Gen. Math. Notes, Vol. 31, No. 2, December 2015, pp. 34-53
ISSN 2219-7184; Copyright © ICSRS Publication, 2015
www.i-csrs.org
Available free online at http://www.geman.in

Physical Reality of Complex Numbers is Proved by Research of Resonance

Alexander Alexandrovich Antonov

Research Center of Information Technologies TELAN Electronics
P.O. Box 73, Kiev, 03142, Ukraine
E-mail: telan@bk.ru

(Received: 29-7-15 / Accepted: 13-10-15)

Abstract

On the example of research of the simplest electric LCR-circuits it is shown that they have different frequencies corresponding to various definitions of resonance at real frequencies. Consequently, the resonance theory is imperfect. Proceeding from the geometric sense of Cassinian oval, the resonance theory based on complex frequencies free of the aforementioned contradictions is suggested. The physical meaning of resonance on complex frequencies is explained. The physical reality of resonance on complex frequencies is concluded.

Keywords: *Resonance, Real Frequencies, Complex Frequencies, LCR electric circuits.*

1 Introduction

Resonance is well-known and widely used natural phenomenon. Nonetheless, it has not been studied in full yet. People often encounter resonance in mechanics, hydraulics, radio electronics etc. The simplest domain of theoretical and practical study of its peculiarities is the electric circuit processes.

2 Hypotheses

By present time for an explanation of a resonance it is offered two hypotheses:

- The resonance takes place on real frequencies.
- The resonance takes place on complex frequencies.

And it is supposed, that first hypothesis corresponds to physical essence of a resonance, and second hypothesis is sometimes used for engineering calculations.

3 Hypotheses Testing

3.1 Testing of Resonance on Real Frequencies

Generally accepted interpretation of resonance in LC circuits is not inconsistent and therefore, does not draw objections.

However, since LC circuits are of marginal practical interest, the aforementioned interpretation of resonance is usually generalized on the examples of LCR circuit. Upon resonance in LCR two pole circuits it is suggested that:

- Complex resistance and complex conductivity of a two-pole circuit is becoming purely active, and as a result the forced constituent of response and pure influence coincide in phase – let's refer to this statement as the first definitive attribute of resonance;
- Complex resistance and complex conductivity of a two pole circuit possess extreme module value, through which the amplitude of forced constituent of extreme module value, through which the amplitude of forced constituent of response is becoming max or min respectively – let's refer to this statement as the second definitive of resonance;
- Resonant frequency coincides with the frequency free oscillations.

And upon resonance in LCR multi pole circuits it is suggested that:

- The complex value of transfer function is becoming purely active, and, respectively, the response and influence coincide in phase – let's refer to this statement as the third definitive attribute of resonance;
- The complex transfer function and, respectively, the amplitude of forced constituent of response possess extreme module value – let's refer to this statement as the fourth definitive attribute of resonance;
- Resonant frequency coincides with frequency of free oscillations.

However, the given interpretation of resonance in LCR-circuits cannot be acknowledged as perfect, for it entails contradictory results even in the simplest cases. Let's prove it on the examples of resonance in the simplest LCR circuits in the form of series RLC circuits.

Complex resistance of series RLC circuit shown in fig. 1a, will be equal to

$$Z(j\omega) = L\frac{\omega\frac{r}{L} + j\left(\omega^2 - \frac{1}{LC}\right)}{\omega} = L\frac{2\sigma_0\omega + j\left(\omega^2 - \omega_0{}^2\right)}{\omega} \quad (1)$$

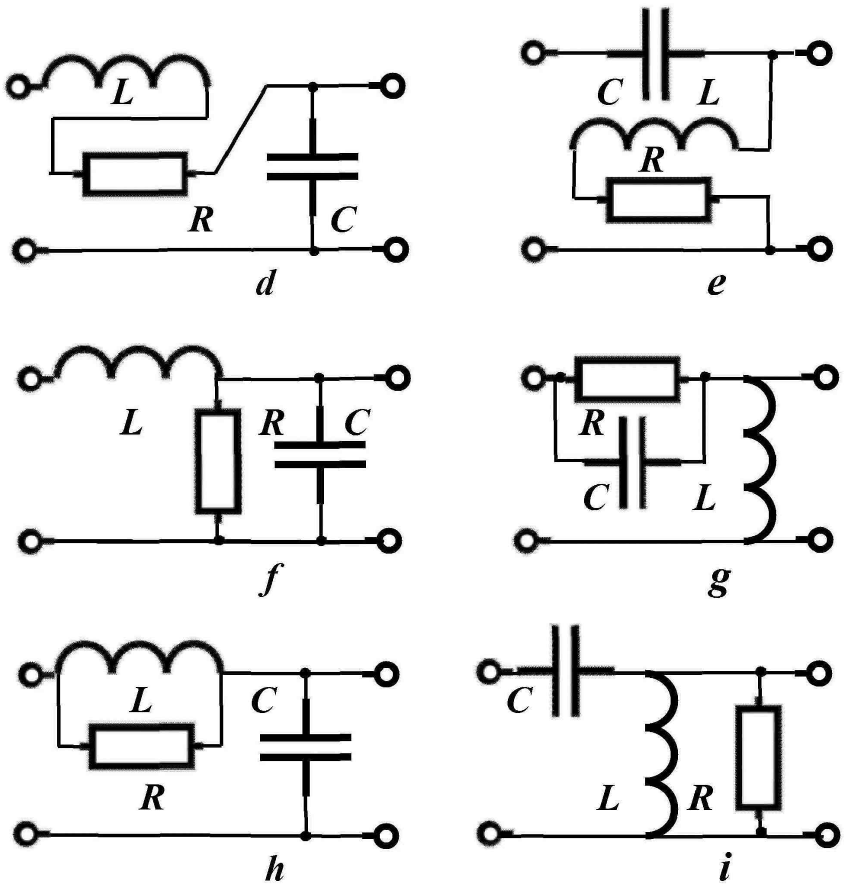

Fig. 1: The electric LCR-circuit under consideration

That’s why it’s reactive constituent in accordance with the first definitive attribute of resonance, acquires zero value on resonant frequency $\omega_{res1} = \omega_0$.

$$ImZ(j\omega) = L\frac{\omega^2 - \omega_0^{\,2}}{\omega} = \rho\frac{\left(\frac{\omega}{\omega_0}\right)^2 - 1}{\frac{\omega}{\omega_0}}$$

On this frequency the complex resistance of LCR circuit is becoming purely active

$$Z(j\omega_{res1}) = r = \frac{\rho}{Q}$$

where $\rho = \sqrt{\frac{L}{C}} = \frac{1}{\omega_0 C} = \omega_0 L$ – wave resistance of LCR-circuit;

$Q = \frac{1}{r}\sqrt{\frac{L}{C}} = \frac{\rho}{r} = \frac{\omega_0}{2\sigma}$ – Q factor of LCR circuit.

The complex resistance module of such LCR-circuit equals to

$$|Z(j\omega)| = L\sqrt{\frac{4\sigma_0^{\,2}\omega^2 + \left(\omega^2 - \omega_0^{\,2}\right)}{\omega^2}} = \rho\sqrt{\frac{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2 + \left[\left(\frac{\omega}{\omega_0}\right)^2 - 1\right]^2}{\left(\frac{\omega}{\omega_0}\right)^2}} \qquad (2)$$

Studying the radicand in the last proportion on the extreme we find the value of resonant frequency, corresponding to the aforementioned second definitive attribute of resonance in LCR two pole circuits $\omega_{res2} = \omega_0$.
In view of the findings in (2) it follows that

$$Z(j\omega_{res2}) = r = \frac{\rho}{Q}$$

As is clear, in the given LCR circuits the values of resonant frequency corresponding to the first and the second definitive attributes of resonance in LCR two pole circuits, are found to be equal. And it seems to be quite natural since they determine the same phenomenon, i.e. resonance, from different points of view. However, it is not always the case, as is shown next.

In fact, for the second series LCR circuit (fig. lb), the complex resistance of which equals to

$$Z(j\omega) = L\frac{\frac{1}{RC}\omega + j\left(\omega^2 - \frac{1}{LC}\right)}{\omega - j\frac{1}{RC}} = L\frac{2\sigma_0\omega + j\left(\omega^2 - \omega_0{}^2\right)}{\omega - j2\sigma_0}$$

from equation

$$Im\, Z(j\omega) = L\frac{\omega\left(4\sigma_0{}^2 + \omega^2 - \omega_0{}^2\right)}{\omega^2 + 4\sigma_0{}^2} = \rho\frac{\left(\frac{\omega}{\omega_0}\right)\left[\frac{1}{Q^2} + \left(\frac{\omega}{\omega_0}\right)^2 - 1\right]}{\left(\frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}} = 0$$

we receive the aggregate of two decisions (not one!), proper to resonant frequencies

$$\left[\begin{array}{l} \omega'_{res1} = 0 \\ \omega''_{res1} = \sqrt{\omega_0{}^2 - 4\sigma_0{}^2} = \omega_0\frac{\sqrt{Q^2 - 1}}{Q} \end{array}\right. \qquad (3)$$

Studying the radicand of circuit complex resistance module on extreme

$$|Z(j\omega)| = L\sqrt{\frac{4\sigma_0{}^2\omega^2 + \left(\omega^2 - \omega_0{}^2\right)^2}{\omega^2 + 4\sigma_0{}^2}} = \rho\sqrt{\frac{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2 + \left[\left(\frac{\omega}{\omega_0}\right)^2 - 1\right]}{\left(\frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}} \qquad (4)$$

i.e. solving equation

$$\frac{d}{d\omega}\left[\frac{4\sigma_0{}^2\omega^2 + \left(\omega^2 - \omega_0{}^2\right)^2}{\omega^2 + 4\sigma_0{}^2}\right] = \frac{2\omega\left[\omega^4 + \omega^2 8\sigma_0{}^2 + \left(16\sigma_0{}^4 - 8\sigma_0{}^2 - \omega_0{}^4\right)\right]}{\omega^2 + 4\sigma_0{}^2} = 0$$

We receive another aggregate of two decisions proper to resonant frequencies

$$\left[\begin{array}{l} \omega'_{res2} = 0 \\ \omega''_{res2} = \sqrt{\omega_0\sqrt{\omega_0{}^2 + 8\sigma_0{}^2} - 4\sigma_0{}^2} = \omega_0\frac{\sqrt{Q\sqrt{Q^2 + 2} - 1}}{Q} \end{array}\right. \qquad (5)$$

Circuit resistance on resonant frequencies of the first aggregate will be

$$Z\left(j\omega'_{res1}\right)=R=\rho Q \qquad (6a)$$

$$Z\left(j\omega''_{res1}\right)=\frac{L}{RC}=\frac{\rho}{Q} \qquad (6b)$$

and complex circuit resistance on resonant frequencies of the second aggregate will be

$$Z\left(j\omega'_{res2}\right)=R=\rho Q \qquad (7a)$$

$$Z\left(j\omega''_{res2}\right)=\rho\frac{\sqrt{2Q\sqrt{Q^2+2}-(1+2Q^2)}}{Q} \qquad (7b)$$

From comparison of expressions (6) and (7) it is clear that $Z\left(j\omega''_{res1}\right)\neq\left|Z\left(j\omega''_{res2}\right)\right|$.
Complex resistance of another LCR circuit (fig. lc) will be equal to

$$Z\left(j\omega\right)=L\frac{\omega\frac{1}{RC}+j\left(\omega^2-\frac{1}{LC}\right)}{\omega+j\omega^2\frac{L}{R}}=L\frac{\omega_0{}^2\left[2\sigma_0\omega+j\left(\omega^2-\omega_0{}^2\right)\right]}{\omega\left(\omega_0{}^2+j2\sigma_0\omega\right)}$$

Its reactive constituent

$$Im\,Z\left(j\omega\right)=L\frac{\omega_0{}^2\left[\omega^2\left(\omega_0{}^2-4\sigma_0{}^2\right)-\omega_0{}^4\right]}{\omega\left(\omega_0{}^4+4\sigma_0{}^2\omega^2\right)}=\rho\frac{\left(\frac{\omega}{\omega_0}\right)^2\frac{Q^2-1}{Q^2}-1}{\left(\frac{\omega}{\omega_0}\right)\left[1+\left(\frac{\omega}{\omega_0}\right)\frac{1}{Q^2}\right]}$$

Possesses zero value on resonant frequency

$$\omega_{res1}=\frac{\omega_0{}^2}{\sqrt{\omega_0{}^2-4\sigma_0{}^2}}=\omega_0\frac{Q}{\sqrt{Q^2-1}} \qquad (8)$$

On this frequency the complex circuit resistance is becoming active

$$Z\left(j\omega_{res1}\right)=\frac{L}{RC}=\frac{\rho}{Q} \qquad (9)$$

Let’s calculate the module of complex resistance of such LCR circuit:

$$\left|Z(j\omega)\right| = \rho\sqrt{\frac{4\sigma_0{}^2\omega_0{}^2\omega^2 + \left(\omega^2-\omega_0{}^2\right)^2\omega_0{}^2}{4\sigma_0{}^2\omega^4 + \omega_0{}^4\omega^2}} = \rho\sqrt{\frac{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2 + \left[\left(\frac{\omega}{\omega_0}\right)^2 - 1\right]^2}{\left(\frac{\omega}{\omega_0}\right)^2\left[1+\left(\frac{\omega}{\omega_0}\right)^2\frac{1}{Q^2}\right]}} \quad (10)$$

Studying of the radicand (10) on extreme allows finding resonant frequency proper to the second resonant frequency

$$\omega_{res2} = \omega_0\sqrt{\frac{\omega_0{}^3\sqrt{\omega_0{}^2+8\sigma_0{}^2} + 4\sigma_0{}^2\omega_0{}^2}{\omega_0{}^4 + 8\sigma_0{}^2\omega_0{}^2 - 16\sigma_0{}^4}} = \omega_0\sqrt{\frac{Q^3\sqrt{Q^2+2}+Q^2}{Q^4+2Q^2-1}} \quad (11)$$

On this resonant frequency the complex resistance module of studies circuit looks as follows

Table 1

lg Q	Q	ω''_{res1}/ω_0 (3)	ω''_{res2}/ω_0 (5)	ω_{res1}/ω_0 (8)	ω_{res2}/ω_0 (11)
0	1,000 000 000	0,000 000 000	0,855 599 677	∞	1,168 770 894
0,1	1,258 925 412	0,607 488 811	0,934 349 493	1,646 120 853	1,070 263 331
0,2	1,584 893 192	0,755 817 523	0,970 629 714	1,288 962 894	1,030 259 002
0,3	1,995 262 315	0,865 338 868	0,987 181 020	1,155 616 645	1,012 985 440
0,4	2,511 886 432	0,917 338 913	0,984 538 877	1,090 109 649	1,005 491 111
0,5	3,162 277 660	0,948 683 298	0,997 719 958	1,054 092 553	1,002 285 252
0,6	3,981 071 706	0,967 938 152	0,999 062 537	1,033 123 860	1,000 938 343
0,7	5,011 872 336	0,979 892 486	0,999 618 734	1,020 520 123	1,000 381 411
0,8	6,309 573 445	0,987 360 692	0,999 846 091	1,012 801 105	1,000 153 933
0,9	7,943 282 347	0,992 043 884	0,999 938 177	1,008 019 923	1,000 061 827
1,0	10,000 000 000	0,994 987 437	0,999 975 247	1,005 037 815	1,000 024 754
1,1	12,589 254 118	0,996 840 221	0,999 990 110	1,003 169 795	1,000 009 891
1,2	15,848 931 925	0,998 007 479	0,999 996 053	1,001 996 499	1,000 003 947
1,3	19,952 623 150	0,998 743 267	0,999 998 427	1,001 258 314	1,000 001 573
1,4	25,118 864 315	0,999 207 239	0,999 999 373	1,000 793 390	1,000 000 627
1,5	31,622 776 602	0,999 499 875	0,999 999 750	1,000 500 375	1,000 000 250
1,6	39,810 717 055	0,999 684 472	0,999 999 901	1,000 315 628	1,000 000 099
1,7	50,118 723 363	0,999 800 927	0,999 999 960	1,000 199 113	1,000 000 040
1,8	63,095 734 448	0,999 874 398	0,999 999 984	1,000 125 618	1,000 000 016
1,9	79,432 823 472	0,999 920 752	0,999 999 994	1,000 079 254	1,000 000 006
2,0	100,000 000 000	0,999 949 999	0,999 999 998	1,000 050 004	1,000 000 002

$$\left|Z(j\omega_{res2})\right| = \rho\sqrt{\frac{2Q^5\sqrt{Q^2+2}-2Q^5-Q^4+4Q^2-1}{2Q^3\sqrt{Q^2+2}+Q^6+2Q^4+Q^2}} \tag{12}$$

Table 1 contains the results of formula (3), (5), (8) and (11) valuation.

Comparison of expressions (9) and (12) allows drawing conclusion that

$Z(j\omega_{res1}) \neq \left|Z(j\omega_{res2})\right|$.

The similar results are produced by study of parallel LCR circuits, since they are dual to series LCR circuits.

From the cites analysis it proceeds that the generally accepted interpretation of resonant on real frequencies in LCR two pole circuits cannot be accepted as satisfactory, since the first and the second definitive attributes of resonance in some circuits have different corresponding resonant frequencies. Moreover, even in LCR two pole circuits with equal complexity level and similar structure the number of resonant frequencies may vary.

Let's now pass to the simplest LCR multi pole circuits. Fig. 1d – 1i shows inclusion of series LCR circuits as four pole circuits. Let's calculate the transfer coefficient under voltage of LCR four pole circuit (fig. 1d).

$$k_{UC} = \frac{U_C}{U_C+U_L} = \frac{-j\frac{1}{LC}}{\omega\frac{r}{L}+j\left(\omega^2-\omega_0{}^2\right)} = \frac{-j\omega_0{}^2}{2\sigma_0\omega+j\left(\omega^2-\omega_0{}^2\right)} \tag{13}$$

Its reactive constituent

$$Im\,k_{UC}(j\omega) = \frac{-2\sigma_0\omega\omega_0{}^2}{4\sigma_0{}^2\omega^2+\left(\omega^2-\omega_0{}^2\right)^2} = \frac{-\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)}{\left(\frac{1}{Q}\right)^2\left(\frac{\omega}{\omega_0}\right)^2+\left[\left(\frac{\omega}{\omega_0}\right)^2-1\right]^2}$$

possesses zero value on resonant frequency $\omega_{res3}=0$, on which the complex transfer function is becoming purely active and equal to $k(j\omega_{res3})=1$.

As a result of study of transfer function module of the given circuit

$$\left|k_{UC}\left(j\omega\right)\right| = \frac{\omega_0{}^2}{\sqrt{4\sigma_0{}^2\omega^2 + \left(\omega^2 - \omega_0{}^2\right)}} = \frac{1}{\sqrt{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2 + \left[\left(\frac{\omega}{\omega_0}\right)^2 - 1\right]^2}}$$

on extreme we find the following aggregate of resonant frequencies for it

$$\left[\begin{array}{l} \omega'_{res4} = 0 \\ \omega''_{res4} = \sqrt{\omega_0{}^2 - 2\sigma_0{}^2} = \omega_0\sqrt{\frac{2Q^2 - 1}{2Q^2}} \end{array}\right. \tag{14}$$

On the first resonant frequency ω'_{res3} the transfer function module of such LCR circuit possesses value $\left|k_{UC}\left(j\omega'_{res4}\right)\right| = 1$, and on the second resonant frequency ω''_{res4} – it possesses value

$$\left|k_{UC}\left(j\omega''_{res4}\right)\right| = \frac{\omega_0{}^2}{2\sigma_0\sqrt{\omega_0{}^2 - \sigma_0{}^2}} = \frac{2Q^2}{\sqrt{4Q^2 - 1}}$$

The voltage transfer coefficient in LCR four pole circuit shown in fig. le, looks as follows

$$k_{UL}\left(j\omega\right) = \frac{U_L}{U_C + U_L} = \frac{\omega\frac{r}{L} + j\omega^2}{\omega\frac{r}{L} + j\left(\omega^2 - \frac{1}{LC}\right)} = \frac{2\sigma_0\omega + j\omega^2}{2\sigma_0\omega + j\left(\omega^2 - \omega_0{}^2\right)} \tag{15}$$

Its reactive constituent

$$Im\, k_{UL}\left(j\omega\right) = \frac{2\sigma_0\omega\omega_0{}^2}{4\sigma_0{}^2\omega^2 + \left(\omega^2 - \omega_0{}^2\right)^2} = \frac{\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)}{\left(\frac{1}{Q}\right)^2\left(\frac{\omega}{\omega_0}\right)^2 + \left[\left(\frac{\omega}{\omega_0}\right)^2 - 1\right]^2}$$

possesses zero value on resonant frequency $\omega_{res3} = 0$, on which the complex transfer function becomes purely active and equal to $k\left(j\omega_{res3}\right) = 0$.

Study of transfer function module of this LCR circuit

$$\left|k_{UL}(j\omega)\right| = \sqrt{\frac{4\sigma_0{}^2\omega^2 + \omega^4}{4\sigma_0{}^2\omega^2 + \left(\omega^2 - \omega_0{}^2\right)}} = \sqrt{\frac{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2 + \left(\frac{\omega}{\omega_0}\right)^4}{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2 + \left[\left(\frac{\omega}{\omega_0}\right)^2 + 1\right]}}$$

On extreme allows finding the following aggregate of resonant frequencies

$$\left[\begin{array}{l} \omega'_{res4} = 0 \\ \omega''_{res4} = \sqrt{\dfrac{\omega_0\sqrt{\omega_0{}^2 + 8\sigma_0{}^2} + \omega^{\,2}}{2}} = \omega_0\sqrt{\dfrac{\sqrt{Q^2+2}+Q}{2Q}} \end{array}\right. \tag{16}$$

On these resonant frequencies the transfer function module possesses values

$$\left|k_{UL}(j\omega'_{res4})\right| = 0$$

$$\left|k_{UL}(j\omega''_{res4})\right| = \sqrt{\frac{\omega_0\sqrt{\omega_0{}^2 + 8\sigma_0{}^2} + 4\sigma_0{}^2 + \omega_0{}^2}{\omega_0\sqrt{\omega_0{}^2 + 8\sigma_0{}^2} + 4\sigma_0{}^2 - \omega_0{}^2}}$$

The complex transfer function of LCR four pole circuit shown in fig. If, looks as follows

$$k_{UC}(j\omega) = \frac{-j\dfrac{1}{LC}}{\omega\dfrac{1}{RC} + j\left(\omega^2 - \dfrac{1}{LC}\right)} = \frac{-j\omega_0{}^2}{2\sigma_0\omega + j\left(\omega^2 - \omega_0{}^2\right)} \tag{17}$$

and complex transfer function of LCR four pole circuit shown in fig. lg, looks as follows

$$k_{UL}(j\omega) = \frac{\omega\dfrac{1}{RC} + j\omega^2}{\omega\dfrac{1}{RC} + j\left(\omega^2 - \dfrac{1}{LC}\right)} = \frac{2\sigma_0\omega + j\omega^2}{2\sigma_0\omega + j\left(\omega^2 - \omega_0{}^2\right)} \tag{18}$$

As is clear, expressions (17) and (18) are completely the same as formulas (13) and (15) evaluated above.

The transfer coefficient of LCR four pole circuit shown in fig. lh, will look as follows

$$k_{UC}(j\omega)=\frac{\omega\frac{1}{RC}-j\frac{1}{LC}}{\omega\frac{1}{RC}+j\left(\omega^2-\frac{1}{LC}\right)}=\frac{2\sigma_0\omega-j\omega_0{}^2}{2\sigma_0\omega+j\left(\omega^2-\omega_0{}^2\right)}$$

Its reactive constituent

$$Im\ k_{UC}(j\omega)=\frac{2\sigma_0\omega^3}{4\sigma_0{}^2\omega^2+\left(\omega^2-\omega_0{}^2\right)^2}=\frac{\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)^3}{\left(\frac{1}{Q}\right)^2\left(\frac{\omega}{\omega_0}\right)^2+\left[\left(\frac{\omega}{\omega_0}\right)^2-1\right]^2}$$

possesses zero value on resonant frequency $\omega_{res3}=0$, on which the complex transfer function becomes purely active and equal to $k(j\omega_{res3})=1$.

This transfer function module looks as follows

$$|k_{UC}(j\omega)|=\sqrt{\frac{4\sigma_0{}^2\omega^2+\omega_0{}^4}{4\sigma_0{}^2\omega^2+\left(\omega^2-\omega_0{}^2\right)}}=\sqrt{\frac{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2+1}{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2+\left[\left(\frac{\omega}{\omega_0}\right)^2-1\right]}}$$

Study of the last expression on extreme allows determining the aggregate of its resonant frequencies

$$\left[\begin{array}{l}\omega'_{res4}=0\\ \omega''_{res4}=\omega_0\sqrt{\dfrac{\omega_0\sqrt{\omega_0{}^2+8\sigma_0{}^2}-\omega_0{}^2}{4\sigma_0{}^2}}=\omega_0\sqrt{Q\sqrt{Q^2+2}-Q^2}\end{array}\right. \qquad (19)$$

The transfer function module of studied LCR circuit on these resonant frequencies possesses values

$$|k_{UC}(j\omega'_{res4})|=0$$

$$|k_{UC}(j\omega''_{res4})|=\sqrt{\frac{8\sigma_0{}^4}{\omega_0{}^3\sqrt{\omega_0{}^2+8\sigma_0{}^2}+8\sigma_0{}^4-4\sigma_0{}^2\omega_0{}^2-\omega_0{}^4}}=$$

$$=\frac{1}{\sqrt{2Q^3\sqrt{Q^2+2}+1-2Q^2-2Q^4}}$$

The transfer function of LCR circuit shown in fig. li, looks as follows

$$k_{UL}(j\omega) = \frac{j\omega^2}{\omega\frac{1}{RC} + j\left(\omega^2 - \frac{1}{LC}\right)} = \frac{j\omega^2}{2\sigma_0\omega + j\left(\omega^2 - \omega_0{}^2\right)}$$

Its reactive constituent

$$Im\ k_{UC}(j\omega) = \frac{2\sigma_0\omega^3}{4\sigma_0{}^2\omega^2 + \left(\omega^2 - \omega_0{}^2\right)^2} = \frac{\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)^3}{\left(\frac{1}{Q}\right)^2\left(\frac{\omega}{\omega_0}\right)^2 + \left[\left(\frac{\omega}{\omega_0}\right)^2 - 1\right]^2}$$

possesses zero value on resonant frequency $\omega_{res3} = 0$, on which the complex transfer function becomes purely active and equal to $k(j\omega_{res3}) = 0$.

Study of this transfer function module

$$|k_{UL}(j\omega)| = \frac{\omega^2}{\sqrt{4\sigma_0{}^2\omega^2 + \left(\omega^2 - \omega_0{}^2\right)}} = \frac{\left(\frac{\omega}{\omega_0}\right)^2}{\sqrt{\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2 + \left[\left(\frac{\omega}{\omega_0}\right)^2 - 1\right]}}$$

On extreme allows finding the following resonant frequencies for it

$$\left[\begin{array}{l} \omega'_{res4} = 0 \\ \omega''_{res4} = \frac{\omega_0{}^2}{\sqrt{\omega_0{}^2 - 2\sigma_0{}^2}} = \omega_0\sqrt{\frac{2Q^2}{2Q^2 - 1}} \end{array}\right. \tag{20}$$

The transfer coefficient module on these frequencies equals to

$$|k_{UL}(j\omega'_{res4})| = 0$$

$$|k_{UL}(j\omega''_{res4})| = \frac{\omega_0{}^2}{2\sigma_0\sqrt{\omega_0{}^2 - \sigma_0{}^2}} = \sqrt{\frac{2Q^2}{4Q^2 - 1}}$$

The above-stated analysis of LCR four pole circuits allows claiming that the generally accepted interpretation of resonance of real frequencies in LCR multi pole circuits is unsatisfactory too, since the same phenomenon (resonance) in one-type circuits has multiple various resonant frequencies.

The existing interpretation of resonance in LCR circuits is imperfect yet because, as is noted in writing [1], its does not allow explaining the variance in resonant frequencies and free oscillation frequencies in the same LCR circuits.

Table 2 shows the results of formula (14), (16), (19) and (20) valuation.

Table 2

lg Q	Q	ω''_{res4}/ω_0 (14)	ω''_{res4}/ω_0 (16)	ω''_{res4}/ω_0 (19)	ω''_{res4}/ω_0 (20)
0	1,000 000 000	0,707 106 781	1,168 770 894	0,855 599 677	1,414 213 562
0,1	1,258 925 412	0,827 358 041	1,118 920 533	0,893 718 517	1,208 666 563
0,2	1,584 893 192	0,894 956 097	1,081 718 358	0,924 455 051	1,117 373 248
0,3	1,995 262 315	0,935 096 614	1,054 920 616	0,947 938 627	1,069 408 214
0,4	2,511 886 432	0,959 559 972	1,036 242 466	0,965 025 110	1,042 144 346
0,5	3,162 277 660	0,974 679 434	1,023 583 195	0,976 960 158	1,025 978 352
0,6	3,981 071 706	0,984 099 656	1,015 190 053	0,985 037 232	1,016 157 250
0,7	5,011 872 336	0,989 997 294	1,099 714 893	0,990 378 578	1,010 103 771
0,8	6,309 573 445	0,993 700 442	1,006 183 649	0,993 854 354	1,006 339 494
0,9	7,943 282 347	0,996 029 886	1,003 923 625	0,996 091 709	1,003 985 939
1,0	10,000 000 000	0,997 496 867	1,002 484 537	0,997 521 620	1,002 509 414
1,1	12,589 254 118	0,998 421 361	1,001 571 214	0,998 431 251	1,001 581 135
1,2	15,848 931 925	0,999 004 236	1,000 992 802	0,999 008 182	1,000 996 756
1,3	19,952 623 150	0,999 371 831	1,000 626 988	0,999 373 404	1,000 628 564
1,4	25,118 864 315	0,999 603 698	1,000 395 831	0,999 604 324	1,000 396 459
1,5	31,622 776 602	0,999 749 969	1,000 249 844	0,999 750 209	1,000 250 094
1,6	39,810 717 055	0,999 842 248	1,000 157 677	0,999 842 343	1,000 157 777
1,7	50,118 723 363	0,999 900 468	1,000 099 502	0,999 900 485	1,000 099 542
1,8	63,095 734 448	0,999 937 201	1,000 062 787	0,999 937 188	1,000 062 803
1,9	79,432 823 472	0,999 960 377	1,000 039 618	0,999 960 379	1,000 039 625
2,0	100,000 000 000	0,999 975 000	1,000 024 998	0,999 975 000	1,000 025 001

In fact, e.g., upon stimulation of LCR two pole circuit shown in fig. 1a, by voltage jump

$$U(t) = U_m \cdot 1(t),$$

The Laplas mapping of which looks as follows

$$U(p) = {U_m}/{p},$$

There occur free oscillations, the Laplas mapping of which looks as follows

$$I(p) = \frac{U_m}{\omega_0 L} \cdot \frac{\omega_0}{p^2 + 2\sigma_0 p + \omega_0{}^2},$$

And original equals to

$$I(t) = \frac{U_m}{\omega_0 L}\left[exp(-\sigma_0 t)sin(\omega_{free} t)\right]\cdot 1(t) \qquad (21)$$

On frequency $\omega_{free} = \sqrt{\omega_0{}^2 - \sigma_0{}^2}$ other than resonant frequency $\omega_{res} = \omega_0$.

However, it should be noted that variance of all resonant frequencies and free oscillation frequencies, found above, from co_0 value, as proceeds from aforementioned tables, in most practical cases is rather insignificant. That's why it is often ignored. But still, this variance always exists. And that's why it should be explained.

Therefore, the generally accepted interpretation of resonance in LCR circuits cannot be recognized as perfect because of its shortcomings. Consequently, it is necessary to develop consistent interpretation of resonance in LCR circuits and to explain the detected conflicts in existing resonance interpretation on its basis.

3.2 Testing of Resonance on Complex Frequencies

One cannot help noticing that the formulas for immittance function module of all considered simplest LCR circuits contain the same expression

$$\left(\omega^2 - \omega_0{}^2\right)^2 + 4\sigma_0{}^2\omega^2 \qquad (23)$$

which in view of (22) can be re-written as follows

$$\left(\omega^2 - \omega_{free}{}^2 - \sigma_0{}^2\right)^2 + 4\sigma_0{}^2\omega^2$$

This expression is the determinant of resonant properties of respective LCR circuits. But the same expression is available on Cassinian oval equation [2]

$$\left(\omega^2 - \omega_{free}{}^2 - \sigma_0{}^2\right)^2 + 4\sigma_0{}^2\omega^2 = d^4 \qquad (24)$$

Which can be written as follows

$$\begin{aligned}&\left(p_1 - p_{free1}\right)\left(p_2 - p_{free2}\right)\left(p_1 - p_{free2}\right)\left(p_2 - p_{free1}\right) = [j\omega - (-\sigma_0 + j\omega_{free})]\times \\ &\times[-j\omega - (-\sigma_0 - j\omega_{free})][j\omega - (-\sigma_0 - j\omega_{free})][-j\omega - (-\sigma_0 + j\omega_{free})] = \\ &= d_1{}^2 d_2{}^2 = d^4\end{aligned} \qquad (25)$$

where $p_{free1,2} = -\sigma_0 \pm j\omega_{free}$ are the complex associated frequencies of free oscillations (21) in studied LCR circuits;

$p_{1,2} = \pm j\omega$ are the complex frequencies of pure influence.

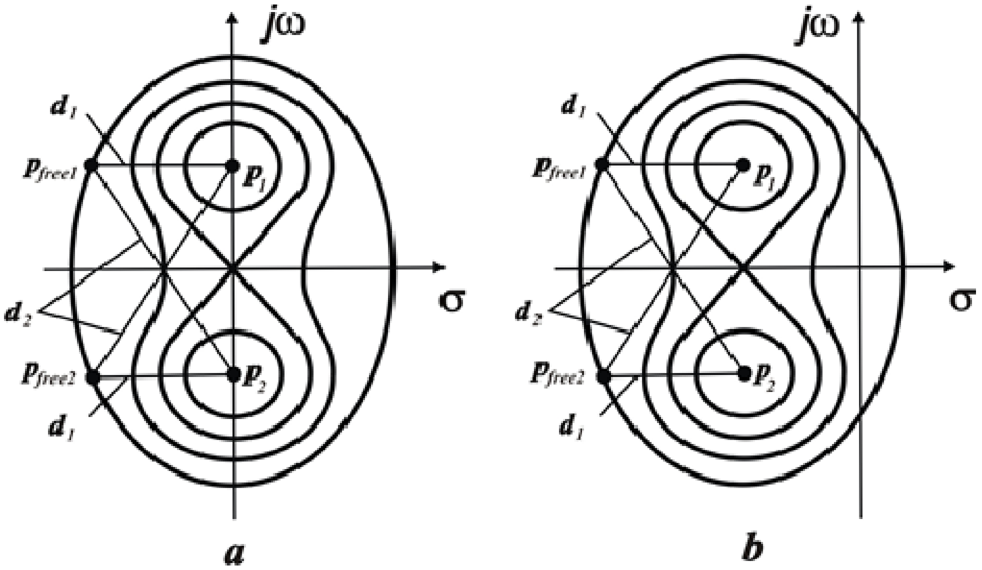

Fig. 2: Cassinian ovals

Consequently, Cassinian ovals represent a geometric place of points (fig. 2a) $p_{free1}(-\sigma_0,+j\omega_{free})$ and $p_{free2}(-\sigma_0,-j\omega_{free})$ on complex plane $\sigma, j\omega$ the production of square distance of which $d_1^{\,2}$ and $d_2^{\,2}$ from two other points $p_1(0,+j\omega)$ and $p_2(0,-j\omega)$ is a constant value equal to d^4. And since the physical mapping of points p_{free1} and p_{free2} in fig. 2a are free damped oscillations, and the physical mapping of points p_1 and p_2 – input continuous oscillations, one can draw a conclusion that immitance function module of respective LCR circuits upon resonance possesses extreme value as a result of decrease of d^4 value, i. e. at the expense of approaching of pairs of complex associated frequencies of pure influence and free oscillations on the complex plane the in studied LCR circuits. As is clear, upon pure influence with LCR circuit one cannot have $d = 0$.

One can have $d = 0$ when $\sigma_0 = 0$ and $\omega_{free} = \omega_0$, i. e. upon pure influence with LC-circuit, when point $p_1(0,+j\omega)$ coincides with point $p_{free1}(0,+j\omega_{free})$, and point $p_2(0,-j\omega)$ coincides with point $p_{free2}(0,-j\omega_{free})$. Meanwhile Cassiman ovals are degenerated into two points.

One can also have $d=0$ upon influence of damped oscillations with LCR-circuit. In this case Cassiman ovals may be viewed as a geometric place of points $p_{free1,2}=-\sigma_0 \pm j\omega_{free}$ on the plane of complex frequencies $\sigma, j\omega$ equally remote in the aforementioned sense from two other points $p_{1,2}=-\sigma \pm j\omega$.

The equation corresponding to such approach is the following:

$$\begin{aligned}&(p_3-p_{free1})(p_4-p_{free2})(p_3-p_{free2})(p_4-p_{free4})=[-\sigma+j\omega-(-\sigma_0+j\omega_{free})]\times\\&\times[-\sigma-j\omega-(-\sigma_0-j\omega_{free})][-\sigma+j\omega-(-\sigma_0-j\omega_{free})][-\sigma-j\omega-(-\sigma_0+j\omega_{free})]=\\&=d_1^2d_2^2=d^4\end{aligned} \quad (26)$$

Cassiman ovals, corresponding to equation (26), are shown in fig. 2b.

On the grounds of the foresaid one can formulate the following definition of resonance. *Resonance if a phenomenon of extreme change in parameter values* (amplitude, phase) *of forced constituent of response* (for electric circuits these are voltage, current, capacity) *of oscillating systems upon approaching of complex influence frequencies and free component of response.*

Let's explain this definition with example. Let the considered series oscillating LCR circuit (fig. 1a) be influenced with damped sinusoidal oscillations

$$U(t)=U_m[exp(-\sigma_0 t)\sin(\omega_{free}t+\varphi_0)]\times 1(t)$$

Then the complex resistance of circuit on complex associated influence frequencies $p=\sigma \pm j\omega$ will be equal to

$$Z(p)=L\frac{p^2+p\frac{r}{L}+\frac{1}{LC}}{p}=L\frac{p^2+p2\sigma_0+\omega_0^2}{p}=L\frac{(p-p_{free1})(p-p_{free2})}{p}$$

or

$$Z(\sigma,\omega)=L\frac{(\sigma\pm j\omega)^2+2\sigma_0(\sigma\pm j\omega)+\omega_0^2}{\sigma\pm j\omega}$$

Reactive constituent of this complex resistance

$$Im\,Z(p)=Im\,Z(\sigma,\omega)=\pm\omega L\frac{(\sigma^2-\sigma_0^2)+(\omega^2-\omega_0^2)}{\sigma^2+\omega^2}$$

on complex associated influence frequencies $p=\sigma \pm j\omega$ equal to complex associated frequencies $p_{free1,2}=-\sigma_0 \pm j\omega_{free}$ of free oscillations (21), possesses zero value, i.e. $\lim Im\, Z(p)_{p_{1,2} \to p_{free\,1,2}} = 0$ due to which the forced constituent of response, which in the given case is represented by current flowing through LCR circuit, and current coincide in phase.

Complex resistance module of this LCR circuit

$$|Z(p)| = |Z(\sigma,\omega)| = L\sqrt{\frac{\left[\left(\omega^2 - \omega_{free}{}^2\right) - (\sigma+\sigma_0)^2\right]^2 + 4\omega^2(\sigma+\sigma_0)^2}{\sigma^2 + \omega^2}}$$

On complex associated frequencies $p_{free\,1,2} = -\sigma_0 \pm j\omega_{free}$ of free oscillations (21) also possesses zero value $\lim|Z(p)|_{p \to p_{free\,1,2}} = 0$, as a result of which the amplitude of forced constituent of response possesses infinitely large values.

Therefore, the first and the second definitive properties of resonance are available in the same complex frequencies.

The same results are shown by study of other electric LCR circuits.

Consequently, the theory of resonance in electric LCR circuits on complex frequencies, unlike its interpretation on real frequencies, is consistent.

4 Physical Meaning of Resonance on Complex Frequencies

The resonance on complex frequencies have been dealt with in writings both without regard to the physical content [3], and in association with some data explaining its physical meaning [4-11]. Though, as far as the issue of physical nature of resonance on complex frequencies remains unresolved, it is expedient to study its further.

Let's give a description of a series of simple experiments proving the fact that resonance exists exactly on complex frequencies. For this purpose let's consider the processes in LC two pole circuit (fig. 3a) and RL two pole circuit (fig. 3b).

As is clear, in accordance with known frequency characteristics of LC two pole circuit, the poles of forced constituents of output voltage $U_{outforc}$ on the frequencies of input voltage $\omega > \omega_0$ and $\omega < \omega_0$ have the opposite signs.

Similarly, in accordance with frequency characteristics of RL - two pole circuit of forced constituents of output voltage $U_{outforc}$ on complex frequencies of input exponential voltage $\sigma > \sigma_0$ and $\sigma < \sigma_0$ also have the opposite signs. Consequently, the resonance is also available upon exponential influence in the studies electric circuit.

Such result from the point of view of the spectral analysis on real frequencies is far not obvious, since in the studied RL two pole circuit each spectral constituent is shifted in phase for no more than 90°.

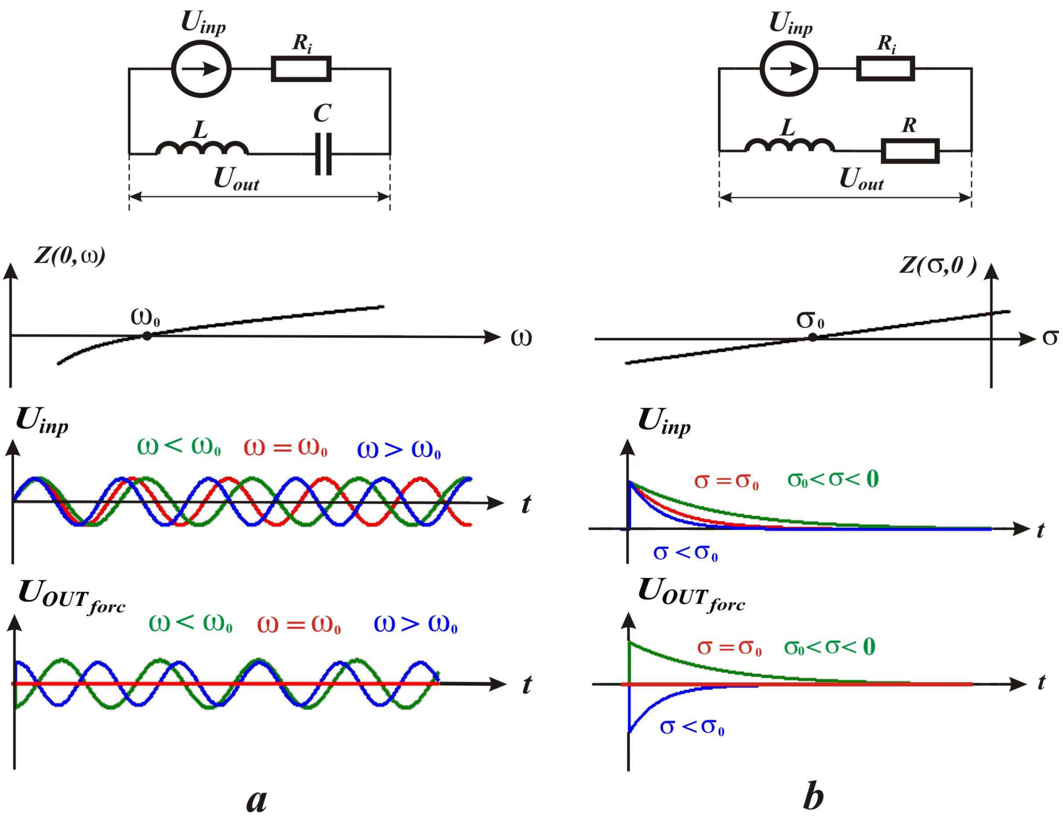

Fig. 3: Resonance in LC- and RL- two-pole circuits

That's why the achieved result encourages more detailed consideration of resonance on complex frequencies $\boldsymbol{p_{res}} = \pm \boldsymbol{j\omega_0}$ in LC two pole circuit (fig. 4a), on the complex frequency $\boldsymbol{p_{res}} = -\boldsymbol{\sigma_0}$ in RL two pole circuit (fig. 4b) and on complex frequencies $\boldsymbol{p_{res}} = -\boldsymbol{\sigma_0} \pm \boldsymbol{j\omega_0}$ in LCR two pole circuit (fig. 4c).

As is clear, in all cases the forced constituent of output voltage at studied two pole circuits $\boldsymbol{U_{outforc}}$ possesses zero value upon non-zero values of forced constituent of voltage on separate mapped elements of these two pole circuits. That's why upon resonance the output voltage at these two pole circuits contains only free constituent $\boldsymbol{U_{outfree}}$.

The aforesaid allows explaining why resonance up to now has been studied mainly for the case of pure influence. As is clear from fig. 4a, 4b, 4c, it is caused by the fact that upon pure influence with steady line circuits only the excretion of forced continuous oscillations out of their sum with free damped oscillations with time

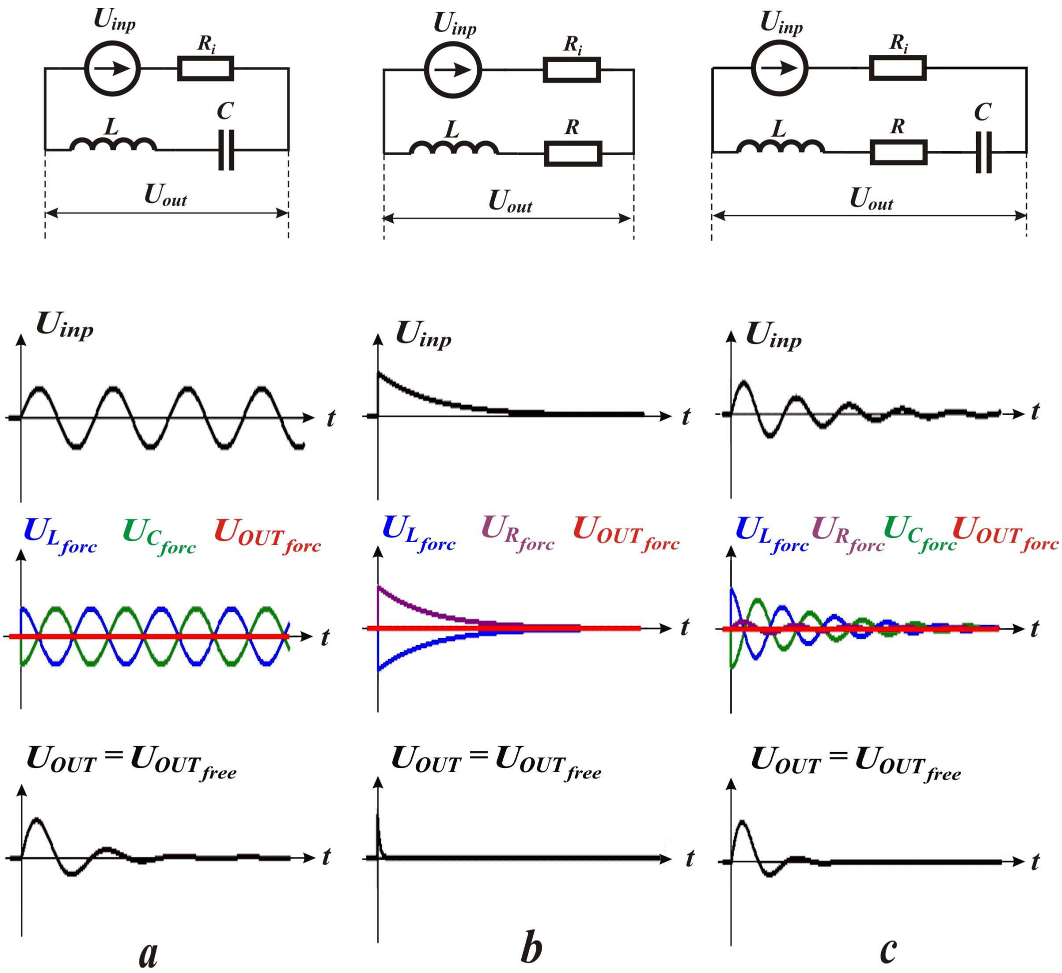

Fig. 4: Resonance in LC-, RL- and LCR-two-pole circuits

passing by is self-forced. In other cases the experiment becomes more difficult, because for excretion of forced constituent of response it is required to assume specific measures: select the respective parameters of the studied circuit, introduce non-zero initial values etc.

5 Conclusions

On the basis of the aforesaid it is possible to claim that:

- resonance as a physical phenomenon does take place on complex frequencies;
- complex frequencies are physical reality as both real and imaginary components influence a resonance similarly;

- complex resistance and complex conductivity as their value depends on complex frequency are physically real;
- also any other complex numbers as the resonance exists not only in electric circuits are physically real.

References

[1] L.I. Mandelshtam, *Lectures on Oscillation (Vol.4)*, Academy of Sciences of USSR. Moscow, (1955).

[2] G.A. Korn and T.M. Korn, *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems and Formulas for Reference and Review*, Courier Dover Publications, NY, (2000).

[3] S.P. Strelkov, *Introduction to the Theory of Oscillations*, Gostekhizdat, Moscow, Leningrad, (1951).

[4] A.I. Dolginov, *Resonance in Electric Circuits and Systems*, Gosenergoizdat, Moscow, (1957).

[5] A.A. Antonov and V.M. Bazhev, Means of formation of deviating currents for spiral scanning of the beam on the screen of the CRT, *USSR Pat*, #433650 (1974).

[6] A.A. Antonov, Output scanning cascade, *USSR Pat.*, # 879818 (1981).

[7] A.A. Antonov, *Research of a Resonance (Preprint # 67)*, Institute of Energetics Modeling Problems of Academy of Sciences of USSR, Kiev, (1987).

[8] A.A. Antonov, Physical reality of resonance on complex frequencies, *European Journal of Scientific Research*, 21(4) (2008), 627-641.

[9] A.A. Antonov, Resonance on real and complex frequencies, *European Journal of Scientific Research*, 28(2) (2009), 193-204.

[10] A.A. Antonov, Oscillation processes as a tool of physics cognition, *American Journal of Scientific and Industrial Research*, 1(2) (2010), 342-349.

[11] A.A. Antonov, Physical reality of complex numbers, *International Journal of Management, IT and Engineering*, 3(4) (2013), 219-230.