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A New Approach to the Accretive Operators Arising from 2-Banach Spaces

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Abstract

In the present paper, our objective is to investigate the accretive operators arising from 2-Banach spaces and to derive not only new but also interesting links between the classes of nonexpansive and of accretive mappings which give rise to a strong connection between the fixed point theory of nonexpansive mappings and the mapping theory of accretive maps.

Keywords: *Accretive operators, m-accretive operators, 2-normed spaces, quasi normed space, fixed point, nonexpansive mappings.*

1 Introduction, Definitions and Notations

In 1928, K. Menger in [1] introduced the notion called n-metrics (or generalized metric) But many mathematicians had not paid attentions to Menger's theory about generalized metrics. But several mathematicians, A. Wald, L. M. Blumenthal, W. A. Wilson etc. have developed Menger's idea.

In 1963, S. Gähler in [2] limited Menger's considerations to $n = 2$. Gähler's study is more complete in view of the fact that he developes the topological properties of the spaces in question. Gähler also proves that if the space is a linear normed space, then it is possible to define 2-norm.

Since 1963, S. Gähler, Y. J. Cho, R. W. Frees, C. R. Diminnie, R. E. Ehret, K. Iséki, A. White and many others have studied on 2-normed spaces and 2-metric spaces (for more [3]).

The origins of the fixed point theory based on the use of good approximations to construct the existence and uniqueness of solutions, especially to differential equations. This method is associated with the names of such celebrated mathematicans as Cauchy, Liouville, Lipschitz, Peano, Fredholm and, especially, Picard. In fact the precursors of a fixed point theoretic approach are explicit in the work of Picard. However, it is the Polish mathematician Stefan Banach who is credited with placing the underlying ideas into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations. In spite of their being a long years old, the study in metric fixed point theory was limited to minor extensions of Banach's contraction mapping principal and its manifold applications. The theory gained new impetus largely as a result of the pioneering work of Felix Browder in the mid-nineteen sixties and the development of nonlinear functional analysis as an active and vital branch of mathematics. Pivotal in this development were the 1965 existence theorems of Browder, Göhde, and Kirk and the early metric results of Edelstein. By the end of the decade, a rich fixed point theory for nonexpansive mappings was clearly emerging and it was equally clear that such mappings play a main role in many aspects of nonlinear functional analysis with links to variational inequalities and the theory of monotone and accretive operators ([4]).

There are some important connections between the classes of nonexpansive and of accretive mappings which give rise to a strong connection between the fixed point theory of nonexpansive mappings and the mapping theory of accretive maps (for more information about this subject, see [5]).

Definition 1 *([3]) Let X be a real linear space with dim ≥ 2 and $\|.,.\| : X^2 \to [0,\infty)$ be a function. Then $(X, \|.,.\|)$ is called linear 2-normed spaces if*

Body Math $2N_1$) $\|x, y\| = 0 \Longleftrightarrow x$ *and* y are *linearly dependent,*
Body Math $2N_2$) $\|x, y\| = \|y, x\|$,
Body Math $2N_3$) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$,
Body Math $2N_4$) $\|x + y, z\| = \|x, z\| + \|y, z\|$,
Body Math *for all* $\alpha \in \mathbb{R}$ *and all* $x, y, z \in X$.

Example 1 *([3])Let E_3 denotes Euclidean vector three spaces. Let $x = ai + bj + ck$ and $y = di + ej + fk$ define*

$$\begin{aligned} \|x, y\| &= |x \times y| = abs \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} \\ &= \left| (bf - ce)^2 i + (cd - af)^2 j + (ae - db)^2 k \right|^{\frac{1}{2}} . \end{aligned}$$

Then $(E_3, \|.,.\|)$ is a 2-normed space and this space is complete.

Definition 2 *([3]) Let $(X, \|.,.\|)$ be a 2-normed space.*

a) *A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|.,.\|)$ is called a Cauchy sequence if there exist two points y,z $\in X$ such that y and z are linearly independent,*

$$\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0, \quad \lim_{m,n\to\infty} \|x_n - x_m, z\| = 0.$$

b) *A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|.,.\|)$ is called a convergent sequence if there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, z\| = 0$ for every z in X.*

c) *A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called 2-Banach space .*

Definition 3 *([3]) A linear 2-normed space $(X, \|.,.\|)$ is said to be uniformly convex if for any sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X ,$\|x_n, c\| \leq 1$, $\|y_n, c\| \leq 1$, $n = 1, 2, 3, ...,$ $\lim\limits_{n\to\infty} \left\|\frac{1}{2}(x_n + y_n), c\right\| = 1$ and $V(c) \cap (\cap_{n=1}^{\infty} V(x_n, y_n)) = 0$ imply that $\lim\limits_{n\to\infty} \|x_n - y_n, c\| = 0$.*

Definition 4 *([6])Let $(X, \|.,.\|)$ be a linear 2-normed space, E be a nonempty subset of X and $e \in E$ then E is said to be $e-$bounded if there exist some $M > 0$ such that $\|x, e\| \leq M$ for all $x \in E$. If for all $e \in E$, E is $e-$bounded then E is called a bounded set.*

Definition 5 *([7]) $F : X \to 2^{X^*}$ duality map and X^* is dual of X.*

$$D(A) = \{x \in X : \ Ax \neq \theta\}$$

$$R(A) = \cup_{x\in D(A)} Ax,$$

Let A be a nonlinear operator mapping a subset of Banach space X to X. A is said to be *accretive* provided that,

$$\|x - y\| \leq \|x - y + \lambda Ax - \lambda Ay\|$$

for all $\lambda \geq 0$ and for all $x, y \in D(A)$.

Definition 6 *([6]) Let $(X, \|.,.\|)$ be a linear 2-Banach space, Let A be a nonlinear operator mapping a subset of X. $A : D(A) \subset X \to X$ is said to be accretive if for every $x, y, z \in D(A)$ and $\lambda > 0$*

$$\|x - y, z\| \leq \|(x - y) + \lambda(Ax - Ay), z\|$$

Also, an accretive operator is said to m-accretive provided that $R(I+\lambda A)=X$.

Definition 7 *([6])Let $(X, \|.,.\|)$ be a linear 2-normed space then an operator T on X said to be a nonexpansive if for each $x, y, z \in D(T) \subset X$*

$$\|Tx - Ty, z\| \leq \|x - y, z\|$$

2 Main Results

Throughout this paper, our applications in the accretive operator arising from 2-Banach spaces seem to be interesting and worthwhile for further works in 2-Banach spaces and other areas.

In this section, we introduce resolvent operator of an accretive operator and derive numerous links between nonexpansive and accretive operators in 2-Banach space.

Let $(X, \|.,.\|)$ be linear 2-normed space and A be an m-accretive operator in X. Define the resolvent of A as $J_n(x) = (I + n^{-1}A)^{-1}(x)$ and Yosida's approximation $A_n(x) = n(I - J_n)(x)$, $n = 1, 2, 3, ...$, for all $x \in X$ (see [6]).

Proposition 1 *Let $(X, \|.,.\|)$ be a linear 2-normed space, Let A be a accretive operator mapping a subset of X. $A : D(A) \subset X \to X$, then $(I + n^{-1}A)$ is expansive operator and $(I + n^{-1}A)^{-1}$ is nonexpansive.*

It is not difficult to show the following:

$$\begin{aligned} \left\|\left(I + n^{-1}A\right)x - \left(I + n^{-1}Ay\right), z\right\| &= \left\|x - y + n^{-1}(Ax - Ay), z\right\| \\ &\geq \|x - y, z\|. \end{aligned}$$

Hence $(I + n^{-1}A)$ is nonexpansive, so $(I + n^{-1}A)^{-1}$ is nonexpansive.

Example 2 *Let $(X, \|.,.\|)$ be a linear 2-normed space, if T is nonexpansive mapping of $D(T)$ into X and if we set $A = I - T$, $D(T) = D(A)$, then A is an accretive mapping of $D(A)$ into X.*

Solution 1 *For all x, $y \in D(A)$ and $z \in X$, $\lambda > 0$, then we readily see that*

$$\begin{aligned} \|x - y + \lambda(Ax - Ay), z\| &\geq \|x - y, z\| + \lambda \|Ax - Ay, z\| \\ &\geq \|x - y, z\|. \end{aligned}$$

Corollary 1 *Let $(X, \|.,.\|)$ be a 2-normed space and A be a accretive operator on $D_n = R(I + n^{-1}A)$ then $\{J_n(x)\}$ is Cauchy sequence in X.*

$$\begin{aligned}\|J_m(x)-J_n(x),z\| &= \left\|\left(I+m^{-1}A\right)^{-1}x-\left(I+n^{-1}A\right)^{-1}x,z\right\| \\ &= \left\|m^{-1}A^{-1}x-n^{-1}A^{-1}x,z\right\| \\ &= \left|\frac{m-n}{mn}\right|\left\|A^{-1}x,z\right\| \\ &\to 0 \text{ as } m,n\to\infty\end{aligned}$$

Proposition 2 *$J_n(x)$ is defined as above, then $\|J_n(x)-x,z\| \leq n^{-1}\|Ax,z\|$ for $x\in D(A)\cap D_n$, and fixed $z\in X$.*

Let A be a accretive operator and $z\in X$

$$\begin{aligned}\|J_n(x)-x,z\| &= \left\|J_n(x)-J_n(J_n^{-1})x,z\right\| \\ &\leq \left\|x-J_n^{-1}x,z\right\| \\ &= \left\|x-(I+n^{-1}A)x,z\right\| \\ &= \left\|x-(I+n^{-1}A)x,z\right\| \\ &= \left\|n^{-1}A,z\right\| \\ &= n^{-1}\|Ax,z\|.\end{aligned}$$

Thus, $\|J_n(x)-x,z\|\leq n^{-1}.\|Ax,z\|$.

Corollary 2 *Let $(X,\|.,.\|)$ be a 2-normed space and A be a accretive operator on $D_n=R\left(I+n^{-1}A\right)$ then $\{A_n(x)\}$ is Cauchy sequence in X.*

Let A be a accretive operator and $z\in X$

$$\begin{aligned}\|A_m(x)-A_n(x),z\| &= \|m(I-J_m)(x)-n(I-J_n)(x),z\| \\ &= \|(m-n)x+J_m(x)-J_n(x),z\| \\ &\leq \|J_m(x)-J_n(x),z\| \\ &= \left|\frac{m-n}{mn}\right|\left\|A^{-1}x,z\right\| \\ &\to 0 \text{ as } m,n\to\infty\end{aligned}$$

Proposition 3 *Let $(X,\|.,.\|)$ be a linear 2-Banach space, Let A be a accretive operator mapping a subset of X. $A: D(A)\subset X\to X$, then $\|A_nx,z\|\leq\|Ax,z\|$ all $x\in X$ and fixed $z\in X$.*

For all $x \in X$ and fixed $z \in X$, then we compute

$$\begin{aligned} \|A_n x, z\| &= \|n(I - J_n)x, z\| = \|n(x - J_n x), z\| \\ &= n \|x - J_n x, z\| \leq n n^{-1} \|Ax, z\| \\ &= \|Ax, z\| . \end{aligned}$$

Hence we conclude that $\|A_n x, z\| \leq \|Ax, z\|$.

Theorem 1 *Let $(X, \|.,.\|)$ be a uniformly convex 2-Banach space and C be a closed bounded convex subset of X. A be a accretive operator on C. $A : D(A) \subset C \to C$, define*

$$J_n x = (I + n^{-1}A)^{-1} x, \quad \text{for all } x \in D_n = R\left(I + n^{-1}A\right).$$

Then J_n have a fixed point and

$$\lim_{n \to \infty} J_n x = Ix.$$

For all $x, y \in D_n = R(I + n^{-1}A)$, we have

$$\begin{aligned} \|J_n x - J_n y, z\| &= \left\|(I + n^{-1}A)^{-1}x - (I + n^{-1}A)^{-1}y, z\right\| \\ &\leq \|x - y, z\| . \end{aligned}$$

Then J_n is nonexpansive. Also,

$$\begin{aligned} \|J_n x - Ix, z\| &= \left\|J_n x - J_n(J_n^{-1})x, z\right\| \\ &\leq \left\|x - J_n^{-1}x, z\right\| \\ &\to 0 \text{ (as } n \to \infty \text{).} \end{aligned}$$

Thus $\lim_{n \to \infty} J_n x = Ix$.

Remark 1 *As a result of this paper, is it possible to introduce the accretive operator in n-Banach spaces?*

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