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The Differential Form of the Tau Method and its Error Estimate for Third Order Non-Overdetermined Differential Equations

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Abstract

This paper is concerned with a variant of the tau methods for initial value problems in non-overdetermined third order ordinary differential equation. The differential formulation is are considered here. The corresponding error estimate for this variant is obtained and numerical results for some selected examples are provided. The numerical evidences show that the order of the tau approximant is closely captured.

Keywords: *Tau method, Formulation, Variant, Approximant, Error estimate.*

1 Introduction

Accurate approximate solution of initial value problems and boundary value problems in linear ordinary differential equations with polynomial coefficients can be obtained by the tau method introduced by Lanczos [10] in 1938. Techniques based on this method have been reported in literature with application to more general equation including non-liner ones as well as to both deferential and integral equations. We review briefly here some of the variants of the method.

Differential Form of the Tau Method

Consider the following boundary value problem in the class of m-th order ordinary differential equations:

$$Ly(x) \equiv \sum_{r=0}^{m} p_r(x) y^{(r)}(x) = f(x), a \le x \le b \qquad \ldots(1.1a)$$

$$L^* y(x_{rk}) \equiv \sum_{r=0}^{m} a_{rk} y^{(r)}(x_{rk}) = \rho_k, k = 1(1)m \qquad \ldots(1.1b)$$

where $|a| < \infty, |b| < \infty, a_{rk}, x_{rk}, \rho_k, r = 0(1)m, k = 0(1)m,$ are given real numbers, and the functions $f(x)$ and

$$p_r(x) = \sum_{k=0}^{N_r} p_{r,k} x^k, r = 0(1)m \qquad \ldots.(1.2)$$

are polynomial functions or sufficiently close polynomial approximants of given real functions .

Definition 1.1 *The number of over-determination, s, of equation (1.1a) is defined as*

$s = max\{ N_r - r : 0 \le r \le m\}$ *(1.3)*
for $N_r \ge r$ *and* $0 \le r \le m$.

Definition 1.2 *Equation (1.1a) is said to be non-overdetermined if s, given by (1.3) is zero, i.e. if s = 0. Otherwise it is over-determined.*

For the solution of (1.1) by the tau method (see [2], [3],[8], [10], [11]), we seek an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \; n < +\infty \qquad \ldots(1.4)$$

of $y(x)$ *which satisfies exactly the perturbed problem*

$$Ly_n(x) = f(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x), a \le x \le b, \qquad \ldots(1.5a)$$

$$L^* y_n(x_{rk}) = \rho_k, k = 1(1)m \qquad (1.5b)$$

where τ_r, r = 1(1)m +s, are fixed parameters to be determined along with a_r, r = 0(1)n, in (1.4)by equating the coefficients of power of x in (1.5). The polynomial

$$T_r(x) = \cos\left\{r\ Cos^{-1}\left[\frac{2x-2a}{b-a}-1\right]\right\} \equiv \sum_{k=0}^{r} C_k^{(r)} x^k \qquad \ldots(1.6)$$

is the r-th degree Chebyshev polynomial valid in [a, b] (see [2],[6] and [12]).

2 Error Estimation of the Tau Method

We review briefly here error estimation of the tau method for the variant of the preceding section and which we had earlier reported in [2], [5] and [6].

2.1 Error Estimation for the Differential Form

While the error function

$$e_n(x) = y(x) - y_n(x) \qquad \ldots(2.1)$$

satisfies the error problem

$$Le_n(x) = \sum_{r=0}^{m+s-1} \hat{\tau}_{m+s-r}\ T_{n-m+r+1}(x) \qquad \ldots(2.2a)$$

$$Le_n(x_{rk}) = 0,\ k = 1(1)m \quad \ldots(2.2b)$$

The polynomial error approximant

$$(e_n(x))_{n+1} = \frac{v_m(x)\phi_n T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \qquad \ldots(2.3)$$

of$e_n(x)$ satisfies the perturbed error problem

$$L(e_n(x))_{n+1} = \sum_{r=0}^{m+s-1} (-\tau_{m+s-r}\ T_{n-m+r+1}(x)\ +\ \hat{\tau}_{m+s-r}\ T_{n-m+r+2}(x)) \ldots(2.4a)$$

$$L^*(e_n(x_{rk}))_{n+1} = 0 \qquad (2.4b)$$

where the extra parameters $\hat{\tau}_r$ r = 1(1)m + s, and φ_n in (2.3) – (2.4) are to be determined and $v_m(x)$ in (2.3) is a specified polynomial of degree in which ensures that $(e_n(x))_{n+1}$ satisfies the homogenous conditions (2.4b).

With 2.3) in (2.4) we get a linear system of m + s +1 equations, obtained by equating the coefficients of x^{n+s+1}, x^{n+s}, …x^{n-m+1}, for the determination of φ_n by forward elimination, since we do not need the $\hat{\tau}$'s in (2.3) consequently, we obtain an estimate

$$\varepsilon = \max_{a \le x \le b} \left| (e_n(x))_{n+1} \right| = \frac{|\phi_n|}{\left| C_{n-m+1}^{(n-m+1)} \right|} \cong \max_{a \le x \le b} \left| (e_n(x) \right| \qquad \ldots(2.5)$$

3 A Class of Non-Overdetermined Third Order Differential Equations

We consider here the two variants of the tau methods of preceding sections for the tau approximants and their error estimates for the class of problems:

$$LY(x) := (\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)\, y'''(x) + (\beta_0 + \beta_1 x + \beta_2 x^2)\, y''(x)$$
$$+ (\gamma_0 + \gamma_1 x) y'(x) + \lambda_0\, y(x) = \sum_{r=0}^{n} f_r x^r,\; a \le x \le b \qquad \ldots(3.1a)$$
$$y(a) = \rho_0,\; y'(a) = \rho_1,\; y''(a) = \rho_2 \qquad \ldots(3.1b)$$

that is, the case when $m = 3$ and $s = 0$ in (1.1)

Without loss of generality, we shall assume that $a = 0$ and $b = 1$, since the transformation

$$u = \frac{(x-a)}{(b-a)},\; a \le x \le b \qquad \ldots(3.2)$$

takes (3.1) into the closed interval [0, 1].

3.1 Tau Approximant by the Differential Form

From (1.3)-(1.4), for $m = 3$ and $s = 0$, that is, corresponding to (3.1), we have

$$(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) \sum_0^n r(r-1)(r-2) a_r x^{r-3}$$
$$+ (\beta_0 + \beta_1 x + \beta_2 x^2) \sum_0^n r(r-1) a_r x^{r-2} + (\gamma_0 + \gamma_1 x) \sum_0^n r\, a_r x^{r-1} + \lambda_0 \sum_0^n a_r x^r$$
$$= \sum_{r=0}^{F} f_r x^r + \tau_1 T_n(x) + \tau_2 T_{n-1}(x) + \tau_3 T_{n-2}(x)$$

This leads to:

$$\sum_{k=0}^{n} [\alpha_3 k(k-1)(k-2) + \beta_2 k(k-1) + \gamma_1 k + \lambda_0]\, a_k x^k$$

$$+\sum_{k=0}^{n-1}[\alpha_2 k(k+1)k(k-1)+\beta_1 k(k-1)k+\gamma_0(k+1)]\,a_{k+1}x^{k1}$$
$$+\sum_{k=0}^{n-2}[\alpha_1(k+2)k(k+1)k+\beta_0 k(k+2)(k+1)]\,a_{k+2}x^{k}$$
$$+\sum_{k=0}^{n-3}[\alpha_0(k+3)k(k+2)k+1)]\,a_{k+3}x^{k}$$
$$=\sum_{k=0}^{F} f_r x^r + \tau_1\sum_{k=0}^{n} C_k^{(n)}x^k + \tau_2\sum_{k=0}^{n-1} C_k^{(n-1)}x^k + \tau_3\sum_{k=0}^{n-2} C_k^{(n-2)}x^k$$

Hence,

$$\{\alpha_3 n(n-1)(n-2)+\beta_2 n(n-1)+\gamma_1 n+\lambda_0\}a_n - f_n - \tau_1 C_n^{(n)}\}x^n$$
$$+\{\alpha_3(n-1)(n-2)(n-3)+\beta_2 n(n-1)(n-2)+\gamma_1(n-1)+\lambda_0\}a_{n-1}-f_{n-1}-\tau_1 C_{n-1}^{(n)}]+$$
$$[\alpha_2 n(n-1)(n-2)+\beta_1 n(n-1)+\gamma_0 n\,]\,a_n - \tau_2 C_{n-1}^{(n-1)}\}x^{n-1}$$
$$+\{\alpha_3 n(n-2)(n-3)(n-4)+\beta_2 n(n-2)(n-3)+\gamma_1(n-2)+\lambda_0\}a_{n-2}-f_{n-2}$$
$$-\tau_1 C_{n-2}^{(n)} + \{\alpha_2(n-1)(n-2)(n-3)+\beta_1 n(n-1)(n-2)+\gamma_0(n-1)]a_{n-1} + [\alpha_1 n(n-$$
$$1)(n-2)+\beta_0 n(n-1)\}a_n - \tau_2 C_{n-2}^{(n-1)} - \tau_3 C_{n-2}^{(n-2)}\}x^{n-2} + \alpha_0 n(n+1)(n-1)a_{n+1} = 0.$$

From this we obtain by equating corresponding coefficients the linear system

$$\{\alpha_3 k(k-1)(k-2)+\beta_2 k(k-1)+\gamma_1 k+\lambda_0\}a_k + \{\alpha_2 k(k+1)k(k-1)+\beta_1 k(k+1)k + \gamma_0(k+1)]a_{k+1} + [\alpha_1 k(k-1)(k-2)k+\beta_0(k+1)(k+2)]a_{k+2} + \alpha_0(k+3)(k+2)(k+1)a_{k+3} - \tau_1 C_k^{(n)} - \tau_2 C_k^{(n-1)} - \tau_3 C_k^{(n-2)} - f_k = 0\,,\quad k = 0\,(1)\,n-3$$

$$[\alpha_3(n-2)(n-3)(n-4)+\beta_2(n-2)(n-3)+\gamma_1(n-2)+\lambda_0\}a_{n-2}$$
$$+\{\alpha_2(n-1)(n-2)(n-3)+\beta_1(n-1)(n-2)+\gamma_0(n-1)]a_{n-1}$$
$$+[\alpha_1 n(n-1)(n-2)+\beta_0 n(n-1)]a_n - \tau_1 C_{n-2}^{(n)} - \tau_2 C_{n-2}^{(n-1)} - \tau_3 C_{n-2}^{(n-2)} - f_{n-2} = 0$$

$$[\alpha_3(n-1)(n-2)(n-3)+\beta_2(n-1)(n-2)+\gamma_1(n-1)+\lambda_0\}a_{n-1}$$
$$+\{\alpha_2 n(n-1)(n-2)+\beta_1 n(n-1)+\gamma_0 n]\,a_n - \tau_1 C_{n-1}^{(n)} - \tau_2 C_{n-1}^{(n-1)} - f_{n-1}$$
$$= 0 \qquad \ldots$$

(3.3)

$$[\alpha_3 n(n-1)(n-2)+\beta_2 n(n-1)+\gamma_1 n+\lambda_0]a_n - \tau_1 C_n^{n} - f_n = 0$$

The solution of this system together with the two equations arising from the condition (3.1b) for a_r, $r = 0\ (1)n$ and τ_1, τ_2, τ_3. Subsequently leads to the approximant $Y_n(x)$.

3.1.1 Error Estimation for the Differential Form

For problem (3.1).we have from (2.4)

$L(e_n(x)_n)$: = $\hat{\tau}_1 T_{n+1}(x)$ + $(\hat{\tau}_2-\tau_1)$ $T_n(x)$ + $(\hat{\tau}_3-\tau_2)T_{n-1}(x)$ $-\tau_3 T_{n-2}(x)$ ....(3.4a)

$(e_n(0))_{n+1}=0$, $(e'_n(0))_{n+1}=0$ ...(3.4b)

where

$$(e_n(x))_{n+1}=\frac{x^3\phi_n T_{n-2}(x)}{C_{n-2}^{(n-2)}}=\frac{\phi_n\sum_{r=0}^{n-2}C_r^{(n-2)}x^{r+3}}{C_{n-2}^{(n-2)}}$$

....(3.5)

From the coefficients of x^{n+1}, x^n, x^{n-1}andx^{n-2}, we get the system

$\hat{\tau}_1\left[C_{n+1}^{(n+1)}\right]=\frac{\phi_n}{C_{n-2}^{(n-2)}}$ [$\alpha_3(n+1)(n)(n-1)C_{n-2}^{(n-2)}$ + $\beta_2(n+1)C_{n-2}^{(n-2)}$ + $\gamma_1(n+1)C_{n-2}^{(n-2)}$ + $\lambda_0 C_{n-2}^{(n-2)}$]

$\hat{\tau}_1 C_n^{(n+1)}$ + $(\hat{\tau}_1-\tau_1)C_n^n$ = $\frac{\phi_n}{C_{n-2}^{(n-2)}}$ [$\alpha_2(n+1)(n)(n-1)C_{n-2}^{(n-2)}$ +$\alpha_3 n(n-1)(n-2)C_{n-3}^{(n-2)}$ + $\beta_1 n(n+1)$ + $\beta_2 n(n-1)C_{n-3}^{(n-2)}$ + $\gamma_0(n+1)C_{n-2}^{(n-2)}$ + $\gamma_1 nC_{n-3}^{(n-2)}$+ $\lambda_0 C_{n-3}^{(n-2)}$]

$\hat{\tau}_1 C_{n-1}^{(n+1)}$ + $(\hat{\tau}_2-\tau_1)C_{n-1}^{(n)}$ + $(\hat{\tau}_3-\tau_3)C_{n-1}^{(n-1)}$ = $\frac{\phi_n}{C_{n-2}^{(n-2)}}$ [$\alpha_1(n+1)(n)(n-1)C_{n-2}^{(n-2)}$ + $\alpha_2 n(n-1)(n-2)C_{n-3}^{(n-2)}$ + $\alpha_3(n-1)(n-2)(n-3)C_{n-4}^{(n-2)}$ + $\beta_0 n(n+1)C_{n-2}^{(n-2)}$ +$\beta_1 n(n-1)C_{n-3}^{(n-2)}$ + $\beta_2(n-1)(n-2)C_{n-4}^{(n-2)}$ + $\gamma_0 nC_{n-3}^{(n-2)}$ + $\gamma_1(n-1)C_{n-4}^{(n-2)}$+ $\lambda_0 C_{n-4}^{(n-2)}$]

$\hat{\tau}_1 C_{n-2}^{(n+1)}$ + $(\hat{\tau}_2-\tau_1)C_{n-2}^{(n)}$ + $(\hat{\tau}_3-\tau_2)C_{n-2}^{(n-1)}$ $-\tau_3 C_{n-2}^{(n-2)}$ = $\frac{\phi_n}{C_{n-2}^{(n-2)}}$ [$\alpha_0(n+1)(n)(n-1)C_{n-2}^{(n-2)}$ + $\alpha_1 n(n-1)(n-2)C_{n-3}^{(n-2)}$ + $\alpha_2(n-1)(n-2)(n-3)C_{n-4}^{(n-2)}$ + $\alpha_3(n-2)(n-3)(n-4)C_{n-5}^{(n-2)}$ + $\beta_0 n(n-1)C_{n-3}^{(n-2)}$ +$\beta_1 n(n-1)(n-2)C_{n-4}^{(n-2)}\gamma_0(n-1)C_{n-4}^{(n-2)}$ + $\gamma_1(n-2)C_{n-5}^{(n-2)}$+ $\lambda_0 C_{n-5}^{(n-2)}$]

$\hat{\tau}_1\left[C_{n+1}^{(n+1)}\right]$ = ϕ [$\alpha_3(n+1)(n)(n-1)C_{n-2}^{(n-2)}$ + $\beta_2 n(n+1)C_{n-2}^{(n-2)}$ + $\gamma_1(n+1)C_{n-2}^{(n-2)}$ + $\lambda_0 C_{n-2}^{(n-2)}$]

$\hat{\tau}_1 C_n^{(n+1)}$ + $(\hat{\tau}_1-\tau_1)C_n^n$ = [$\alpha_2(n+1)(n)(n-1)C_{n-2}^{(n-2)}$ + $\alpha_3 n(n-1)(n-2)C_{n-3}^{(n-2)}$ + $\beta_1 n(n+1)C_{n-2}^{(n-2)}$ + $\beta_2 n(n-1)C_{n-3}^{(n-2)}$ + $\gamma_0(n+1)C_{n-2}^{(n-2)}$ + $\gamma_1 nC_{n-3}^{(n-2)}$+ $\lambda_0 C_{n-3}^{(n-2)}$]

$\hat{\tau}_1 C_{n-1}^{(n+1)} + (\hat{\tau}_2 - \tau_1) C_{n-1}^{(n)} + (\hat{\tau}_3 - \tau_3) C_{n-1}^{(n-1)} = \phi\,[\alpha_1(n+1)(n)(n-1) C_{n-2}^{(n-2)} + \alpha_2 n(n-1)(n-2) C_{n-3}^{(n-2)} + \alpha_3(n-1)(n-2)(n-3) C_{n-4}^{(n-2)} + \beta_0 n(n+1) C_{n-2}^{(n-2)} + \beta_1 n(n-1) C_{n-3}^{(n-2)} + \beta_2(n-1)(n-2) C_{n-4}^{(n-2)} + \gamma_0 n C_{n-3}^{(n-2)} + \gamma_1(n-1) C_{n-4}^{(n-2)} + \lambda_0 C_{n-4}^{(n-2)}]$ ..(3.6)

$\hat{\tau}_1 C_{n-2}^{(n+1)} + (\hat{\tau}_2 - \tau_1) C_{n-2}^{(n)} + (\hat{\tau}_3 - \tau_2) C_{n-2}^{(n-1)} - \tau_3 C_{n-2}^{(n-2)} = \phi\,[\alpha_0(n+1)n(n-1) C_{n-2}^{(n-2)} + \alpha_1 n(n-1)(n-2) C_{n-3}^{(n-2)} + \alpha_2(n-1)(n-2)(n-3) C_{n-4}^{(n-2)} + \alpha_3(n-2)(n-3)(n-4) C_{n-5}^{(n-2)} + \beta_0 n(n-1) C_{n-3}^{(n-2)} + \beta_1(n-1)(n-2) C_{n-4}^{(n-2)} + \gamma_0(n-1) C_{n-4}^{(n-2)} + \gamma_1(n-2) C_{n-5}^{(n-2)} + \lambda_0 C_{n-5}^{(n-2)}]$

where $\phi = \phi_n (C_{n-1}^{(n-1)})^{-1}$

By using the well-known relations (see [5] and [7]).

$$C_n^n = 2^{2n-1},\quad C_{n-1}^{(n)} = -\frac{1}{2} n C_n^{(n)},\quad C_{n-1}^{(n)} = -n\, 2^{2n-2} \qquad ...(3.7)$$

we solve this by forward elimination for ϕ_n to obtain

$$\phi_n = \frac{2^{2n-5}\tau_3}{p_7} \qquad ..(3.8)$$

where

$$p_7 = \{[(n+1)(n-1)\alpha_0 - (n-1)(n-3)\alpha_2](n-1)(n-2)\beta_1 + (n-1)\gamma_0) C_{n-4}^{(n-2)} + ((n-2)(n-3)(n-4)\,\alpha_3 + (n-2)\,\gamma_1 + \lambda_0) C_{n-5}^{(n-2)} - n(n-1)(n-2)^2\alpha_1 + n(n-1)(n-2)\,\beta_0] \qquad ...(3.9)$$

Thus, from (2.5) we obtain, as our error estimate

$$\varepsilon = \frac{2^{2-10n}\,|\tau_3|}{p_7} \qquad ... (3.10)$$

4 Numerical Experiments

We consider here two selected problems for experimentation with our results of the preceding sections. The exact errors are obtained as

$\varepsilon^* = \max_{0\le x\le 1} \{|y(x_k) - y_n(x_k)|\},\ 0 \le x \le 1\quad \{x_k\} = \{0.01k\},$ for $k = 0\,(1)100$

Example 4.1

$Ly(x) = y'''(x) - 5y''(x) + 6y'(x) = 0\ ,\ 0 \le x \le 1$

$y(0) = 0, y'(0) = 1, y''(0) = 0$

The exact solution is

$$y(x) = \frac{5}{6} + \frac{3}{2}e^{2x} - \frac{2}{3}e^{3x}$$

The numerical results are presented in Table 4.1 below.

Example 4.2

$Ly(x) = y'''(x) - y''(x) + 2y'(x) = 2, \ 0 \leq x \leq 1$
$y(0) = 0, y'(0) = 1, \ y''(0) = 0$

The exact solution is

$$y(x) = \tfrac{1}{2}\,(5e^{x} - 4e^{x} - e^{4x}).$$

The numerical results are presented in Table 4.2 below.

Table 4.1 Error and Error Estimates for Example 4.1

Degree(n) / **Error**	**5**	**6**	**7**
ε	1.67×10^{-2}	3.13×10^{-3}	1.66×10^{-5}
ε^*	1.96×10^{-4}	2.29×10^{-5}	4.25×10^{-7}

Table 4.2 Error and Error Estimates for Example 4.2

Degree(n) / **Error**	**5**	**6**	**7**
ε	1.57×10^{-4}	1.40×10^{-6}	1.56×10^{-8}
ε^*	1.09×10^{-5}	1.69×10^{-7}	1.07×10^{-9}

5 Conclusion

The tau method for the solution of initial value problems (IVPs) for third order differential equations with non-overdetermination has been presented.
The error involved in the approximants thus obtained was closely estimated. The effectiveness of the method was demonstrated as the order of the tau approximant estimated.

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