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A Generalization of Badshah and Singh's Result through Compatibility

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Abstract

Using the idea of compatibility of self-maps, due to Gerald Jungck, we obtain a modest generalization of Badshah and Singh's result.

Keywords: *Compatible self-maps, continuity and common fixed point.*

1 Introduction

In this paper, X denotes a complete metric space with metric d. If f and g are self-maps on X, we write fg for their composition $f \circ g$, f^n for the composition of f of order n, and fx for the f-image of a point x in X.

Badshah and Singh [1] proved the following result for commuting self-maps:

Theorem 1.1 *Let f and g be self-maps on X satisfying the inclusion*

$$f(X) \subset g(x) \tag{1}$$

and the inequality

$$\begin{aligned}[d(fx,fy)]^2 \leq \ & \alpha\,[d(fx,gx)d(fy,gy)+d(fy,gx)d(fx,gy)] \\ & +\beta\,[d(fx,gx)d(fx,gy)+d(fy,gx)d(fy,gy)] \\ & \quad for\ all \quad x,y \in X,\end{aligned} \tag{2}$$

where

(a) *α and β are non negative constants with $\alpha + 2\beta \leq 1$,*

(b) *(f,g) is a commuting pair,*

(c) *f and g are continuous.*

Then f and g have a unique common fixed point.

We prove a generalization of Theorem 1.1 by replacing the condition (b) with a weaker condition, namely the compatibility, and dropping the continuity of f. In fact according to Gerald Jungck [2], self-maps f and g on X form a compatible pair, if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0 \tag{3}$$

whenever $\langle x_n \rangle_{n=0}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \tag{4}$$

for some $t \in X$.

It is easy to observe that every commuting pair of self-maps is necessarily compatible. However, one can refer to [2], [3], and [4] for compatible self-maps which are not commuting.

Our result is

Theorem 1.2 *Let f and g be self-maps on X satisfying the inclusion (1), and the inequality (2) with the choice (a). If g is continuous, and (f, g) is a compatible pair, then f and g have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary.

In view of (1), we can choose points $x_1, x_2, \ldots, x_n, \ldots$ in X inductively such that

$$fx_{n-1} = gx_n = y_n \quad \text{forall} \quad n \geq 1. \tag{5}$$

We now prove that $\langle y_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence.

Writing $x = x_{n-1}$ and $y = x_n$ in (2) and using (5), we get

$$\begin{aligned}
[d(y_n, y_{n+1})]^2 &= [d(fx_{n-1}, fx_n)]^2 \\
&\leq \alpha[d(fx_{n-1}, gx_{n-1})d(fx_n, gx_n) + d(fx_n, gx_{n-1})d(fx_{n-1}, gx_n)] \\
&\quad +\beta[d(fx_{n-1}, gx_{n-1})d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})d(fx_n, gx_n)] \\
&= \alpha[d(y_n, y_{n-1})d(y_{n+1}, y_n) + d(y_{n+1}, y_{n-1}).0] \\
&\quad\quad +\beta[d(y_n, y_{n-1}).0 + d(y_{n+1}, y_{n-1})d(y_{n+1}, y_n)] \\
&= [d(y_n, y_{n+1})]\,[\alpha d(y_n, y_{n-1}) + \beta d(y_{n+1}, y_{n-1})]
\end{aligned}$$

or

$$\begin{aligned}
d(y_n, y_{n+1}) &= \alpha d(y_n, y_{n-1}) + \beta d(y_{n+1}, y_{n-1}) \\
&\leq \alpha d(y_n, y_{n-1}) + \beta[d(y_{n-1}, y_n) + d(y_n, y_{n+1})
\end{aligned}$$

so that $d(y_n, y_{n+1}) \leq \left(\frac{\alpha+\beta}{1-\beta}\right) d(y_n, y_{n-1})$.

Repeating this argument, we get

$$d(y_n, y_{n+1}) \leq q^{n-2} d(y_n, y_{n-1}), \tag{6}$$

where $q = \frac{\alpha+\beta}{1-\beta}$.

Now from (a), we see that $\alpha + \beta < 1 - \beta$ or $q < 1$.

Thus for any positive integer k, (6), and the triangle inequality give

$$\begin{aligned} d(y_n, y_{n+k}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+k-1}, d_{n+k}) \\ &\leq d(y_2, y_1)\left(q^{n-2} + q^{n-1} + \cdots + q^{n+k-3}\right) \\ &= q^{n-2}\left(1 + q + \cdots + q^{k-1}\right) d(y_2, y_1). \end{aligned}$$

Proceeding the limit as $n \to \infty$, this gives $d(y_n, y_{n+k}) \to 0$, since $q^{n-2} \to 0$. Hence $\langle y_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in X, and hence converges in it.

That is there is a point $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} y_n = z. \tag{7}$$

Now the compatibility of f and g, and (7) imply that

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0, \tag{8}$$

while the sequenctial property of the continuiy of g and (7) give

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} g^2x_n = gz. \tag{9}$$

Hence it follows from (8) and (9), that

$$\lim_{n\to\infty} d(fgx_n, gz) = 0 \quad \text{or} \quad \lim_{n\to\infty} fgx_n = gz. \tag{10}$$

But the use of (2) yields

$$\begin{aligned} [d(fgx_n, fz)]^2 &\leq \alpha[d(fgx_n, g^2x_n)d(fz, gz) + d(fz, g^2x_n)d(fgx_n, gz)] \\ &\quad +\beta[d(fgx_n, g^2x_n)d(fgx_n, gz) + d(fz, g^2x_n)d(fz, gz)]. \end{aligned}$$

Now applying the limit as $n \to \infty$ in this, and using (9) and (10),

$$\begin{aligned} [d(gz, fz)]^2 &\leq \alpha[d(gz, gz)d(fz, gz) + d(fz, gz)d(gz, gz) \\ &\quad +\beta[d(gz, gz)d(gz, gz) + d(fz, gz)d(fz, gz)] \end{aligned}$$

or

$$[d(gz, fz)]^2 \leq \beta\,[d(fz, gz)]^2$$

so that

$$gz = fz. \tag{11}$$

Finally again from (2), we see that

$$\begin{aligned}[d(fx_n, fz)]^2 \leq \; & \alpha[d(fx_n, gx_n)d(fz, gz) + d(fz, gx_n)d(fx_n, gz)] \\ & +\beta[d(fx_n, gx_n)d(fx_n, gz) + d(fz, gx_n)d(fz, gz)].\end{aligned}$$

The limiting case of this as $n \to \infty$, (7), and (9) would imply that

$$[d(z, fz)]^2 \leq \alpha\,[d(fz, z)]^2 \quad \text{or} \quad fz = z.$$

Thus $gz = fz = z$, that is z is a common fixed point of f and g.

The uniqueness of the common fixed point follows easily from the inequality (2).

Remark 1: Theorem 1.2 does not require the continuity of f.

Remark 2: Since every commuting pair is compatible, Theorem 1.1 follows as a particular case of *our result*.

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