Q-P quantale module morphism.

Since $(f \circ \delta_k)(x) = f(\delta_k(x)) = \bigvee\limits_{l\in K} f_l(\delta_k(x))_l = f_k(x)$ for all $k \in K, x \in M_k$, then$f \circ \delta_k = f_k$ for all $k \in K$.

It's easy to prove that there is a only one morphism satisfy the condition. Thus $(\prod\limits_{k\in K} M_k, T)$ is the coproducts of $\{(M_k, T_k) \mid k \in K\}$.

Definition 3.11 *Let Q,P are quantales, (M, T_M) is a Q-P quantale module, $R \subseteq M \times M$. The set R is said to be a congruence of Q-P quantale module on the M. If R satisfy*

(1) R is an equivalence relation on M.

(2) If $(m_i, n_i) \in R$ for all $i \in I$, then $(\bigvee\limits_{i\in I} m_i, \bigvee\limits_{i\in I} n_i) \in R$;

(3) If$(m, n) \in R$, then $(T_M(a, m, b), T_M(a, n, b)) \in R$ for all $a \in Q, b \in P$.

Let Q,P is a quantale, M is a Q-P quantale module, R is a congrence of Q-P quantale module on M, define order on M/R is that $[m] \leq [n]$if and only if $[m \vee n] = [n]$ for all $[m], [n] \in M/R$.

Theorem 3.12 *Let Q,P are quantales, M is a Q-P quantale module, R is a congruence of Q-P quantale module on M, define $T_{M/R} : Q \times M/R \times P \longrightarrow M/R$ such that $T_{M/R}(a, [m], b) = [T_M(a, m, b)]$ for all $a \in Q, b \in P$, $[m] \in M/R$, then $(M/R, T_{M/R})$is a Q-P quantale module, and $\pi : m \mapsto [m] : M \longrightarrow M/R$is a Q-P quantale module morphism.*

Proof: We will prove that " $\leq$ "is a partial order on M/R, and $T_{M/R}$is well defined. In fact, for all $[m],[n],[l]\in M/R$,

(i) It's clearly that $[m]\leq[m]$;

(ii) Let$[m]\leq[n]$, $[n]\leq[m]$, then $[m\vee n]=[n]$ and $[n\vee m]=[m]$, thus $[m]=[n]$;

(iii) Let$[m]\leq[n]$, $[n]\leq[l]$,then $[m\vee n]=[n]$ and $[n\vee l]=[l]$, therefore$[m\vee l]=[m\vee(n\vee l)]=[(m\vee n)\vee(n\vee l)]=[n\vee l]=[l]$;

If $[m_1]=[m_2]$, then $(m_1,m_2)\in R$, $(T_M(a,m,b),T_M(a,n,b))\in R$ for all $a\in Q,b\in P$,i.e.,$[T_M(a,m,b)]=[T_M(a,n,b)]$, thus$T_{M/R}$is well defined.

(2)We will prove that $(M/R,\leq)$ is a complete lattice. Let $\{[m_i]\mid i\in I\}\subseteq M/R$, we have

(i) Since $[m_i\vee(\bigvee\limits_{i\in I}m_i)]=[\bigvee\limits_{i\in I}m_i]$ for all $i\in I$, then $[m_i]\leq[\bigvee\limits_{i\in I}m_i]$;

(ii) Let$[m]\in M/R$and $[m_i]\leq[m]$ for all $i\in I$, then $[m_i\vee m]=[m]$ for all $i\in I$, therefore $[(\bigvee\limits_{i\in I}m_i)\vee m]=[\bigvee\limits_{i\in I}(m_i\vee m)]=[m]$, i.e.,$[\bigvee\limits_{i\in I}m_i]\leq[m]$.

Thus $\bigvee\limits_{i\in I}^{M/R}[m_i]=[\bigvee\limits_{i\in I}m_i]$.

(3) For all $\{a_i\mid i\in I\}\subseteq Q$, $\{b_j\mid j\in J\}\subseteq P$, $\{[m_l]\mid l\in H\}\subseteq M/R$, $a,b\in Q,c,d\in P$, $[m]\in M/R$,we have that

(i) $T_{M/R}(\bigvee\limits_{i\in I}a_i,[m],\bigvee\limits_{j\in J}b_j)=[T_M(\bigvee\limits_{i\in I}a_i,m,\bigvee\limits_{j\in J}b_j)]=[\bigvee\limits_{i\in I}\bigvee\limits_{j\in J}T_M(a_i,m,b_j)]=\bigvee\limits_{i\in I}\bigvee\limits_{j\in J}T_M[a_i,m,b_j]=\bigvee\limits_{i\in I}\bigvee\limits_{j\in J}T_{M/R}(a_i,[m],b_j)$;

(ii) $T_{M/R}(a,(\bigvee\limits_{j\in J}[m_j]),c)=T_{M/R}(a,[\bigvee\limits_{j\in J}m_j],c)=[T_M(a,(\bigvee\limits_{j\in J}m_j),c)]$
$=[\bigvee\limits_{j\in J}T_M(a,m_j,c)]=\bigvee\limits_{j\in J}[T_M(a,m_j,c)]=\bigvee\limits_{j\in J}T_{M/R}(a,[m_j],c)$;

(iii) $T_{M/R}(a\&b,[m],c\&d)=[T_M(a\&b,m,c\&d)]=[T_M(a,T_M(b,m,c),d)]$
$=T_{M/R}(a,[T_M(b,m,c)],d)=T_{M/R}(a,T_{M/R}(b,[m],c),d)$.

Then is a Q-P quantale module.

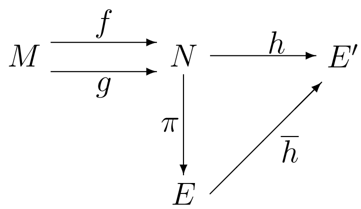
(4) For all $\{[m_i]\mid i\in I\}\subseteq M/R$, $a\in Q,b\in P$, $[m]\in M/R$,

$\pi(\bigvee\limits_{i\in I}m_i)=[\bigvee\limits_{i\in I}m_i]=\bigvee\limits_{i\in I}[m_i]=\bigvee\limits_{i\in I}\pi(m_i)$;

$\pi(T_M(a,m,b))=[T_M(a,m,b)]=T_{M/R}(a,[m],b)=T_{M/R}(a,\pi(m),b)$.
So $\pi:m\mapsto[m]:M\longrightarrow M/R$ is a Q-P quantale module morphism.

Theorem 3.13 *Let Q,P are quantales, M is a Q-P quantale module, then* $\triangle=\{(x,x)\mid x\in M\}$ *is a congrence of Q-P quantale module on M.*

Theorem 3.14 *The category* $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *has coequalizer.*



Proof: Let Q,P are quantales, (M, T_M) and (N, T_N) are Q-P quantale modules, f and g are Q-P quantale module morphisms. Suppose R is the smallest congrence of the Q-P quantale modules on N, which contain $\{(f(x), g(x)) \mid x \in M\}$. Let $E = N/R$, $\pi : N \longrightarrow N/R$is the canonical epimorphsim, by the theorem 2.11 that $(N/R, T_{N/R})$is a Q-P quantale module, $\pi$is a Q-P quantale module morphism. We will prove $(\pi, E)$is the coequalier of f and g. In fact,

(1) $\pi \circ f = \pi \circ g$is clearly.

(2) $(E', T_{E'})$is a Q-P quantale module, $h : N \longrightarrow E'$ is a Q-P quantale module morphism, and $h \circ f = h \circ g$. Let $R_1 = h^{-1}(\triangle)$,$\triangle = \{(x, x) \mid x \in E'\}$. By the theorem 2.12, we can see that $R_1$is a congrence of Q-P quantale module on N. Since $h(f(x)) = h(g(x))$ for all $x \in M$, then $(f(x), g(x)) \in R_1$, therefore R is the smallest congrence which contain $\{(f(x), g(x)) \mid x \in M\}$. Define $\overline{h} : N/R \longrightarrow E'$ such that $\overline{h}([n]) = h(n)$ for all $[n] \in Q/R$. Let $n_1, n_2 \in N$ and $(n_1, n_2) \in R$, then$(n_1, n_2) \in R_1$, we have that$h(n_1) = h(n_2)$, thereore$\overline{h}$is wll defined.

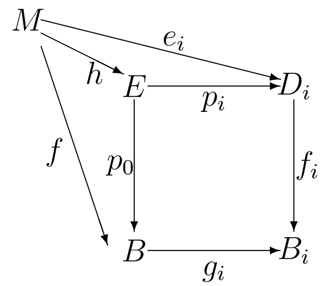
For all $\{[n_i] \mid i \in I\} \subseteq N/R$, $a \in Q, b \in P$, $[n] \in N/R$, we have that

$\overline{h}(\bigvee_{i\in I}[n_i]) = \overline{h}([\bigvee_{i\in I} n_i]) = h(\bigvee_{i\in I} n_i) = \bigvee_{i\in I} h(n_i) = \bigvee_{i\in I} \overline{h}([n_i])$,

$\overline{h}(T_{N/R}(a, [n], b)) = \overline{h}([T(a, n, b)]) = h(T(a, n, b)) = T_{E'}(a, h(n), b)$
$= T_{E'}(a, \overline{h}([n]), b)$,

thus $\overline{h}$is a Q-P quantale module morphism. It's easy to prove that $\overline{h} \circ \pi = h$ and $\overline{h}$is the only one morphism which satisfy the above condition. Therefore (π, E) is the coequalizer of f and g.

Theorem 3.15 *The category*$_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ *has mutiple pullback.*



Proof: Let I is a set, (B, T_B) and $(D_i, T_{D_i})_{i\in I}$ are Q-P quantale modules. $g_i : B \longrightarrow B_i, f_i : D_i \longrightarrow B_i$are Q-P quantale modules morphisms for all $i \in I$.

Suppose $E = \{x \in B \times \prod_{i\in I} D_i \mid \forall i \in I, g_i(x_0) = f_i(x_i), x_0 \in B\}$. We will prove that $E$is the submodule of $B \times \prod_{i\in I} D_i$.

(1) For all $\{x_j \mid j \in J\} \subseteq B \times \prod_{i\in I} D_i$, we have $g_i((\bigvee_{j\in J} x_j)_0) = g_i(\bigvee_{j\in J}(x_j)_0) = \bigvee_{j\in J} g_i((x_j)_0) = \bigvee_{j\in J} f_i((x_j)_i) = f_i(\bigvee_{j\in J}(x_j)_i) = f_i((\bigvee_{j\in J} x_j)_i)$;

(2) For all $x \in B \times \prod_{i\in I} D_i$, $a \in Q, b \in P$, we have $g_i((T(a,x,b)_0) = g_i(T_B(a,x_0,b)) = T_{B_i}(a, g_i(x_0), b) = T_B(a, f_i(x_i), b) = f_i(T_{D_i}(a, x_i, b))$;

then E is a submodule of $B \times \prod_{i\in I} D_i$.

Let $p_0, p_i (i \in I)$ are projects from $B \times \prod_{i\in I} D_i (i \in I)$ to B and D_i restrict on E respectively, then$g_i \circ p_0 = f_i \circ p_i$, for all$i \in I$, we have gained a family commutative squares.

Let M is a Q-P quantale module,suppose $(x_q)_0 = f(q), (x_q)_i = e_i(q)$, for all $q \in M$, then $x_q \in B \times \prod_{i\in I} D_i$. Since $f_i \circ e_i = g_i \circ f$, for all $i \in I$, then $x_q \in E$.

Define $h : M \longrightarrow E$such that$h(q) = x_q$ for all $q \in Q$,we will prove that h is a double quantale module morphism. For all $m \in M$, $a \in Q, b \in Q$, $\{a_j\}_{j\in J} \subseteq M$, $i \in I$, then

(1) since $(h(\bigvee_{j\in J} a_j))_0 = f(\bigvee_{j\in J} a_j) = \bigvee_{j\in J} f(a_j) = \bigvee_{j\in J}(h(a_j))_0$,

$(h(\bigvee_{j\in J} a_j))_i = e_i(\bigvee_{j\in J} a_j) = \bigvee_{j\in J} e_i(a_j) = \bigvee_{j\in J}(h(a_j))_i$, then$h(\bigvee_{j\in J} a_j) = \bigvee_{j\in J} h(a_j)$;

(2) $(h(T_M(a,m,b))_0 = f(T_M(a,m,b)) = T_B(a, f(m), b) = T_B(a, (h(m))_0, b)$,

$(h(T_M(a,m,b))_i = e_i(T_M(a,m,b)) = T_{D_i}(a, e_i(m), b) = T_{D_i}(a, (h(m))_i, b)$;

hence h is a Q-P quantale module morphism, and $f = p_0 \circ h, e_i = p_i \circ h$. It's easy to prove that h is the only Q-P quantale module morphism which satisfy the conditions, therefore the category${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{Q}}$has mutiple pullback.

Theorem 3.16 *The category ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ has kernel.*

Proof: Let Q,P are quantales, M and N are Q-P quantale modules, $f : M \longrightarrow N$is a Q-P quantale modules morphism, $0_{M,N} : M \longrightarrow N$ such that f(m)=0 for all $m \in M$. Suppose $E = \{x \in M \mid f(x) = 0\}$, then $(E, i : E \hookrightarrow M)$is a equalizer of f and$0_{M,N}$, then f has kernel.

Theorem 3.17 *The category${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$has cokernel.*

Proof: Let Q,P are quantales, M and N are Q-P quantale modules, $f : M \longrightarrow N$is a Q-P quantale modules morphism, $0_{M,N} : M \longrightarrow N$ such that f(m)=0 for all $m \in M$. Let R is the smallest congrvence which contain $\{(f(m), 0) \mid m \in M\}$, by the theorem 3.14 we know that $(E = N/R, \pi : N \hookrightarrow E)$is the coequalizer of f and$0_{M,N}$, then f has cokernel.

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References

[1] A. Joyal and M. Tiernry, An extension of the Galois theory of Grothendieck, *Amer. Math. Soc. Memoirs*, 309(1984)108-118.

[2] S. Abramsky and S. Vickers, Quantales, observational logic and process semantics, *Math. Struct. Comput. Sci.*, 3(1993), 161-227.

[3] D. Kruml, Spatial quantales [J], *Applied Categorical Structures*, 10(2002), 49-62.

[4] P. Resende, Sup-lattice 2-forms and quantales [J], *Journal of Algebra*, 276(2004), 143-167.

[5] J. Paseka, A note on Girard bimodules [J], *International Journal of Theoretical Physics*, 39(3) (2000), 805-812.

[6] J. Paseka, Hilbter Q-modules and nuclear ideals in the category of $\vee$-semilattices with a duality, *CTCS'99: Conference on Category Theory and Computer Science (Edinburgh)*, Elsevier, Amsterdam, Paper No. 29019(1999), 19 (electronic).

[7] Y.H. Zhou and B. Zhao, The free objects in the category of involutive quantales and its property of well-powered, *Chinese Journal of Engineering Mathematics*, 23(2) (2006), 216-224 (In Chinese).

[8] F. Miraglia and U. Solitro, Sheaves over right sided idempotent quantales, *Logic J. IGPL*, 6(4) (1998), 545-600.

[9] M.E. Coniglio and F. Miraglia, Modules in the category of sheaves over quantales, *Annals of Pure and Applied Logic*, 108(2001), 103-136.

[10] K.I. Rosenthal, *Quantales and their Applications*, Longman Scientific and Techical, London, (1990).

[11] Z. Bin, The inverse limit in the category of topological molecular lattices [J], *Fuzzy and Systems*, 118(2001), 574-554.

[12] S. Abramsky and S. Vickers, Quantales, observational logic and process semantics, *Math. Struct. Comput. Sci.*, 3(1993), 161-227.

[13] D. Kruml, Spatial quantales, *Applied Categorial Structures*, 10(2002), 49-62.

[14] P. Resende, Sup-lattice 2-forms and quantales, *Journal of Algebra*, 276(2004), 143-167.

[15] P. Resende, Tropological systems are points of quantales, *Journal of Pure and Applied Algebra*, 173(2002), 87-120.

[16] J. Paseka, A note on Girard bimodules, *International Journal of Theoretical Physics*, 39(3) (2000), 805-812.

[17] J. Paseka, Morita equivalence in the context of hilbert modules, *Proceedings of the Ninth Prague Topological Symposium Contribution Papers from the Symposium Held in Prague*, Czech Republic, August 19-25 (2001), 223-251.

[18] H. Herrilich and E. Strecker, *Category Theory*, Berlin: Heldermann Verlag, (1979).