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On a Particular Condition for Regular Coequality Relations

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Abstract

Setting of this research is Bishop's constructive mathematics, the mathematics developed on Intuitionistic logic. If $(X, =, \neq, \theta)$ is an anti-ordered set, for a coequality q on X we say that it is strongly regular if it is regular and $\theta \circ q^C \subseteq q^C \circ \theta$ holds. In this case, $\theta \circ q^C$ is a quasi-antiorder relation on X such that the relation $\Theta = \pi \circ \theta \circ \pi^{-1}$ on X/q is the maximal anti-order on X/q.

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1 Introduction and Preliminaries

This short investigation in the spirit of Bishop's constructive mathematics (see, e.g. books [1]-[4], [10] and papers [5]-[8]) is a continuation of forthcoming the author's papers [9]. Bishop's constructive mathematics is developed on Constructive logic ([10]) - logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal logic. Since the axiom system for Constructive logic is part of the axiom system for classical logic, then the mathematical development based on the

Constructive Logic is acceptable in the Mathematics developed on the Classical logic.

Let $(X, =, \neq)$ be a set, where the relation $\neq$ is a binary relation on X, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq z \Longrightarrow x \neq y \vee y \neq z,\ x \neq y \wedge y = z \Longrightarrow x \neq z$$

for any $x, y, z \in X$. Following Heyting a relation with such properties is called *apartness*. A relation q on X is a *coequality relation* on X if and only if it is consistent, symmetric and cotransitive ([5]-[8]):

$$q \subseteq \neq,\ \ q^{-1} = q,\ \ q \subseteq q * q,$$

where "$*$ " is a *filed product* between relations ([5]) defined by the following way: If R and S are relation on set X, then $S * R$ is the relation

$$\{(x, z) \in X \times X : (\forall y \in X)((x, y) \in R \ \vee\ (y, z) \in S)\}.$$

Let β be a relation on X. We put ${}^0\beta = \nabla = \{(x, y) \in X \times X : x \neq y\}$, ${}^1\beta = \beta$ and ${}^n\beta = \beta * ... * \beta$ (n factors, $n \in \mathbf{N}$). Then ([5]), the relation $c(\beta) = \bigcap_{n \in N \cup \{0\}} {}^n\beta$ is the maximal consistent and cotransitive relation on set X under β.

A relation θ on X is an *antiorder* ([6], [7]) on X if and only if

$$\theta \subseteq \neq,\ \theta \subseteq \theta * \theta,\ \neq \subseteq \theta \cup \theta^{-1} (linearity)$$

and a relation τ on X is a *quasi-antiorder* ([6], [7], [8]) on X if

$$\tau \subseteq \neq,\ \tau \subseteq \tau * \tau.$$

Let x be an element of X and A a subset of X. We write $x \bowtie A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in X : x \bowtie A\}$. Note that the relation θ^C is an order relation on the set $(X, \neg \neq, \neq)$. Recall that a relation on set X is a order relation if it is reflexive, antisymmetric and transitive relation on X. If the relation $\neg\theta$ is an order relation on set $(X, =, \neq)$, where the apartness is tight, then the relation θ is called *excise relation* on X. (For apartness '$\neq$' we say that it is tight if the following implication is true

$$(\forall x, y \in X)(\neg(x \neq) \Longrightarrow x = y).)$$

If q is a coequality relation on a set X, then the relation $q^C = \{(x, y) \in X \times X : (x, y) \bowtie q\}$ is an equality on X compatible with q, in the following sense $q \circ q^C \subseteq q \wedge q^C \circ q \subseteq q$. Here the operation '$\circ$' is standard composition of relations. If T and U are relations on set X, then $U \circ T$ is the relation defined by $\{(x,z) \in X \times X : (\exists y \in X)((x,y) \in T \wedge (y,z) \in U)\}$. We can construct factor-set $X/(q^C, q) = \{aq^C : a \in S\}$, where $aq^C = \{x \in X : (a,x) \in q^C\}$, with

$$aq^C =_1 bq^C \Longleftrightarrow (a,b) \bowtie q, \ aq^C \neq_1 bq^C \Longleftrightarrow (a,b) \in q.$$

We can also construct the factor-set $X/q = \{aq : a \in S\}$, where $aq = \{x \in X : (a,x) \in q\}$, with

$$aq =_1 bq \Longleftrightarrow (a,b) \bowtie q, \ aq \neq_1 bq \Longleftrightarrow (a,b) \in q.$$

It is easy to check that there exists the strongly extensional and embedding bijection $h : X/(q^C, q) \cong X/q$. The mapping $\pi(q^C) : X \longrightarrow X/(q^C, q)$, defined by $\pi(q^C)(a) =_1 aq^C$ for any $a \in X$, and the mapping $\pi : X \longrightarrow X/q$, defined by $\pi(a) =_1 aq$ for any $a \in X$, are strongly extensional surjective functions. Recall that the function φ from set X into set Y is:
a *strongly extensional* if holds $(\forall x, y \in X)(\varphi(x) \neq \varphi(y) \Longrightarrow x \neq y)$;
an *embedding* if holds $(\forall x, y \in X)(x \neq y \Longrightarrow \varphi(x) \neq \varphi(y))$.
Connections between mappings $\pi(q^C)$ and π are given by the following relations

$$\pi = h \circ \pi(q^C) \quad \text{and} \quad \pi(q^C) = h^{-1} \circ \pi.$$

It is easy to check that

$$q^C = (\pi(q^C))^{-1} \circ \pi(q^C) = (h^{-1} \circ \pi)^{-1} \circ (h^{-1} \circ \pi) = \pi^{-1} \circ \pi.$$

For a given anti-ordered set $(X, =, \neq, \theta)$ is essential to know if there exists a coequality relation q on X such that X/q be an anti-ordered set. This plays an important role for studying the structure of anti-ordered sets. The following questions arise:

(1) Is there coequality relation q on X for which X/q is anti-ordered set?
(2) When the relation $\Theta = \pi \circ \theta \circ \pi^{-1}$ is an anti-order relation on X/q?

The concept of quasi-antiorder relation was introduced by this author in his papers [6], [7] and [8]. According to [6] and [7], if $(X, =, \neq, \theta)$ is an anti-ordered set and σ a quasi-antiorder on X under θ, then the relation q on X, defined by $q = \sigma \cup \sigma^{-1}$, is a coequality relation on X and the set X/q is an anti-ordered set under anti-order θ_1 defined by $(xq, yq) \in \theta_1 \Longleftrightarrow (x,y) \in \sigma$. So, according to results in [7], each quasi-antiorder σ on an ordered set X under anti-order induces a coequality relation $q = \sigma \cup \sigma^{-1}$ on X such that X/q is an ordered set under antiorder θ_1. In paper [8] we prove that the converse of this statement also holds: If $(X, =, \neq, \theta)$ is an anti-ordered set and q coequality on X and if there exists an antiorder relation Θ_1 on X/q such that the $(X/q, =_1, \neq_1, \Theta_1)$ is an ordered set under antiorder Θ_1 such that the mapping $\pi : X \longrightarrow X/q$ is a reverse isotone, then there exists a quasi-antiorder τ on X such that $q = \tau \cup \tau^{-1}$ and $\Theta_1 = \theta_1$. (A function $f : (X, =, \neq, \theta) \longrightarrow (Y, =, \neq, \Theta)$ is an anti-order reverse isotone if $(f(a), f(b)) \in \Theta \Longrightarrow (a, b) \in \theta$ holds for any $a, b \in X$.) So, each coequality q on a set $(X, =, \neq, \theta)$ such that X/q is an anti-ordered semigroup induces a quasi-antiorder on X. This is the motive for introduction a new notion: A coequality relation q on X is called *regular* with respect to θ if there an antiorder "θ_1" on X/q satisfying the following conditions:

(i) $(X/q, =_1, \neq_1, \theta_1)$ is a anti-ordered set;

(ii) The mapping $\pi : X \ni a \longmapsto aq \in X/q$ is a reverse isotone function.

We call the antiorder "θ_1" on X/q is a *regular antiorder* with respect to the regular coequality q on X and to the anti-order θ . It is known that the regular antiorder on X/q with respect to a regular coequality q and to the antiorder θ on X is in general not unique. In the paper [9] we got a construction of a maximal quasi-antiorder $\tau_{max} = c(q \cap \theta)$ on X with respect to q and such that $q = \tau_{max} \cup \tau_{max}^{-1}$.

Let q be a regular anti-order on the anti-ordered set $(X, =, \neq, \theta)$. Then there exists anti-order θ_1 on X/q such that the natural homomorphism $\pi : X \longrightarrow X/q$ is reverse isotone. Hence, by [9], there exists a quasi-antiorder σ under θ such that $q = \sigma \cup \sigma^{-1}$ and $\theta_1 = \{(aq, bq) \in X/q \times X/q : (a, b) \in \sigma\}$. In the following theorem we show that there exists such maximal quasi-antiorder τ under θ and we prove that there such construction of that relation.

Theorem 1.1 ([8], Theorem 3) *Let q be a regular coequality relation on an anti-ordered set $(X, =, \neq, \theta)$. Then there exists the maximal quasi-antiorder relation τ under θ such that $q = \tau \cup \tau^{-1}$ and $\theta_1 \subseteq \{(aq, bq) \in X/q \times X/q : (a, b) \in \tau\}$. That relation is exactly the following relation*

$$c(q \cap \theta) = \bigcap_{n \in N} {}^n(q \cap \theta).$$

Let $(X, =, \neq, \theta)$ be an anti-ordered set and q a regular coequality on X. In the following assertion we show a construction of the maximal regular anti-order relation on X/q. For that we need an auxiliary result.

Theorem 1.2 ([9], Theorem 3) *Let $(X, =, \neq, \beta)$ be an anti-ordered set and q a coequality on X. Then holds*

$$c(\pi \circ \beta \circ \pi^{-1}) = \pi \circ c(\beta) \circ \pi^{-1}.$$

Theorem 1.3 ([9], Theorem 4) *Let $(X, =, \neq, \theta)$ be an anti-ordered set and q a regular coequality on X. Then $\pi \circ c(\theta \cap q) \circ \pi^{-1}$ is the maximal anti-order relation on X/q with respect to θ.*

2 The Main Results

In [8] giving an answer on question (2) we find necessary and sufficient conditions that the relation $\Theta = \pi \circ \theta \circ \pi^{-1}$ is an anti-order relation on X/q.

Theorem 2.1 ([8], Theorem 4) Let q be a coequality relation in anti-ordered set $(X, =, \neq, \theta)$. Then the relation $\Theta = \pi \circ \theta \circ \pi^{-1}$ is an anti-order relation on factor-set X/q if and only if the relation $\tau = q^C \circ \theta \circ q^C$ is a quasi-antiorder relation on X such that $\tau \cup \tau^{-1} = q$.

In this section we analyze a special case of regular coequality relation on anti-ordered set X, when we not need cotransitive fulfillment operation. For a regular coequality q we say that it is a *strongly regular* coequality relation on X if

$$\theta \circ q^C \subseteq q^C \circ \theta.$$

For a strongly regular coequality q we have the following assertion: If coequality relation q on a set $(X, =, \neq)$ is a strongly regular, then the relation $\theta \circ q^C$ is a quasi-antiorder relation on X? In this section we start with the following result important for our main result of this paper and interesting by itself:

Lemma 2.2 *For any three relations $\alpha_1 \subseteq X_1 \times X_2$, $\alpha_2 \subseteq X_2 \times X_3$ and $\alpha_3 \subseteq X_3 \times X_4$ the following inclusion*

$$\alpha_3 * (\alpha_2 \circ \alpha_1) \supseteq (\alpha_3 * \alpha_2) \circ \alpha_1 \text{ and } (\alpha_3 \circ \alpha_2) * \alpha_1 \supseteq \alpha_3 \circ (\alpha_2 * \alpha_1)$$

are valid in the set $X_1 \times X_4$.

Proof: Let $a_1 \in X_1$ and $a_4 \in X_4$ such that $(a_1, a_4) \in (\alpha_3 * \alpha_2) \circ \alpha_1$. Then there exists an element $a_2 \in X_2$ such that

$$(a_1, a_2) \in \alpha_1 \wedge (a_2, a_4) \in (\alpha_3 * \alpha_2)$$

and

$$(\exists a_2 \in X_2)((a_1, a_2) \in \alpha_1 \wedge (\forall z \in X_3)((a_2, z) \in \alpha_2, \vee (z, a_4) \in \alpha_3))).$$

Thus

$$(\exists a_2 \in X_2)(\forall z \in X_3)((((a_1, a_2) \in \alpha_1 \wedge (a_2, z) \in \alpha_2)) \vee ((a_1, a_2) \in \alpha_1 \wedge (z, a_4) \in \alpha_3))$$

and hence

$$(\forall z \in X_3)(((\exists a_2 \in X_2)((a_1, a_2) \in \alpha_1 \wedge (a2, z) \in \alpha_2)) \vee ((a_1, a_2) \in \alpha_1 \wedge (z, a_4) \in \alpha_3)).$$

From above formula , we have

$$(\forall z \in X_3)(((\exists a_2 \in X_2)((a_1, a_2) \in \alpha_1 \wedge (a_2, z) \in \alpha_2)) \vee (z, a_4) \in \alpha_3).$$

Last is equivalent with

$$(\forall z \in X_3)((a_1, z) \in \alpha_2 \circ \alpha_1 \ \vee \ (z, a_4) \in \alpha_3)).$$

So, the last means

$$(a_1, a_4) \in \alpha_3 * (\alpha_2 \circ \alpha_1).$$

Analogously, we proof the following the inclusion

$$(\alpha_3 \circ \alpha_2) * \alpha_1 \ \supseteq \ \alpha_3 \circ (\alpha_2 \alpha_1).$$

For a strongly regular coequality q on S, we have:

Theorem 2.3 *If the coequality relation q is a strongly regular, then the relation* $\theta \circ q^C$ *is a quasi-antiorder relation on* X *and the relation* $\Theta = \pi \circ \theta \circ \pi^{-1}$ *is the maximal anti-order relation on* X/q.

Proof: (I) Let q be a regular coequality relation on the set $(X, =, \neq, \theta)$. Then there exists an anti-order θ_1 on X/q such that the natural mapping $\pi : X \longrightarrow X/q$ is anti-order reverse isotone function. So, it holds

$$(\forall aq, bq \in X/q)((aq, bq) \in \theta_1 \Longrightarrow (a, b) \in \theta).$$

Hence, there exists a quasi-antiorder $\pi^{-1}(\theta_1)$ on X, defined by

$$(aq, bq) \in \theta_1 \Longleftrightarrow (a, b) \in \pi^{-1}(\theta_1) \, (\subseteq \theta),$$

such that

$$\begin{aligned} q = \{(a, b) \in X \times X : \pi(a) \neq_1 \pi(b)\} &= \{(a, b) \in X \times X : (aq, bq) \in \theta_1 \cup {\theta_1}^{-1}\} \\ = \{(a, b) \in X \times X : (aq, bq) \in \theta_1\} &\cup \{(a, b) \in X \times X : (aq, bq) \in {\theta_1}^{-1}\} \\ &= \pi^{-1}(\theta_1) \cup (\pi^{-1}(\theta_1))^{-1}. \end{aligned}$$

On the other hand, we have

$$\pi^{-1} \circ \theta_1 \circ \pi = \pi^{-1}(\theta_1) \ \subseteq \theta.$$

Therefore, we have the inclusion

$$\theta_1 \subseteq \pi \circ \theta \circ \pi^{-1}.$$

Besides, we have $\pi^{-1}(\theta_1) \subseteq q^C \circ \theta \circ q^C$. Indeed: Let (a, b) be an arbitrary element of $\pi^{-1}(\theta 1)$. Then $(aq, bq) \in \theta_1 \subseteq \pi \circ \theta \circ \pi^{-1}$. Thus we conclude that there exist elements $x, y \in X$ such that $(aq, x) \in \pi^{-1}$, $(x, y) \in \theta$ and $(y, bq) \in \pi$. Since $(a, aq) \in \pi$ and $(bq, b) \in \pi^{-1}$, we have $(a, b) \in \pi^{-1} \circ \pi \circ \theta \circ \pi^{-1} \circ \pi = q^C \circ \theta \circ q^C$.
Expect that, we have:

(1) $\theta \circ q^C \subseteq \theta \circ q^C \circ q^C \subseteq q^C \circ \theta \circ q^C \subseteq q^C \circ (\theta * \theta) \circ q^C \subseteq (q^C \circ \theta) * (\theta \circ q^C)$
$\subseteq (\theta \circ q^C) * (\theta \circ q^C)$;

(2) Let us prove that the implication $\theta \circ q^C \subseteq q^C \circ \theta \Longrightarrow \theta \circ q^C = q^C \circ \theta \circ q^C$ is valid. In fact:

(i) $\theta \circ q^C = Id_X \circ \theta \circ q^C \subseteq q^C \circ \theta \circ q^C$;

(ii) $q^C \circ \theta \circ q^C \subseteq q^C \circ q^C \circ \theta \subseteq q^C \circ \theta$.

Therefore, if the relation q is a strongly regular coequality relation on set $(X, =, \neq, \theta)$, then holds $\theta \circ q^C = q^C \circ \theta \circ q^C$.

(II) Let Ξ be a anti-order relation on the factor set X/q such that the mapping $\pi' : X \longrightarrow X/q$ is a reverse isotone surjection. Then there exists a quasi-antiorder $\sigma = q^C \circ \theta \circ q^C$ on X such that $\Xi = \pi' \circ \sigma \circ (\pi')^{-1}$.

In the next example we show that there difference between regular and strongly regular coequalities in anti-ordered set.

Example: We consider the anti-ordered set $X = \{a, b, c, d, e, f\}$ under the relation

$$\theta = X \times X \backslash \{(a,a), (a,d), (a,e), (b,b), (b,e), (c,c),$$
$$(c,b), (c,e), (d,d), (d,e), (e,e), (f,f), (f,a), (f,b), (f,c), (f,d), (f,e)\}.$$

Let q_1, q_2 be coequality relations on X defined as follows:

$$q_1 = X \times X \backslash \{(a,a), (b,b), (b,c), (b,d), (c,c), (c,b), (c,d), (d,d),$$
$$(d,c), (d,b), (e,e), (f,f)\},$$
$$q_2 = X \times X \backslash \{(a,a), (a,c), (a,d), (b,b), (b,e), (c,c), (c,a), (c,d),$$
$$(d,d), (d,a), (d,c), (e,b), (e,e), (f,f)\}$$

Then

$$X/q_1 = \{aq_1 = \{b,c,d,e,f\}, bq_1 = \{a,e,f\}, cq_1 = \{a,e,f\}, dq_1 = \{a,e,f\},$$
$$eq_1 = \{a,b,c,d,f\}, fq_1 = \{a,b,c,d,e\}\},$$
$$X/q_2 = \{aq_2 = \{b,e,f\}, bq_2 = \{a,c,d,f\}, cq_2 = \{b,e,f\}, dq_2 = \{b,e,f\},$$
$$eq_2 = \{a,c,d,f\}, fq_2 = \{a,b,c,d,e\}\}.$$

The following relations are anti-order relation on X/q_1 and X/q_2 :

$$\theta_1 = \wp(X) \times \wp(X) \ \backslash$$
$$\{(\{f\},\{f\}), (\{f\},\{a\}), (\{f\},\{b,d,c\}), (\{f\},\{e\}), (\{a\},\{a\}),$$
$$(\{a\},\{b,d,c\}), (\{a\},\{e\}), (\{b,d,c\},\{b,d,c\}), (\{b,d,c\},\{e\}), (\{e\},\{e\}) \ \}.$$
$$\theta_2 =$$
$$\wp(X) \times \wp(X) \backslash \{(\{f\},\{f\}), (\{f\},\{d,a,c\}), (\{f\},\{e,b\}), (\{d,a,c\},\{d,a,c\}),$$
$$(\{d,a,c\},\{e,b\}), (\{e,b\},\{e,b\})\}.$$

Then $(X/q_1, =_1, \neq_1, \theta_1)$ and $(X/q_2, =_2, \neq_2, \theta_2)$ are anti-ordered sets, q_1 and q_2 are strongly regular coequalities on X, $q_1 \cup q_2$ is a regular coequality with respect to $\theta_1 \cup \theta_2$ but $q_1 \cup q_2$ is not a strongly regular coequality relation on X. The proof of these facts is technical.

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