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On Some Integrals Involving Laguerre Polynomials of Several Variables

Fadhle B.F. Mohsen[1], Ahmed Ali Atash[2] and Salem Saleh Barahma[3]

[1]Department of Mathematics
Faculty of Education-Zingibar, Aden University, Yemen
E-mail: mfazalmohsen@yahoo.com
[2]Department of Mathematics
Faculty of Education-Shabowh, Aden University, Yemen
E-mail: ah-a-atash@hotmail.com
[3]Department of Mathematics
Faculty of Education-Aden, Aden University, Yemen
E-mail: salemalqasemi@yahoo.com

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Abstract

The main object of the present work is to derive some general integral formulas (single, double and multiple) involving Laguerre polynomials of several variables. A number of known and new integral formulas involving Laguerre polynomials of two and three variables are obtained as special cases of our general formulas.

Keywords: *Laguerre polynomials, Hypergeometric functions, Integral formulas, Lauricella's function, Kampé de Fériet function, Exton's functions, Chandel function.*

1 Introduction

In 1991, Ragab [7] defined the Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ as follows:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r\, L_{n-r}^{(\alpha)}(x)}{r!\Gamma(\alpha+n-r+1)\Gamma(\beta+r+1)} \tag{1.1}$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomials of one variable [8]

The definition (1.1) is equivalent to the following explicit representation of $L_n^{(\alpha,\beta)}(x,y)$, given by Ragab [7]:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n\,(\beta+1)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\,y^r\,x^s}{(\alpha+1)_s\,(\beta+1)_r\,r!s!} \tag{1.2}$$

It may be remarked that (1.2) can be written as

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\,\Psi_2\left[-n;\,\alpha+1,\ \beta+1;x,y\right] \tag{1.3}$$

where Ψ_2 is the confluent hypergeometric function of two variables [11, p.62]

$$\Psi_2^{(m)}\left[a;b,c;x,y\right]=\sum_{r,s=0}^{\infty}\frac{(a)_{r+s}}{(b)_r\,(c)_s}\,\frac{x^r}{r!}\,\frac{y^s}{s!}, \tag{1.4}$$

$$\text{where } (\lambda)_n=\begin{cases}1 & ,\ if\ \ n=0\\ \lambda(\lambda+1)(\lambda+2)....(\lambda+n-1) & ,\ if\ \ n=1,2,3,......\end{cases} \tag{1.5}$$

Khan and Shukla [4,p. 163] defined the Laguerre polynomials of several variables $L_n^{(\alpha_1,\cdots,\alpha_m)}(x_1,\cdots,x_m)$ as follows:

$$L_n^{(\alpha_1,\cdots\cdots,\alpha_m)}(x_1,\cdots,x_m)$$

$$=\frac{\prod_{j=1}^{m}(\alpha_j+1)_n}{(n!)^m}\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\cdots\cdots\sum_{r_m=0}^{n-r_1-\cdots\cdots-r_{m-1}}\frac{(-n)_{r_1+\cdots\cdots+r_m}\prod_{j=1}^{m}x_{m+1-j}^{r_j}}{\prod_{j=1}^{m}r_j!\prod_{j=1}^{m}(\alpha_j+1)_{m+1-j}} \tag{1.6}$$

$$= \frac{\prod_{j=1}^{m} (\alpha_j + 1)_n}{(n!)^m} \Psi_2^{(m)} \left[-n;\, \alpha_1 + 1, \cdots\cdots, \alpha_m + 1; x_1, \cdots\cdots, x_m \right], \tag{1.7}$$

where $\Psi_2^{(m)}$ is the confluent hypergeometric function of m-variables [11, p.62]

$$\Psi_2^{(m)} \left[a; c_1, \cdots, c_m; x_1, \cdots, x_m \right] = \sum_{r_1, \cdots, r_m = 0}^{\infty} \frac{(a)_{r_1 + \cdots + r_m}}{(c_1)_{r_1} \cdots (c_m)_{r_m}} \frac{x_1^{r_1}}{r_1!} \cdots \frac{x_m^{r_m}}{r_m!} \tag{1.8}$$

The object of this paper is to obtain certain integral formulas involving Laguerre polynomials of several variables ,these integrals are evaluated in terms of Chandel function (c.f.[2, p.90]) and the generalized Kampé de Fériet function of several variables [3, p.28] which are defined as follows:

$${}^{(k)}_{(1)}E_C^{(n)} \left[a, a', b\, ; c_1 \cdots, c_n;\, x_1, \cdots, x_n \right]$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_k} (a')_{m_{k+1} + \cdots + m_n} (b)_{m_1 + \cdots + m_n}\, x_1^{m_1} \cdots x_n^{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}\, m_1! \cdots m_n!} \tag{1.9}$$

and

$$F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}} \left[x_1, \cdots, x_n \right] = F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}} \left[\begin{matrix} (a):(b');\cdots;(b^{(n)})\,; \\ (c):(d');\cdots;(d^{(n)})\,; \end{matrix} x_1, \cdots, x_n \right]$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{((a))_{m_1 + \cdots + m_n} ((b'))_{m_1} \cdots ((b^{(n)}))_{m_n}\, x_1^{m_1} \cdots x_n^{m_n}}{((c))_{m_1 + \cdots + m_n} ((d'))_{m_1} \cdots ((d^{(n)}))_{m_n}\, m_1! \cdots m_n!} \tag{1.10}$$

where $((a))_m$ mean the product $\prod_{j=1}^{A} (a_j)_m$.

2 Integral Formulas

For Re (λ) > 0; Re (σ) > 0, we have the following integral formulas involving Laguerre polynomials of several variables:

$$\int_0^\infty e^{-\sigma x} x^{\lambda - 1} L_m^{(\alpha_1, \cdots\cdots, \alpha_r)} (\gamma_1 x, \cdots\cdots, \gamma_r x) L_n^{(\beta_1, \cdots\cdots, \beta_s)} (\delta_1 x, \cdots\cdots, \delta_s x)\, dx$$

$$= \frac{\Gamma(\lambda) (\alpha_1 + 1)_m \cdots (\alpha_r + 1)_m (\beta_1 + 1)_n \cdots (\beta_s + 1)_n}{\sigma^\lambda (m!)^r (n!)^s}$$

$${}^{(r)}_{(1)}E_C^{(r+s)} \left[-m, -n, \lambda\, ; \alpha_1 + 1, \cdots, \alpha_r + 1, \beta_1 + 1, \cdots, \beta_s + 1;\, \frac{\gamma_1}{\sigma}, \cdots, \frac{\gamma_r}{\sigma}, \frac{\delta_1}{\sigma}, \cdots, \frac{\delta_s}{\sigma} \right] \tag{2.1}$$

$$\int_0^t x^{\sigma-1}(t-x)^{\lambda-1}L_n^{(\alpha_1,\cdots\cdots,\alpha_r)}(\beta_1 x,\cdots\cdots,\beta_r x)\,dx = \frac{(\alpha_1+1)_n\cdots(\alpha_r+1)_n B(\sigma,\lambda)t^{\sigma+\lambda-1}}{(n!)^r}$$

$$F\begin{matrix}2:0;\cdots\cdots;0\\1:1;\cdots\cdots;1\end{matrix}\left[\begin{matrix}-n,\sigma: & - & ;\cdots\cdots; & - & ;\\ \sigma+\lambda:\alpha_1+1; & & \cdots\cdots; & \alpha_r+1 & ;\end{matrix}\beta_1 t,\cdots\cdots,\beta_r t\right] \tag{2.2}$$

$$\int_0^t x^{\sigma-1}(t-x)^{\lambda-1}L_n^{(\alpha_1,\cdots\cdots,\alpha_r)}(\gamma_1(t-x),\cdots\cdots,\gamma_r(t-x))\,dx$$

$$= \frac{(\alpha_1+1)_n\cdots(\alpha_r+1)_n B(\sigma,\lambda)\,t^{\lambda+\sigma-1}}{(n!)^r}$$

$$F\begin{matrix}2:0;\cdots;0\\1:1;\cdots;1\end{matrix}\left[\begin{matrix}-n,\lambda: & -- & ;\cdots; & -- & ;\\ \lambda+\sigma:1+\alpha_1 & & ;\cdots;1+\alpha_r & & ;\end{matrix}\gamma_1 t,\cdots,\gamma_r t\right] \tag{2.3}$$

$$\int_0^t\int_0^s\int_0^r x^{\alpha}(r-x)^{\lambda-1}y^{\beta}(s-y)^{\mu-1}z^{\gamma}(t-z)^{\nu-1}L_n^{(\delta_1,\cdots\cdots,\delta_m)}(xyz,\cdots\cdots,\ xyz)\,dx\,dy\,dz$$

$$= \frac{(\delta_1+1)_n\cdots(\delta_m+1)_n B(\alpha+1,\lambda)B(\beta+1,\mu)B(\gamma+1,\nu)r^{\alpha+\lambda}s^{\beta+\mu}\,t^{\gamma+\nu}}{(n!)^r}$$

$$F\begin{matrix}4:0;\cdots\cdots;0\\3:1;\cdots\cdots;1\end{matrix}\left[\begin{matrix}-n,\alpha+1,\beta+1,\gamma+1 & : -- ;\cdots; -- ;\\ \alpha+\lambda+1,\beta+\mu+1,\gamma+\nu+1 & :\delta_1+1;\cdots;\delta_m+1;\end{matrix}rst,\cdots,rst\right] \tag{2.4}$$

$$\int_0^{t_r}\cdots\int_0^{t_1} x_1^{\mu_1}(t_1-x_1)^{\lambda_1-1}\cdots x_1^{\mu_r}(t_r-x_r)^{\lambda_r-1}L_n^{(\alpha_1,\cdots,\alpha_r)}(\,_1x_1,\cdots,x_r)\,dx_1\cdots dx_r$$

$$= \frac{(\alpha_1+1)_n\cdots(\alpha_r+1)_n B(\mu_1+1,\lambda_1)\cdots B(\mu_r+1,\lambda_r)t_1^{\mu_1+\lambda_1}\cdots t_r^{\mu_r+\lambda_r}}{(n!)^r}$$

$$F\begin{matrix}1:1;\cdots\cdots;1\\0:2;\cdots\cdots;2\end{matrix}\left[\begin{matrix}-n: & \mu_1+1 & ;\cdots\cdots; & \mu_r+1 & ;\\ - & :\alpha_1+1,\mu_1+\lambda_1+1; & \cdots\cdots; & \alpha_r+1,\mu_r+\lambda_r+1 & ;\end{matrix}t_1,\cdots\cdots,t_r\right] \tag{2.5}$$

$$\int_0^{t_r}\cdots\int_0^{t_1} x_1^{\mu_1}(t_1-x_1)^{\lambda_1-1}\cdots x_1^{\mu_r}(t_r-x_r)^{\lambda_r-1}L_n^{(\alpha_1,\cdots,\alpha_r)}(\gamma_1(t_1-x_1),\cdots,\gamma_r(t_r-x_r))\,dx_1\cdots dx_r$$

$$= \frac{(\alpha_1+1)_n \cdots (\alpha_r+1)_n \, B(\mu_1+1,\lambda_1)\cdots B(\mu_r+1,\lambda_r)\, t_1^{\mu_1+\lambda_1}\cdots t_r^{\mu_r+\lambda_r}}{(n!)^r}$$

$$F\begin{matrix} 1:1;\cdots\cdots;1 \\ 0:2;\cdots\cdots;2 \end{matrix}\left[\begin{matrix} -n: & \lambda_1 & ;\cdots\cdots; & \lambda_r & ; \\ -\ : & \alpha_1+1,\mu_1+\lambda_1+1 & ;\cdots\cdots; & \alpha_r+1,\mu_r+\lambda_r+1 & ; \end{matrix} \gamma_1 t_1,\cdots\cdots,\gamma_r t_r\right] \quad (2.6)$$

Following integral can be obtained readily from (2.6) as follows:

$$\int_0^{t_r}\cdots\int_0^{t_1} x_1^{\mu_1}(t_1-x_1)^{\lambda_1-1}\cdots x_r^{\mu_r}(t_r-x_r)^{\lambda_r-1} L_n^{(\lambda_1-1,\cdots,\lambda_r-1)}(\gamma_1(t_1-x_1),\cdots,\gamma_r(t_r-x_r))\,dx_1\cdots dx_r$$

$$= \frac{(\lambda_1)_n \cdots (\lambda_r)_n \, B(\mu_1+1,\lambda_1)\cdots B(\mu_r+1,\lambda_r)\, t_1^{\mu_1+\lambda_1}\cdots t_r^{\mu_r+\lambda_r}}{(\mu_1+\lambda_1+1)_n\cdots(\mu_r+\lambda_r+1)_n}$$
$$\times L_m^{(\mu_1+\lambda_1,\cdots\cdots,\mu_r+\lambda_r)}(\gamma_1 t_1,\cdots\cdots,\gamma_r t_r)\ . \quad (2.7)$$

To obtain the main integral formula (2.1), we consider the left-hand side of (2.1) and using (1.2), then expressing $\Psi_2^{(m)}$ in series forms and changing the order of integration and summation to get

$$L.H.S = \frac{(\alpha_1+1)_m\cdots(\alpha_r+1)_m(\beta_1+1)_n\cdots(\beta_s+1)_n}{(m!)^r(n!)^s}$$

$$\sum_{p_1,\cdots,p_r,q_1,\cdots,q_s=0}^{\infty} \frac{(-m)_{p_1+\cdots+p_r}(-n)_{q_1+\cdots+q_s}\,\gamma_1^{p_1}\cdots\gamma_r^{p_r}\delta_1^{q_1}\cdots\delta_s^{q_s}}{(\alpha_1+1)_{p_1}\cdots(\alpha_r+1)_{p_r}(\beta_1+1)_{q_1}\cdots(\beta_s+1)_{q_s}\,p_1!\cdots p_r!q_1!\cdots q_s!}$$

$$\times\int_0^\infty e^{-\sigma x}x^{\lambda+p_1+\cdots+p_r+q_1+\cdots+q_s}\,dx \quad (2.8)$$

In (2.8), using the definition of Gamma function and considering the definition (1.4), we get the right- hand side of (2.1).

The integrals (2.2) to (2.6) are similarly established and we using the definition of Beta function .

3 Special Cases

It is important to note that the above integrals are capable of yielding a number of other integrals formulas, these integral are evaluated in terms of certain

hypergeometric function for example *the generalized hypergeometric function functions* ${}_pF_q$ *[8,p.42]*, Appell's function F_2 [8,p. 53] , Lauricella's function $F_C^{(n)}$ [8,p. 60] , Kampé de Fériet function of two variables $F_{E:G;H}^{A:B;D}$ [8,p. 63] Saran's function F_E [8, p. 66] and Exton's functions K_2 and K_5 [2, p.78] .
On setting $r=0$ in (2.1), we get

$$\int_0^\infty e^{-\sigma x} x^{\lambda-1} L_n^{(\beta_1,\cdots\cdots,\beta_s)}(\delta_1 x,\cdots\cdots,\delta_s x)\,dx$$

$$=\frac{\Gamma(\lambda)(\beta_1+1)_n\cdots(\beta_s+1)_n}{\sigma^\lambda (n!)^s}\; F_C^{(s)}\left[-n,\lambda\,;\beta_1+1,\cdots,\beta_s+1;\,\frac{\delta_1}{\sigma},\cdots,\frac{\delta_s}{\sigma}\right] \tag{3.1}$$

On setting $r=s=1$, integral (2.1) reduces to a known result [6, p. 94(12)] see also [9,p. 1132]

$$\int_0^\infty e^{-\sigma x} x^{\lambda-1} L_m^{(\alpha)}(\gamma x) L_n^{(\beta)}(\delta x)\,dx$$

$$=\frac{\Gamma(\lambda)(\alpha+1)_m(\beta+1)_n}{\sigma^\lambda\, m!\,n!}\; F_2\left[\lambda,-m,-n;\alpha+1,\beta+1;\frac{\gamma}{\sigma},\frac{\delta}{\sigma}\right] \tag{3.2}$$

On setting $r=3, s=1$ in (2.1), we get

$$\int_0^\infty e^{-\sigma x} x^{\lambda-1} L_m^{(\alpha_1,\alpha_2,\alpha_3)}(\gamma_1 x,\gamma_2 x,\gamma_3 x)\, L_n^{(\beta)}(\delta x)\,dx$$

$$=\frac{\Gamma(\lambda)(\alpha_1+1)_m(\alpha_2+1)_m(\alpha_3+1)_m(\beta+1)_n}{\sigma^\lambda (m!)^3 n!}$$

$$K_2\left[\lambda,\lambda,\lambda,\lambda;-m,-m,-m,-n\;;\alpha_1+1,\alpha_2+1,\alpha_3+1,\beta+1;\,\frac{\gamma_1}{\sigma},\frac{\gamma_2}{\sigma},\frac{\gamma_3}{\sigma},\frac{\delta}{\sigma}\right] \tag{3.3}$$

On setting $r=s=2$ in (2.1), we get

$$\int_0^\infty e^{-\sigma x} x^{\lambda-1} L_m^{(\alpha_1,\alpha_2)}(\gamma_1 x,\gamma_2 x) L_n^{(\beta_1,\beta_2)}(\delta_1 x,\delta_2 x)\,dx$$

$$=\frac{\Gamma(\lambda)(\alpha_1+1)_m(\alpha_2+1)_m(\beta_1+1)_n(\beta_2+1)_n}{\sigma^\lambda (m!)^2 (n!)^2}$$

$$K_5\left[\lambda,\lambda,\lambda,\lambda;-m,-m,-n,-n,\lambda\ ;\alpha_1+1,\alpha_2+1,\beta_1+1,\beta_2+1;\ \frac{\gamma_1}{\sigma},\frac{\gamma_2}{\sigma},\frac{\delta_1}{\sigma},\frac{\delta_2}{\sigma}\right] \quad (3.4)$$

Further, (3.4) for $\gamma_1=\gamma_2=\gamma$ *and* $\delta_1=\delta_2=\delta$ and use the result [1, p. 64(3.7)]
$K_5(a,a,a,a;b,b,d,d;e,f,h,k;z,z,v,v)$

$$=F^{1:3;3}_{0:3;3}\left[\begin{matrix}a\ :b,\tfrac{1}{2}(e+f),\tfrac{1}{2}(e+f-1);d,\tfrac{1}{2}(h+k),\tfrac{1}{2}(h+k-1);\\ -:\quad e,\ f\quad ,\ e+f-1\ ;\quad h\ ,k\quad ,\ h+k-1\ ;\end{matrix}\ 4z,4v\right] \quad (3.5)$$

We get

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}L_m^{(\alpha_1,\alpha_2)}(\gamma x,\gamma x)\,L_n^{(\beta_1,\beta_2)}(\delta x,\delta x)\,dx$$

$$=\frac{\Gamma(\lambda)\,(\alpha_1+1)_m(\alpha_2+1)_m(\beta_1+1)_n(\beta_2+1)_n}{\sigma^{\lambda}(m!)^2(n!)^2}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{matrix}\lambda:-m\,,\tfrac{1}{2}(\alpha_1+\alpha_2+2),\tfrac{1}{2}(\alpha_1+\alpha_2+1);-n\,,\tfrac{1}{2}(\beta_1+\beta_2+2),\tfrac{1}{2}(\beta_1+\beta_2+1)\,;\\ -:\quad \alpha_1+1,\alpha_2+1,\quad ,\ \alpha_1+\alpha_2+1\ ;\quad \beta_1+1,\beta_2+1\quad ,\ \beta_1+\beta_2+1\ ;\end{matrix}\ \frac{4\gamma}{\sigma},\frac{4\delta}{\sigma}\right] \quad (3.6)$$

On setting $r=1, s=2$ in (2.1), we get

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}L_m^{(\alpha)}(\gamma x)\,L_n^{(\beta_1,\beta_2)}(\delta_1 x,\delta_2 x)\,dx$$

$$=\frac{\Gamma(\lambda)\,(\alpha+1)_m(\beta_1+1)_n(\beta_2+1)_n}{\sigma^{\lambda}m!(n!)^2}$$

$$F_E\left[\lambda,\lambda,\lambda,-m,-n,-n\ ;\alpha+1,\beta_1+1,\beta_2+1;\ \frac{\gamma}{\sigma},\frac{\delta_1}{\sigma},\frac{\delta_2}{\sigma}\right] \quad (3.7)$$

Now, on putting $r=1,\alpha_1=\alpha,\beta_1=\beta$ *and* $\sigma=\alpha+1$, integral (2.2) reduces to

$$\int_0^t x^{\alpha}(t-x)^{\lambda-1}L_m^{(\alpha)}(\beta x)\,dx\ =\frac{(\alpha+1)_m B(\alpha+1,\lambda)t^{\alpha+\lambda}}{m!}\ {}_1F_1\left[-m;\alpha+\lambda+1\ ;\ \beta t\right] \quad (3.8)$$

On setting $r=2,\beta_1=\beta_2=\beta$ and using the result [7, p. 28(33)]

$$F^{A:0;0}_{C:1;1}\left[\begin{matrix}(a):-;-\ ;\\ (c):d\,;d'\ ;\end{matrix}\ x,\ x\right]={}_{A+2}F_{C+3}\left[\begin{matrix}(a),(d+d'-1)/2,(d+d')/2\,;\\ (c),\ d,d',d+d'-1\quad ;\end{matrix}\ 4x\right] \quad (3.9)$$

integral (2.2) reduces to

$$\int_0^t x^{\sigma-1}(t-x)^{\lambda-1} L_n^{(\alpha_1,\alpha_2)}(\beta x,\beta x)\,dx = \frac{(\alpha_1+1)_n(\alpha_2+1)_n B(\sigma,\lambda) t^{\sigma+\lambda-1}}{(n!)^2}$$

$${}_4F_4\left[\begin{matrix} -n,\sigma,(\alpha_1+\alpha_2+1)/2,(\alpha_1+\alpha_2+2)/2\ ; \\ \sigma+\lambda, \quad \alpha_1+1\ ,\alpha_2+1,\alpha_1+\alpha_2+1 \quad ; \end{matrix} 4\beta t\right] \tag{3.10}$$

On setting $m=3,\lambda=\beta-\alpha,\mu=\gamma-\beta,\nu=\alpha-\gamma$ in (2.4), we get

$$\int_0^t\int_0^s\int_0^r x^{\alpha}(r-x)^{\beta-\alpha-1} y^{\beta}(s-y)^{\gamma-\beta-1} z^{\gamma}(t-z)^{\alpha-\gamma-1} L_n^{(\delta_1,\delta_2,\delta_3)}(\sigma_1 xyz,\sigma_2 xyz,\sigma_3 xyz)\,dx\,dy\,dz$$

$$= B(\alpha+1,\beta-\alpha)B(\beta+1,\gamma-\beta)B(\gamma+1,\alpha-\gamma)t^{\alpha}r^{\beta}\,s^{\gamma}L_n^{(\delta_1,\delta_2,\delta_3)}(\sigma_1 rst,\sigma_2 rst,\sigma_3 rst) \tag{3.11}$$

Finally, setting $\mu_j=\alpha_j\ , j=1,2,\cdots,r$ in (2.5) and considering the definition (1.2), we get a known result of Khan and Shukla [5, p. 115(4.1)].

$$\int_0^{t_r}\cdots\int_0^{t_1} x_1^{\alpha_1}(t_1-x_1)^{\lambda_1-1}\cdots\cdots x_1^{\alpha_r}(t_r-x_r)^{\lambda_r-1} L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots\cdots,x_r)\,dx_1\cdots\cdots dx_r$$

$$= \frac{(\alpha_1+1)_n\cdots(\alpha_r+1)_n B(\alpha_1+1,\lambda_1)\cdots B(\alpha_r+1,\lambda_r) t_1^{\alpha_1+\lambda_1}\cdots t_r^{\alpha_r+\lambda_r}}{(\alpha_1+\lambda_1+1)_n\cdots(\alpha_r+\lambda_r+1)_n}$$

$$\times L_n^{(\alpha_1+\lambda_1,\cdots\cdots,\alpha_r+\lambda_r)}(t_1,\cdots\cdots,\,t_r)\,. \tag{3.12}$$

4 Conclusion

The results established in this paper are useful in deriving certain new integral formulas involving Laguerre polynomials of several variables. Further, certain class of known integral formulas involving the product of two Laguerre polynomials $L_m^{(\alpha)}(x)$ can also be obtained in terms of *hypergeometric functions* ${}_2F_1$ *and* ${}_3F_2$ see for example Mavromatis [6], Shawagfeh [9] and Srivastava et al. [12].

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