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On Generalized Solutions of Boundary Value Problems for Some Class of Fourth Order Operator-Differential Equations on the Segment

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Abstract

In the paper we give definition of a generalized solution of boundary value problems for some fourth order operator-differential equations, and prove the existence of generalized solution of this problem by the coefficients of the equations on the segment.

Keywords: *Hilbert spaces, existence of generalized solution, operator-differential equation.*

1 Introduction

Let H be a separable Hilbert space, A be a positive-definite self-adjoint operator in H with domain of definition $D(A)$. By H_γ we denote a scale of Hilbert space that is generated by the operator A, i.e. $H_\gamma = D(A^\gamma)\,,\ (\gamma \ge 0)\,,\ (x,y)_\gamma = (A^\gamma x, A^\gamma y)\,,\ x,y \in D(A^\gamma)$. By $L_{2,\gamma}((a,b);H_\gamma)$ we denote a Hilbert space of the vector-function $f(t)$ defined in (a,b) almost everywhere, with values from H, measurable square integrable in the Bochner sense

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f\|_\gamma^2 \, dt \right)^{1/2}.$$

Then we determine the Hilbert spaces

$$W_2^2((a,b);H) = \left\{u | u'' \in L_2((a,b);H),\ A^2u \in L_2((a,b);H_2)\right\}$$

with the norm

$$\|u\|_{W_2^2((a,b);H)} = \left(\|u''\|_{L_2((a,b);H)} + \left\|A^2u\right\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

Here and in future the derivatives are understood in the sense of distribution [1]. By $D^4((a,b);H)$ we denote a linear set of infinitely-differentiable functions with values from $H_4 = D(A^4)$ having compact supports in (a,b). Further we determine the space $\overset{\circ}{W}{}_2^2((a,b);H) \subset W_2^2((a,b);H)$ in the following way.

$$\overset{\circ}{W}{}_2^2((a,b);H) = \left\{u | u \in W_2^2((a,b);H),\ u^{(j)}(a) = u^{(j)}(b) = 0,\ \ j = 0,1\right\}.$$

In sequel, we'll assume that (a,b) is as $[0,1]$. Assume that

$$L_2((a,b);H) \equiv L_2([0,1];H), \quad W_2^2((a,b);H) \equiv W_2^2([0,1];H),$$

$$\overset{\circ}{W}{}_2^2((a,b);H) \equiv \overset{\circ}{W}{}_2^2([0,1];H), \ \ D^4((a,b);H) \equiv D^4([0,1];H).$$

In the separable Hilbert space H we consider a polynomial operator bundle in the following form

$$P(\lambda) = \left(-\lambda^2 E + A^2\right)^2 + \sum_{j=0}^{4} A_j \lambda^{(4-j)}, \tag{1}$$

and the related boundary value problem

$$P(d/dt)\, u(t) \equiv \left(-\frac{d^2}{dt^2} + A^2\right)^2 u(t) + \sum_{j=1}^{4} A_j u^{(4-j)}(t) = f(t), \ \ t \in [0,1], \tag{2}$$

$$u^{(j)}(0) = \varphi_j,\ u^{(j)}(1) = \psi_j, \ \ j = 0,1 \tag{3}$$

where A is a self-adjoint positive-definite operator, A_j $\left(j = \overline{1,4}\right)$ are linear, generally speaking, unbounded operators in $H, f \in L_2([0,1];H)$, $\varphi_j, \psi_j \in H,\ j = 0,1$.

In the paper we'll give a generalized solution of boundary value problem (2), (3) and prove for it theorems on the existence of a generalized solution in terms of coefficients of the investigated fourth order differential equation on the seqment.

2 Auxiliary Facts

Denote

$$P_0\left(\frac{d}{dt}\right)u(t) = \left(-\frac{d^2}{dt^2} + A^2\right)^2 u(t)\ , \qquad u(t) \in \overset{\circ}{W}{}_2^2([0,1];H),$$

$$P_1\left(\frac{d}{dt}\right)u(t) = \sum_{j=0}^{4} A_j u^{(4-j)}(t)\ \ , \qquad u(t) \in \overset{\circ}{W}{}_2^2([0,1];H),$$

and

$$Pu(t) = P_0 u(t) + P_1 u(t)\ , \qquad u(t) \in \overset{\circ}{W}{}_2^2([0,1];H).$$

Now we formulate a lemma that shows the conditions on operator coefficients (1) under which the expression $P_1\left(\frac{d}{dt}\right)u(t)$ makes sense for the function from the class $\overset{\circ}{W}{}_2^2([0,1];H)$.

Lemma 2.1. *[2]. Let A be a self-adjoint positive-definite operator, $B_0 = A_0$, $B_1 = A_1 A^{-1}, B_2 = A^{-1}A_2A^{-1}, B_3 = A^{-2}A_3A^{-1}$, $B_4 = A^{-2}A_4A^{-2}$ be bounded ope-rators in H. Then the bilinear functional $\left(P_1\left(\frac{d}{dt}\right)u,\xi\right)_{L_2([0,1];H)}$ determined on $D^4([0,1];H) \oplus \overset{\circ}{W}{}_2^2([0,1];H)$ continues to the space $W_2^2([0,1];H) \oplus \overset{\circ}{W}{}_2^2([0,1];H)$ continuously up to the bilinear functional $P_1(u,\xi)$ acting in the following way*

$$P_1(u,\xi) = \sum_{j=0}^{1}(-1)^j\left(A_j u^{(2-j)},\xi''\right)_{L_2([0,1];H)} - \left(A_2 u', \xi'\right)_{L_2([0,1];H)} +$$

$$+\sum_{j=3}^{4}\left(A_j u^{(4-j)},\xi\right)_{L_2([0,1];H)}.$$

Lemma 2.2. *Let all the conditions of lemma 1 be fulfilled. Then a bilinear functional $\left(P\left(\frac{d}{dt}\right)u,\xi\right)_{L_2([0,1];H)}$ determined on the space $D^4([0,1];H) \oplus \overset{\circ}{W}{}_2^2([0,1];H)$ of the bilinear functional*

$$P(u,\xi) = (u,\xi)_{W_2^2([0,1];H)} + P_1(u,\xi) + 2\left(Au',\xi'\right)_{L_2([0,1];H)}$$

where $P_1(u,\xi)$ is determined as in lemma 1.

Proof. Really

$$P\left(\frac{d}{dt}\right)u(t)=P_0u(t)+P_1u(t).$$

Integrating the following expression by parts we get

$$\left(P_0\left(\frac{d}{dt}\right)u,\xi\right)_{L_2([0,1];H)}=\left(\left(-\frac{d^2}{dt^2}+A^2\right)^2u,\xi\right)_{L_2([0,1];H)}=$$

$$=\left(\left(\frac{d^4}{dt^4}-2A^2\frac{d^2}{dt^2}+A^4\right)u,\xi\right)_{L_2([0,1];H)}=(u'',\xi'')_{L_2([0,1];H)}+$$

$$+2\left(Au',A\xi'\right)_{L_2([0,1];H)}+\left(A^2u,A^2\xi\right)_{L_2([0,1];H)}.$$

Since $Au'(t)\in L_2([0,1];H)$, $A\xi'(t)\in L_2([0,1];H)$, the right hand side of the last equality continues by continuity from the space $D^4([0,1];H)\oplus \overset{\circ}{W}{}_2^2([0,1];H)$ to $W_2^2([0,1];H)\oplus \overset{\circ}{W}{}_2^2([0,1];H)$. The lemma is proved.

Determination. The vector function $u(t)\in W_2^2([0,1];H)$ is said to be a generalized solution of boundary value problem (2), (3) if it satisfies conditions of (3) and for any function $\xi(t)\in \overset{\circ}{W}{}_2^2([0,1];H)$ the equality

$$P(u,\xi)=(f,\xi)_{L_2([0,1];H)}$$

is fulfilled.

Now, let's cite a lemma on the solvability of a boundary value problem for an operator-differential equation representing the principal part of equation (3).

Lemma 2.3. *Let $\varphi_0,\psi_0\in H_{3/2}=D\left(A^{3/2}\right)$, $\varphi_1,\psi_1\in H_{1/2}=D\left(A^{1/2}\right)$. Then the boundary value problems*

$$P_0\left(\frac{d}{dt}\right)u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^2u(t)=f(t),\quad t\in[0,1], \tag{4}$$

$$u^{(j)}(0)=\varphi_j,\quad u^{(j)}(1)=\psi_j,\quad j=0,1 \tag{5}$$

has a unique solution.

3 Basic Results

Assuming the above mentioned facts we prove a theorem on the existence of a generalized solution of boundary value problem (2), (3).

Theorem 3.1. *Let A be a self-adjoint positive-definite operator, $B_0 = A_0$, $B_1 = A_1 A^{-1}$, $B_2 = A^{-1} A_2 A^{-1}$, $B_3 = A^{-2} A_3 A^{-1}$, $B_4 = A^{-2} A_4 A^{-2}$ be bounded operators in H and it holds the inequality*

$$\delta = \sum_{j=0}^{4} \gamma_j \|B_j\| < 1, \tag{6}$$

where $\gamma_0 = \gamma_4 = 1$, $\gamma_1 = \gamma_3 = 1/2$, $\gamma_2 = 1/4$. Then boundary value problem (2), (3) has a unique generalized solution.

Proof. Show that by fulfilling inequality (6) for any $\xi \in \overset{\circ}{W}{}_2^2([0,1]; H)$ it holds the inequality

$$\left| \left(P\left(\frac{d}{dt}\right) \xi, \xi \right)_{L_2([0,1];H)} \right| \geq d \|\xi\|^2_{W_2^2([0,1];H)}, \quad (d > 0).$$

Obviously

$$\left(P\left(\frac{d}{dt}\right) \xi, \xi \right)_{L_2([0,1];H)} = \|\xi\|^2_{W_2^2([0,1];H)} +$$

$$+2 \|A\xi'\|^2_{L_2([0,1];H)} + \left(P_1\left(\frac{d}{dt}\right) \xi, \xi \right)_{L_2([0,1];H)}.$$

Therefore

$$\left| \left(P\left(\frac{d}{dt}\right) \xi, \xi \right)_{L_2([0,1];H)} \right| \geq \|\xi\|^2_{W_2^2([0,1];H)} +$$

$$+2 \|A\xi'\|^2_{L_2([0,1];H)} - \left(P_1\left(\frac{d}{dt}\right) \xi, \xi \right)_{L_2([0,1];H)}. \tag{7}$$

Since

$$\left| \left(P_1\left(\frac{d}{dt}\right) \xi, \xi \right)_{L_2([0,1];H)} \right| \leq \left| \sum_{j=0}^{1} \left(A_j \xi^{(2-j)}, \xi'' \right)_{L_2([0,1];H)} \right| +$$

$$+ \left| \left(A_2 \xi', \xi \right)_{L_2([0,1];H)} \right| + \left| \sum_{j=3}^{4} \left(A_j \xi^{(4-j)}, \xi \right)_{L_2([0,1];H)} \right|,$$

then

$$\left| \left(A_2 \xi', \xi' \right)_{L_2([0,1];H)} \right| = \left| \left(B_2 A\xi', A\xi' \right)_{L_2([0,1];H)} \right| \leq$$

$$\leq \|B_2\| \, \|A\xi'\|_{L_2([0,1];H)} \|A\xi'\|_{L_2([0,1];H)} = \|B_2\| \, \|A\xi'\|^2_{L_2([0,1];H)}. \tag{8}$$

On the other hand

$$\|A\xi'\|^2_{L_2([0,1];H)} \leq \frac{1}{2} \left(\|\xi''\|^2_{L_2([0,1];H)} + \left\|A^2 \xi\right\|^2_{L_2([0,1];H)} \right).$$

Then

$$2\left\|A\xi'\right\|^2_{L_2([0,1];H)} \le \frac{1}{2}\left(\left\|\xi''\right\|^2_{L_2([0,1];H)} + 2\left\|A\xi'\right\|^2_{L_2([0,1];H)} + \left\|A^2\xi\right\|^2_{L_2([0,1];H)}\right)$$

or

$$\left\|A\xi'\right\|^2_{L_2([0,1];H)} \le \frac{1}{4}\left(\left\|\xi''\right\|^2_{L_2([0,1];H)} + 2\left\|A\xi'\right\|^2_{L_2([0,1];H)} + \left\|A^2\xi\right\|^2_{L_2([0,1];H)}\right).$$

Allowing for the last inequality in (8) we get

$$\left\|(A_2\xi',\xi')\right\|^2_{L_2([0,1];H)} \le \left\|B_2\right\|\frac{1}{4}\left(\left\|\xi\right\|^2_{W_2^2([0,1];H)} + 2\left\|A\xi'\right\|^2_{L_2([0,1];H)}\right). \quad (9)$$

In the same way, for $j=0$ we have

$$\left|(A_0\xi'',\xi'')_{L_2([0,1];H)}\right| \le \left\|B_0\right\|\left\|\xi''\right\|^2_{L_2([0,1];H)} \le$$
$$\le \left\|B_0\right\|\left(\left\|\xi''\right\|^2_{W_2^2([0,1];H)} + 2\left\|A\xi'\right\|^2_{L_2([0,1];H)}\right), \quad (10)$$

for $j=1$

$$\left|(A_1\xi',\xi')_{L_2([0,1];H)}\right| \le \left|(B_1A\xi',\xi'')_{L_2([0,1];H)}\right| \le$$
$$\le \left\|B_1\right\|\left\|A\xi'\right\|_{L_2([0,1];H)}\left\|\xi'\right\|_{L_2([0,1];H)} \le$$
$$\le \left\|B_1\right\|\frac{1}{2}\left(\left\|\xi''\right\|^2_{L_2([0,1];H)} + \left\|A\xi'\right\|_{L_2([0,1];H)}\right) \le$$
$$\le \frac{1}{2}\left\|B_1\right\|\left(\left\|\xi\right\|^2_{W_2^2([0,1];H)} + 2\left\|A\xi'\right\|^2_{L_2([0,1];H)}\right), \quad (11)$$

for $j=3$

$$\left|(A_3\xi',\xi')_{L_2([0,1];H)}\right| = \left|\left(B_3A\xi',A^2\xi\right)_{L_2([0,1];H)}\right| \le$$
$$\le \left\|B_3\right\|\left\|A\xi\right\|_{L_2([0,1];H)}\left\|A^2\xi\right\|_{L_2([0,1];H)} \le$$
$$\le \left\|B_3\right\|\frac{1}{2}\left(\left\|A\xi'\right\|^2_{L_2([0,1];H)} + \left\|A^2\xi\right\|^2_{L_2([0,1];H)}\right) \le$$
$$\le \frac{1}{2}\left\|B_3\right\|\left(\left\|\xi\right\|^2_{W_2^2([0,1];H)} + 2\left\|A\xi'\right\|^2_{L_2([0,1];H)}\right), \quad (12)$$

for $j=4$

$$\left|(A_4\xi,\xi)_{L_2([0,1];H)}\right| = \left|\left(B_4A^2\xi,A^2\xi\right)_{L_2([0,1];H)}\right| \le \left\|B_4\right\|\left\|A^2\xi\right\|^2_{L_2([0,1];H)} =$$
$$= \left\|B_4\right\|\left(\left\|\xi\right\|^2_{W_2^2([0,1];H)} + 2\left\|A^2\xi'\right\|^2_{L_2([0,1];H)}\right). \quad (13)$$

Thereby, allowing for inequalities (9)-(13) in (7) we get

$$\left| \left(P\left(\frac{d}{dt}\right)\xi, \xi \right)_{L_2([0,1];H)} \right| \geq (1-\delta)\left(\|\xi\|^2_{W_2^2([0,1];H)} + 2\left\|A\xi'\right\|^2_{L_2([0,1];H)} \right).$$

Now, we search for the generalized solution of boundary value problem (2), (3) in the form of

$$u(t) = v_0(t) + v(t)$$

where $v_0(t)$ is a generalized solution of boundary value problem (4), (5) and $v(t) \in \overset{\circ}{W}{}_2^2([0,1];H)$. Then, to determine $v(t)$ we get

$$P(v_0+v,\xi) = (v_0+v,\xi)_{W_2^2([0,1];H)} + P_1(v_0+v,\xi) + 2\left(Av_0' + Av', A\xi\right)_{L_2([0,1];H)} \equiv$$

$$\equiv (v_0,\xi)_{W_2^2([0,1];H)} + (v,\xi)_{W_2^2([0,1];H)} + P_1(v_0,\xi) + P_1(v,\xi) +$$
$$+2\left(Av_0', A\xi'\right)_{L_2([0,1];H)} = (f,\xi)_{L_2([0,1];H)}.$$

Hence we get

$$(v_0,\xi)_{W_2^2([0,1];H)} + P_1(v,\xi) + 2\left(Av', A\xi\right)_{L_2([0,1];H)} = -P_1(v_0,\xi). \qquad (14)$$

The right hand side of relation (14) determines a continuous functional in $W_2^2([0,1];H) \oplus \overset{\circ}{W}{}_2^2([0,1];H)$, the left-hand side uses equality (7) and satisfies the conditions of Lax-Milgram theorem [3]. Therefore there exists a unique vector - function $v(t) \in \overset{\circ}{W}{}_2^2([0,1];H)$ satisfying equality (14), i.e. $u(t) = v(t) + v_0(t)$ is a generalized solution of boundary value problem (2), (3). The theorem is proved.

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