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Lacunary Ideal Convergence of Double Set Sequences

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Abstract

In this paper the relation between lacunary ideal convergent double set sequences and lacunary ideal Cauchy double set sequences has been established. The notions of lacunary ideal limit sets and lacunary ideal cluster sets have been introduced and find the relation between these two notions.

Keywords: *Wijsman convergence, $\mathcal{I}-$convergence, double sequences, Wijsman $\mathcal{I}-$limit sets, Wijsman $\mathcal{I}-$cluster sets.*

1 Introduction

Hill [15] was the first who applied methods of functional analysis to double sequence. A lot of useful developments of double sequences in summability methods can be found in Limayea and Zeltser [21] and Savaş [29].

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [30]. This concept was extended to the double sequences by Mursaleen and Edely [22]. Mursaleen and Edely [22] extended the above idea from single to double sequences of scalars and established relations between statistical convergence and strongly Cesàro summable double sequences.

The concept of lacunary statistical convergence was defined by Fridy and Orhan [13]. Also, Fridy and Orhan gave the relationships between the lacunary statistical convergence and the Cesàro summability. This concept was extended to the double sequences by Savas and Patterson [29].

The concept of $\mathcal{I}-$convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers. P. Kostyrko et al. [18] introduced the concept of $\mathcal{I}-$convergence of sequences in a metric space and studied some properties of this convergence. In [33] the notion of ideal convergent double sequences was introduced.

The concept of convergence of numbers has been extended by several authors to convergence of sequences of sets (see, Baronti and Papini [2]; Beer [3], [4] ; Wijsman [36]; Kişi and Nuray [16]; Nuray and Rhoades [24]) introduced the concept of statistical convergence of sequences of sets. The concept of $\mathcal{I}-$convergence of real sequences was extended to the sequences of sets by Kişi and Nuray [16]. Sever et al. [31] investigated the ideas of Wijsman strongly $\mathcal{I}-$lacunary convergence, Wijsman strongly $\mathcal{I}^*-$lacunary convergence and Wijsman strongly $\mathcal{I}-$lacunary Cauchy sequences of sets. Ulusu and Nuray [34] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades [24]. Ulusu and Nuray [35] introduced the concept of Wijsman strongly lacunary summability for sequences of sets and discussed its relation with Wijsman strongly Cesàro summability.

2 Definitions and Notations

In this section, we recall some definitions and notations, which form the base for the present study.

Definition 2.1 [17] *A family of sets $\mathcal{I} \subseteq 2^N$ is called an ideal if and only if*

(i) $\emptyset \in \mathcal{I}$,
(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,
(iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

Definition 2.2 [17] *A family of sets $\mathcal{F} \subseteq 2^N$ is a filter in N if and only if*

(i) $\emptyset \notin \mathcal{F}$,
(ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,
(iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

Lemma 2.3 [17] *If $\mathcal{I}$ is proper ideal of N (i.e.,$N \notin \mathcal{I}$), then the family of sets*

$$\mathcal{F}(\mathcal{I}) = \{M \subset N : \exists A \in \mathcal{I} : M = N \setminus A\}$$

is a filter of N it is called the filter associated with the ideal.

An ideal is called non-trivial if $N \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in N$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Definition 2.4 [36] *Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A*

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim_{k \to \infty} A_k = A$.

As an example, consider the following sequence of circles in the (x, y)-plane: $A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}$. As $k \to \infty$ the sequence is Wijsman convergent to the y-axis $A = \{(x, y) : x = 0\}$.

Definition 2.5 [24] *Let (X, d) a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

$$WS := \{\{A_k\} : st - \lim{}_W A_k = A\}$$

where WS denotes the set of Wijsman statistical convergence sequences.

Also the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades [24]. As follows: Let (X, ρ) a metric space. For any non-empty closed subsets A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$.

Definition 2.6 [34] *Let (X, ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary statistical convergent to A if $\{d(x, A_k)\}$ is lacunary statistically convergent to $d(x, A)$; i.e., for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_r \frac{1}{h_r} |k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon| = 0.$$

In this case, we write $S_\theta - \lim_W = A$ or $A_k \to A(WS_\theta)$.

Definition 2.7 [35] *Let (X, ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary Summable to A if $\{d(x, A_k)\}$ is lacunary summable to $d(x, A)$; i.e., for each $x \in X$,*

$$\lim_{r\to\infty} 1h_r \sum_{I_r} d(x, A_k) = d(x, A).$$

In this case, we write $S_\theta - \lim_W = A$ or $A_k \to A(WN_\theta)$.

Definition 2.8 [35] *Let (X, ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly lacunary Summable to A if $\{d(x, A_k)\}$ is strongly lacunary summable to $d(x, A)$; i.e., for each $x \in X$,*

$$\lim_{r\to\infty} 1h_r \sum_{I_r} |d(x, A_k) - d(x, A)| = 0.$$

In this case, we write $S_\theta - \lim_W = A$ or $A_k \to A([WN_\theta])$.

Definition 2.9 [16] *Let (X, d) be a metric space and $\mathcal{I} \subseteq 2^N$ be a proper ideal in N. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}-$convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$, the set,*

$$A\,(x, \varepsilon) = \{k \in N : |d\,(x, A_k) - d\,(x, A)| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. In this case, we write $\mathcal{I}_W - \lim A_k = A$ or $A_k \to A(\mathcal{I}_W)$, and the set of Wijsman $\mathcal{I}-$convergent sequences of sets will be denoted by

$$\mathcal{I}_W = \{\{A_k\} : \{k \in N : |d\,(x, A_k) - d\,(x, A)| \geq \varepsilon\} \in \mathcal{I}\}\,.$$

Definition 2.10 [28] *A double sequence $x = (x_{k,l})$ has a Pringsheim limit L (denoted by $P-\lim x = L$) provided that given an $\varepsilon > 0$, there exists a $n \in N$ such that $|x_{k,l} - L| < \varepsilon$, whenever $k, l > n$. We describe such an $x = (x_{k,l})$ more briefly as "P-convergent".*

The double sequence $(x_{k,l})$ is bounded if there exists a positive integer M such that $|x_{k,l}| < M$ for all k and l. We denote all bounded double sequence by l^2_∞.

Throught the paper, A, $A_{k,l}$ be any non-empty closed subsets of X.

Definition 2.11 [25] *The double sequence $\{A_{k,l}\}$ is Wijsman convergent to A, if for each $x \in X$*

$$P - \lim_{k,l\to\infty} d\,(x, A_{k,l}) = d\,(x, A) \quad \textit{or}$$

$\lim_{k,l\to\infty} d\,(x, A_{k,l}) = d\,(x, A)$.

In this case, we write $W_2 - \lim A_{k,l} = A$.

Definition 2.12 [25] *The double sequence $\{A_{k,l}\}$ is Wijsman statistically convergent to A, if for each $x \in X$ and for every $\varepsilon > 0$,*

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\{k \leq m, l \leq n : |d(x, A_{k,l}) - d(x, A)| \geq \varepsilon\}\right| = 0,$$

that is,

$$|d(x, A_{k,l}) - d(x, A)| < \varepsilon, a.a. (k, l).$$

In this case, we write $st_2 - \lim_W A_{k,l} = A$.

The set of Wijsman statistically convergent double sequences will be denoted by

$$W_2S := \left\{\{A_{k,l}\} : st_2 - \lim_W A_{k,l} = A\right\}.$$

Throughout the paper, we shall denoted by $\mathcal{I}$ be an admissible ideal of $N \times N$ and $\theta_{r,s} = \{(k_r.l_s)\}$ a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

A double sequence $\overline{\theta} = \theta_{r,s} = \{(k_r.l_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers (k_r) and (l_s) such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty as r \to \infty$$

and

$$l_0 = 0, \overline{h}_s = l_s - l_{s-1} \to \infty as s \to \infty.$$

Let us denote $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h}_s$ and $\theta_{r,s}$ is determined by

$$J_{r,s} = \{(k, l) : k_{r-1} < k \leq k_r and l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \overline{q}_s = \frac{l_s}{l_{s-1}} and q_{r,s} = q_r \overline{q}_s.$$

For details on double lacunary sequence we refer to [29].

Definition 2.13 [26] *Let $\mathcal{I}_2$ be an admissible ideal of $N \times N$. We say that the double sequence $\{A_{k,l}\}$ is Wijsman $\mathcal{I}_2-$convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$, the set,*

$$A(x, \varepsilon) = \{(k, l) \in N \times N : |d(x, A_{k,l}) - d(x, A)| \geq \varepsilon\}$$

belongs to $\mathcal{I}_2$.

In this case, we write $\mathcal{I}_2 - \lim A_k = A$ or $A_{k,l} \to A(\mathcal{I}_2)$.

Definition 2.14 [9] *Let $\theta_{r,s} = (k_{r,s})$ be a double lacunary sequence. Then, a double set sequence $\{A_{k,l}\}$ is said to be strongly $\mathcal{I}_{\theta_{r,s}}-$convergent is for every $\varepsilon > 0$, and for every $x \in X$ such that*

$$\left\{(r,s) \in N \times N : \frac{1}{h_{r,s}} \sum_{(k,l)\in J_{r,s}} |d(x, A_{k,l}) - d(x, A)| \geq \varepsilon\right\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I}_{\theta_{r,s}} - \lim A_{k,l} = A$.

Definition 2.15 [9] *Let $\mathcal{I}$ be an admissible ideal of $N \times N$. A double set sequence $\{A_{k,l}\}$ is said to be strongly $\mathcal{I}_{\theta_{r,s}}$ − Cauchy if there exists a subsequence $\left\{A_{k'(r),l'(s)}\right\}$ of $\{A_{k,l}\}$ such that $(k'(r), l'(s)) \in J_{r,s}$ for each r, s, $\lim_{r,s\to(\infty,\infty)} A_{k'(r),l'(s)} = A$ and or every $\varepsilon > 0$, and for every $x \in X$ such that*

$$\left\{(r,s) \in N \times N : \frac{1}{h_{r,s}} \sum_{(k,l)\in J_{r,s}} \left|d(x, A_{k,l}) - d\left(x, A_{k'(r),l'(s)}\right)\right| \geq \varepsilon\right\} \in \mathcal{I}.$$

3 Main Results

Theorem 3.1 *Let $\mathcal{I}$ be an admissible ideal of $N \times N$. A double sequence $\{A_{k,l}\}$ is strongly $\mathcal{I}_{\theta_{r,s}}-$convergent if and only if it is strongly $\mathcal{I}_{\theta_{r,s}}-$Cauchy sequence.*

Proof: Let $\{A_{k,l}\}$ be strongly $\mathcal{I}_{\theta_{r,s}}$ − *convergent*. Suppose that $\mathcal{I}_{\theta_{r,s}} - \lim A_{k,l} = A$. Let

$$H_{(i,j)} = \left\{(r, s) \in N \times N : \frac{1}{h_{r,s}} \sum_{(k,l)\in J_{r,s}} |d(x, A_{k,l}) - d(x, A)| < \frac{1}{ij}\right\}$$

for each $i, j \in N$. Hence for each i, j, $H_{(i+1,j+1)} \subseteq H_{(i,j)}$ and

$$\left\{(r, s) \in N \times N : \frac{\left|H_{(i,j)} \cap J_{r,s}\right|}{h_{r,s}} \geq \frac{1}{ij}\right\} \in \mathcal{I}.$$

We choose k_1, l_1 such that $r \geq k_1$, $s \geq l_1$, then,

$$\left\{(r, s) \in N \times N : \frac{\left|H_{(1,1)} \cap J_{r,s}\right|}{h_{r,s}} < 1\right\} \notin \mathcal{I}.$$

Next we choose $k_2 > k_1$, $l_2 > l_1$ such that $r \geq k_2$, $s \geq l_2$, then,

$$\left\{(r, s) \in N \times N : \frac{\left|H_{(2,2)} \cap J_{r,s}\right|}{h_{r,s}} < 1\right\} \notin \mathcal{I}.$$

Proceeding in this way inductively we can choose $k_{p+1} > k_p$, $l_{q+1} > l_s$ such that $r \geq k_{p+1}$, $s \geq l_{q+1}$ implies that $H_{(p+1,q+1)} \cap J_{r,s} \neq \emptyset$. Further for each r, s satisfying $k_1 \leq r \leq k_2$, $l_1 < s < l_2$, choose $(k'(r), l'(s)) \in H_{(p,q)} \cap J_{r,s}$ such that

$$\left| d\left(x, A_{k'(r),l'(s)}\right) - d(x, A) \right| < \frac{1}{pq}.$$

This implies that

$$\lim_{r,s \to (\infty,\infty)} A_{k'(r),l'(s)} = A.$$

Therefore, for every $\varepsilon > 0$, we have

$$\left\{ (r,s) \in N \times N : 1h_{r,s_{(k,l) \in J_{r,s}}} \left| d(x, A_{k,l}) - d\left(x, A_{k'(r),l'(s)}\right) \right| \geq \varepsilon \right\}$$

$$\subseteq \left\{ (r,s) \in N \times N : 1h_{r,s_{(k,l) \in J_{r,s}}} \left| d(x, A_{k,l}) - d(x, A) \right| \geq \varepsilon 2 \right\}$$

$$\cup \left\{ (r,s) \in N \times N : 1h_{r,s_{(k'(r),l'(s)) \in J_{r,s}}} \left| d\left(x, A_{k'(r),l'(s)}\right) - d(x, A) \right| \geq \varepsilon 2 \right\}.$$

i.e.

$$\left\{ (r,s) \in N \times N : 1h_{r,s_{(k,l) \in J_{r,s}}} \left| d(x, A_{k,l}) - d\left(x, A_{k'(r),l'(s)}\right) \right| \geq \varepsilon \right\} \in \mathcal{I}.$$

Then, $\{A_{k,l}\}$ is a strongly $\mathcal{I}_{\theta_{r,s}} - Cauchy$ sequence.

Conversely suppose $\{A_{k,l}\}$ is a strongly $\mathcal{I}_{\theta_{r,s}} - Cauchy$ sequence. Then, for every $\varepsilon > 0$, we have

$$\left\{ (r,s) \in N \times N : 1h_{r,s_{(k,l) \in J_{r,s}}} \left| d(x, A_{k,l}) - d(x, A) \right| \geq \varepsilon \right\}$$

$$\subseteq \left\{ (r,s) \in N \times N : 1h_{r,s_{(k,l) \in J_{r,s}}} \left| d(x, A_{k,l}) - d\left(x, A_{k'(r),l'(s)}\right) \right| \geq \varepsilon 2 \right\}$$

$$\cup \left\{ (r,s) \in N \times N : 1h_{r,s_{(k'(r),l'(s)) \in J_{r,s}}} \left| d\left(x, A_{k'(r),l'(s)}\right) - d(x, A) \right| \geq \varepsilon 2 \right\}.$$

It follows that $\{A_{k,l}\}$ is a strongly $\mathcal{I}_{\theta_{r,s}} - convergent$ sequence.

The definition of double lacunary refinement for sequences of sets is given as follows:

Definition 3.2 *The double index sequence* $\rho = \left(\overline{k}_r, \overline{l}_s\right)$ *is called a double lacunary refinement of the double lacunary sequence* $\theta = (k_r, l_s)$ *if* $(k_r, l_s) \subseteq \left(\overline{k}_r, \overline{l}_s\right)$.

Theorem 3.3 *If* $(\rho_{r,s})$ *is a double lacunary refinement of* $\theta_{r,s}$ *and* $A_{k,l} \to A\left(\mathcal{I}_{\rho_{r,s}}\right)$, *then* $A_{k,l} \to A\left(\mathcal{I}_{\theta_{r,s}}\right)$.

Proof: Suppose each $l_{r,s}$ of $\theta_{r,s}$ contains the points $\left(\overline{k}_{r,i},\bar{l}_{s,j}\right)_{i,j=1}^{v(r),u(s)}$ of $(\rho_{r,s})$ so that

$$k_{r-1}<\overline{k}_{r,1}<\overline{k}_{r,2},...<\overline{k}_{r,v(r)}=k_r,$$

where

$$\bar{l}_{r,i}=\left(\overline{k}_{r,i-1},\overline{k}_{r,i}\right],$$

$$l_{s-1}<\bar{l}_{s,1}<\bar{l}_{s,2},...<\overline{k}_{s,u(s)}=l_s,$$

where

$$\overline{J}_{s,j}=\left(\bar{l}_{s,j-1},\bar{l}_{s,j}\right],$$

and

$$\overline{J}_{r,s,i,j}=\left\{(k,l):\overline{k}_{r,i-1}<k\leq\overline{k}_r;\bar{l}_{s,i-1}<l\leq\bar{l}_s\right\}$$

for all r, s and let $v(r)\geq 1$, $u(s)\geq 1$. This implies that $(k_r,l_s)\subseteq\left(\overline{k}_r,\bar{l}_s\right)$. Let $\left(\overline{J}_{i,j}\right)_{i,j=1,1}^{\infty,\infty}$ be the sequence of abutting blocks of $\left(\overline{J}_{r,s,i,j}\right)$ ordered by increasing a lower right index points. Since $A_{k,l}\to A\left(\mathcal{I}_{\rho_{r,s}}\right)$, we have the following for each $\varepsilon>0$,

$$\left\{(i,j)\in N\times N:1\overline{h}_{i,j\overline{J}_{i,j}\subset J_{r,s}}\left|d\left(x,A_{k,l}\right)-d\left(x,A\right)\right|\geq\varepsilon\right\}.1 \tag{1}$$

As before, we write $h_{r,s}=h_r\overline{h}_s$; $\overline{h}_{r,i}=\overline{k}_{r,i}-\overline{k}_{r,i-1}$, $\overline{h}_{s,j}=\bar{l}_{s,j}-\bar{l}_{s,j-1}$,

For each $\varepsilon>0$ we have

$$\left\{(r,s)\in N\times N:1h_{r,s(k,l)\in J_{r,s}}\left|d\left(x,A_{k,l}\right)-d\left(x,A\right)\right|\geq\varepsilon\right\}$$

$$\subseteq\left\{(r,s)\in N\times N:1h_{r,s(k,l)\in J_{r,s}}\right.$$

$$\left.\left\{(i,j)\in N\times N:1\overline{h}_{i,j\overline{J}_{i,j}\subset J_{r,s};(k,l)\in\overline{J}_{i,j}}\left|d\left(x,A_{k,l}\right)-d\left(x,A\right)\right|\geq\varepsilon\right\}\right\}.$$

By (2.1), for each $\varepsilon>0$ if we define

$$t_{i,j}=\left(1\overline{h}_{i,j(k,l)\in\overline{J}_{i,j}}\left|d\left(x,A_{k,l}\right)-d\left(x,A\right)\right|\geq\varepsilon\right)_{i,j=1}^{\infty,\infty},$$

then, $(t_{i,j})$ is Pringsheim null sequence. The transformation

$$(At)_{r,s}=1h_{r,s(k,l)\in J_{r,s}}\left(\overline{h}_{i,j}1\overline{h}_{i,j(k,l)\in\overline{J}_{i,j}}\left|d\left(x,A_{k,l}\right)-d\left(x,A\right)\right|\geq\varepsilon\right)$$

satisfies all condition for a matrix transformation to map a Pringsheim null sequence into a Pringsheim null sequence. Therefore $A_{k,l}\to A\left(\mathcal{I}_{\theta_{r,s}}\right)$. This completes the proof of the theorem.

3.1 Wijsman $\mathcal{I}-$Limit Sets and Wijsman $\mathcal{I}-$Cluster Sets of Double Sequences of Sets

In this section we introduce Wijsman $\mathcal{I}-$limit sets of double sequences of sets, Wijsman $\mathcal{I}-$cluster sets of double sequences of sets and prove some basic properties of these concepts and establish some basic theorems.

Definition 3.4 *Let $\{A_{k,l}\}$ be a double set sequence. Then,*

(1) A set A is said to be $\mathcal{I}_{\theta_{r,s}}-$limit set of $\{A_{k,l}\}$ if there is a set

$$M = \{(k_1, l_1) < (k_2, l_2) < ... < (k_r, l_s) < ...\} \subset N \times N$$

such that the set

$M' = \{(r, s) \in N \times N : (k_r, l_s) \in J_{r,s}\} \notin \mathcal{I}$ and $\theta_{r,s} - \lim A_{k_r,l_s} = A$.

(ii) A set A is said to be $\mathcal{I}_{\theta_{r,s}}-$cluster set of $\{A_{k,l}\}$ if for every $\varepsilon > 0$ we have

$$\left\{(r, s) \in N \times N : 1h_{r,s_{(k,l)\in J_{r,s}}} |d(x, A_{k,l}) - d(x, A)| \geq \varepsilon\right\} \notin \mathcal{I}.$$

Let $\Lambda^{\mathcal{I}}_{\theta_{r,s}}(A)$ denote the set of all $\mathcal{I}_{\theta_{r,s}}-$limit sets and $\Gamma^{\mathcal{I}}_{\theta_{r,s}}(A)$ denote the set of all $\mathcal{I}_{\theta_{r,s}}-$cluster sets, respectively.

Theorem 3.5 *Let $\{A_{k,l}\}$ be a double set sequence. Then $\Lambda^{\mathcal{I}}_{\theta_{r,s}}(A) \subset \Gamma^{\mathcal{I}}_{\theta_{r,s}}(A)$.*

Proof: Let $A \in \Lambda^{\mathcal{I}}_{\theta_{r,s}}(\{A_{k,l}\})$, then there exists a set $M \subset N \times N$ such that $M' \notin \mathcal{I}$, where M and M' are as in the Definition 16, satisfies $\theta_{r,s} - \lim A_{k_r,l_s} = A$. Thus, for every $\varepsilon > 0$ there exists $m_0, n_0 \in N$ such that

$$1h_{r,s_{(k,l)\in J_{r,s}}} |d(x, A_{k_r,l_s}) - d(x, A)| < \varepsilon$$

for all $r \geq m_0$, $s \geq n_0$. Therefore,

$$B = \left\{(r, s) \in N \times N : 1h_{r,s_{(k,l)\in J_{r,s}}} |d(x, A_{k,l}) - d(x, A)| < \varepsilon\right\}$$

$$\supset M' \setminus \{(k_1, l_1), (k_2, l_2), ..., (k_{m_0}, l_{m_0})\}.$$

Now, with $\mathcal{I}$ being admissible, we must have $M' \setminus \{(k_1, l_1), (k_2, l_2), ..., (k_{m_0}, l_{m_0})\} \notin \mathcal{I}$ and as such $B \notin \mathcal{I}$. Hence $A \in \Gamma^{\mathcal{I}}_{\theta_{r,s}}(\{A_{k,l}\})$.

Theorem 3.6 *Let $\{A_{k,l}\}$ be a double set sequence. Let $\mathcal{I}$ be a nontrivial admissible ideal in $N \times N$. If there is a $\{B_{k,l}\}$ such that $\{(k, l) \in N \times N : B_{k,l} \neq A_{k,l}\} \in \mathcal{I}$ then A is also $\mathcal{I}_{\theta_{r,s}}-$convergent.*

Proof: Suppose that $\{(k,l) \in N \times N : B_{k,l} \neq A_{k,l}\} \in \mathcal{I}$ and $\mathcal{I}_{\theta_{r,s}} - \lim B_{k,l} = A$. Then for every $\varepsilon > 0$, the set

$$\left\{(r,s) \in N \times N : 1h_{r,s_{(k,l)\in J_{r,s}}} \left|d\left(x, B_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\} \in \mathcal{I}.$$

For every $\varepsilon > 0$, we have

$$\left\{(r,s) \in N \times N : 1h_{r,s_{(k,l)\in J_{r,s}}} \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}$$

$$\subseteq \{(k,l) \in N \times N : B_{k,l} \neq A_{k,l}\} \cup$$

$$\left\{(r,s) \in N \times N : 1h_{r,s_{(k,l)\in J_{r,s}}} \left|d\left(x, B_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}.$$

As both the sets of right-hand side of the above relation is in $\mathcal{I}$, therefore we have that

$$\left\{(r,s) \in N \times N : 1h_{r,s_{(k,l)\in J_{r,s}}} \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\} \in \mathcal{I}.$$

This completes the proof of the theorem.

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