Gen. Math. Notes, Vol. 10, No. 1, May 2012, pp. 58-62
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Common Fixed Points of Compatible Mappings of Type (R)

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(Received: 19-4-12/ Accepted: 22-5-12)

Abstract

In this paper we prove a common fixed point theorem of compatible mappings of type(R) by considering four mappings. Our result modify the result of Bijendra and Chouhan[1] and others.

Keywords: *Fixed point, complete metric space, compatible mappings.*

1 Introduction

The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck in 1986[2] as a generalization of commuting mappings. Pathak, Chang and Cho[3] in 1994 introduced the concept of compatible mappings of type(P). In 2004 Rohen, Singh and Shambhu[4] introduced the concept of compatible mappings of type(R) by combining the definitions of compatible mappings and compatible mappings of type(P).
The aim of this paper is to prove a common fixed point theorem of compatible mappings of type(R) in metric space by considering four self mappings.

Following are definition of types of compatible mappings.

Definition 1.1 [2]: *Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be compatible if* $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ *for some* $t \in X$.

Definition 1.2 [3]: *Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be compatible of type (P) if* $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in X such that for* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ *for some* $t \in X$.

Definition 1.3 [4]: *Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be compatible of type (R) if* $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ *and* $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ *for some* $t \in X$.

2 Main Results

We need the following propositions for our main result.

Proposition 2.1[4]: *Let S and T be mappings from a complete metric space (X, d) into itself. If a pair {S, T} is compatible of type (R) on X and* $Sz = Tz$ *for* $z \in X$, *then*
$STz = TSz = SSz = TTz$.

Proposition 2.2[4]: *Let S and T be mappings from a complete metric space (X, d) into itself. If a pair {S, T} is compatible of type (R) on X and* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ *for some* $z \in X$, *then we have*

(i) $d(TSx_n, Sz) \to 0$ as $n \to \infty$ if S is continuous,

(ii) $d(STx_n, Tz) \to 0$ as $n \to \infty$ if T is continuous and

(iii) $STz=TSz$ and $Sz=Tz$ if S and T are continuous at z.

Now we prove the following theorem.

Lemma 2.3[1] *Let A, B, S and T be mapping from a metric space (X, d) into itself satisfying the following conditions:*

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax, By)]^2 \le k_1[d(Ax, Sx)d(By, Ty)+d(By, Sx)d(Ax, Ty)]$
$+ k_2[d(Ax, Sx)d(Ax, Ty)+d(By, Ty)d(By, Sx)]$
Where $0 \le k_1 + 2k_2 < 1$; $k_1, k_2 \ge 0$

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1}=Tx_{2n+1}=Ax_{2n} \text{ and } y_{2n}= Sx_{2n}= Bx_{2n-1}$$

then the sequence $\{y_n\}$ is Cauchy sequence in X.

Proof. By condition (2) and (3), we have

$$[d(y_{2n+1}, y_{2n})]^2=[d(Ax_{2n}, Bx_{2n-1})]^2$$

$$\le k_1[d(Ax_{2n}, Sx_{2n})d(Bx_{2n-1}, Tx_{2n-1})+d(Bx_{2n-1}, Sx_{2n})d(Ax_{2n}, Tx_{2n-1})]$$

$$+ k_2[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n-1})+d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, Sx_{2n})]$$

$$= k_1[d(y_{2n+1}, y_{2n})d(y_{2n}, y_{2n-1}) + 0] + k_2[d(y_{2n+1}, y_{2n})d(y_{2n+1}, y_{2n-1})+0]$$

$$[d(y_{2n+1}, y_{2n})]\le k_1d(y_{2n}, y_{2n-1}) + k_2[d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]$$

$$[d(y_{2n+1}, y_{2n})]\le pd(y_{2n}, y_{2n-1}) \text{ where } p = \frac{k_1+k_2}{1-k_2} < 1.$$

Hence $\{y_n\}$ is Cauchy sequence.

Now we give our main theorem.

Theorem 2.4: *Let A, B, S and T be self maps of a complete metric space (X, d) satisfying the following conditions:*

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax, By)]^2 \le k_1[d(Ax, Sx)d(By, Ty)+d(By, Sx)d(Ax, Ty)]$
$+ k_2[d(Ax, Sx)d(Ax, Ty)+d(By, Ty)d(By, Sx)]$
Where $0 \le k_1 + 2k_2 < 1$; $k_1, k_2 \ge 0$

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1}$$

then the sequence $\{y_n\}$ is Cauchy sequence in X.

(4) One of A, B, S or T is continuous.

(5) $[A, S]$ and $[B, T]$ are compatible of type (R) on X.

Then A, B, S and T have a unique common fixed point in X.

Proof: By lemma 2.3, $\{y_n\}$ is Cauchy sequence and since X is complete so there exists a point $z \in X$ such that $\lim y_n = z$ as $n \to \infty$. Consequently subsequences Ax_{2n}, Sx_{2n}, Bx_{2n-1} and Tx_{2n+1} converges to z.

Let S be continuous. Since A and S are compatible of type (R) on X, then by proposition 2.2. we have $S^2x_{2n} \to Sz$ and $ASx_{2n} \to Sz$ as $n \to \infty$.

Now by condition (2) of lemma 2.3, we have

$$[d(ASx_{2n}, Bx_{2n-1})]^2 \le k_1[d(ASx_{2n}, S^2x_{2n})d(Bx_{2n-1}, Tx_{2n-1}) + d(Bx_{2n-1}, S^2x_{2n})d(ASx_{2n-1}, Tx_{2n-1})]$$

$$+ k_2[d(ASx_{2n}, S^2x_{2n})d(ASx_{2n}, Tx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, S^2x_{2n})]$$

As $n \to \infty$, we have

$$[d(Sz, z)]^2 \le k[d(Sz, z)]^2,$$

which is a contradiction. Hence $Sz = z$,

Now $$[d(Az, Bx_{2n-1})]^2 \le k_1[d(Az, Sz)d(Bx_{2n-1}, Tx_{2n-1}) + d(Bx_{2n-1}, Sz)d(Az, Tx_{2n-1})]$$

$$+ k_2[d(Az, Sz)d(Az, Tx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, Sz)]$$

Letting $n \to \infty$, we have $[d(Az, z)]^2 \le k_2[d(Az, z)]^2$. Hence $Az = z$.

Now since $Az = z$, by condition (1) $z \in T(X)$. Also T is self map of X so there exists a point $u \in X$ such that $z = Az = Tu$. More over by condition (2), we obtain,

$$[d(z, Bu)]^2 = [d(Az, Bu)]^2 \le k_1[d(Az, Sz)d(Bu, Tu) + d(Bu, Sz)d(Az, Tu)]$$

$$+ k_2[d(Az, Sz)d(Az, Tu) + d(Bu, Tu)d(Bu, Sz)]$$

i.e., $[d(z, Bu)]^2 \le k_2[d(z, Bu)]^2$.

Hence $Bu = z$ i.e., $z = Tu = Bu$.

By condition (5), we have

$$d(TBu, BTu) = 0.$$

Hence $d(Tz, Bz) = 0$ i.e., $Tz = Bz$.

Now,

$$[d(z, Tz)]^2=[d(Az, Bz)]^2\leq k_1[d(Az, Sz)d(Bz, Tz) + d(Bz, Sz)d(Az, Tz)]$$

$$+ k_2[d(Az, Sz)d(Az, Tz) + d(Bz, Tz)d(Bz, Sz)]$$

i.e., $[d(z, Tz)]^2 \leq k_1[d(z, Tz)]^2$ which is a contradiction. Hence $z = Tz$ i.e, $z = Tz = Bz$.

Therefore z is common fixed point of A, B, S and T. Similarly we can prove this any one of A, B or T is continuous.

Finally, in order to prove the uniqueness of z, suppose w be another common fixed point of A, B, S and T Then we have,

$$[d(z, w)]^2=[d(Az, Bw)]^2\leq k_1[d(Az, Sz)d(Bw, Tw) + d(Bw, Sz)d(Az, Tw)]$$

$$+ k_2[d(Az, Sz)d(Az, Tw) + d(Bw, Tw)d(Bw, Sz)]$$

which gives

$$[d(z, Tw)]^2 \leq k_1[d(z, Tw)]^2. \text{ Hence } z = w.$$

This completes the proof.

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