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Some Generating Functions of Two Variables Laguerre Polynomials $L_n^{\alpha}(x,y)$ from the View Point of Lie-Algebra

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Abstract

Laguerre polynomials have special importance in engineering, science and good model for many systems in various fields. In this paper we consider a six parameters lie-group for these polynomials, which doesn't seem to appear before. By means of Weisner's group theoretic method some new generating functions of two variable and one parameter Laguerre polynomials $L_n^{\alpha}(x,y)$ *are obtained from which several generating functions can be easy derived.*

Keywords: *Two variables Laguerre polynomials, Recurrence relations, Group theoretic, Method and generating relation.*

1 Introduction

Group theoretic method was proposed by Louis Weisner in 1955 and he employed this method to find generated relations for a large class of special functions. Weisner discussed the group-theoretic significance of generating functions for hyper geometric, Hermite and Bessel functions [4,5 and 6] respectively. Miller, McBride, Srivastava and Monocha [3,7 and 8] respectively reported group theoretic method for obtaining generating relations in their books.

Two variables and one parameter Laguerre polynomials $L_n^\alpha(\mathrm{x},\mathrm{y})$ have been defined in [9] and specified by the series

$$L_n^\alpha(x,y)=\sum_{k=0}^{n}\frac{(-1)^k(1+\alpha)_n x^k y^{n-k}}{k!(n-k)!(1+\alpha)_k}, \tag{1.1}$$

where $(\alpha)_n=\dfrac{\Gamma(\alpha=n)}{\Gamma(\alpha)}$ is the pochammer symbol and $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt, \quad (\mathrm{Re}(\alpha)>0)$, and the generating function for $L_n^\alpha(\mathrm{x},\mathrm{y})$ is given by

$$\sum_{n=0}^{\infty}L_n^\alpha(\mathrm{x},\mathrm{y})\mathrm{t}^{\mathrm{n}}=\frac{1}{(1-yt)^{1+\alpha}}\exp\left(\frac{-xt}{1-yt}\right),\quad |yt|<1 \tag{1.2}$$

where α is a non-negative integer.

The differential equation satisfied by one parameter and two variables Laguerre polynomials $L_n^\alpha(x,y)$ is:

$$\left[x\frac{d^2}{dx^2}+\left(1+\alpha-\frac{x}{y}\right)\frac{dy}{dx}+\frac{n}{y}\right]L_n^\alpha(\mathrm{x},\mathrm{y})=0 \tag{1.3}$$

These polynomials satisfy the following differential and pure recurrence relations:

$$\frac{\partial}{\partial y}L_n^\alpha(\mathrm{x},\mathrm{y})=(n+\alpha)L_{n-1}^\alpha(\mathrm{x},\mathrm{y}) \tag{1.4}$$

$$\frac{\partial}{\partial x}L_n^\alpha(\mathrm{x},\mathrm{y})=\frac{n}{x}L_n^\alpha(\mathrm{x},\mathrm{y})-\frac{(\alpha+\mathrm{n})}{\mathrm{x}}yL_{n-1}^\alpha(\mathrm{x},\mathrm{y}) \tag{1.5}$$

$$nL_n^\alpha(\mathrm{x},\mathrm{y})=\{(2n-1+\alpha)y-x\}L_{n-1}^\alpha(\mathrm{x},\mathrm{y})-(\mathrm{n}-1+\alpha)y^2L_{n-2}^\alpha(\mathrm{x},\mathrm{y}) \tag{1.6}$$

$$(n+\alpha)L_{n-1}^\alpha(\mathrm{x},\mathrm{y})=\{(2n+\alpha+1)y-x\}L_n^\alpha(\mathrm{x},\mathrm{y})-(\alpha+n)y^2L_{n-1}^\alpha(\mathrm{x},\mathrm{y}) \tag{1.7}$$

2 Linear Differential Operators

Replacing d/dx by $\partial/\partial x$, α by $x(\frac{1}{y}-1)+y\frac{\partial}{\partial x}$, n by $y^{-1}z\frac{\partial}{\partial z}$ and y by $u(x,y,z)$. We obtain from (1.3) the following partial differential equation:

$$x\frac{\partial^2 u}{\partial x^2}+(1-x)\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial y\partial x}+z\frac{\partial u}{\partial z}=0 \qquad (2.1)$$

Thus, $u_1(x,y,z)=L_n^{\alpha}(\mathrm{x},\mathrm{y})z^n$ is a solution of the differential equation (2.1) since $L_n^{\alpha}(\mathrm{x},\mathrm{y})$ is a solution of equation (1.3). From (2.1), we define the infinitesimal operators A_{ij} $(i=1,2,\ j=1,2,3)$

$$A_{ij}=A_{ij}^{(1)}\frac{\partial}{\partial x}+A_{ij}^{(2)}\frac{\partial}{\partial y}+A_{ij}^{(3)}\frac{\partial}{\partial z}+A_{ij}^{(0)} \quad ,i=1,2,3$$

As follows:

$$\left.\begin{aligned}
A_{11}&=y\frac{\partial}{\partial y}; & A_{21}&=z\frac{\partial}{\partial z}\\
A_{12}&=xy^{-1}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}; & A_{22}&=xy^{-1}z\frac{\partial}{\partial x}-z\frac{\partial}{\partial y}-xy^{-1}z\\
A_{13}&=y\frac{\partial}{\partial x}-y; & A_{23}&=yz^{-1}\frac{\partial}{\partial x}
\end{aligned}\right\} \qquad (2.2)$$

Which satisfy the following rules

$$A_{11}\left[L_n^{\alpha}(\mathrm{x},\mathrm{y})z^n\right]=(n+\alpha)L_{n-1}^{\alpha}(\mathrm{x},\mathrm{y})\mathrm{y}z^n$$
$$A_{12}\left[L_n^{\alpha}(\mathrm{x},\mathrm{y})z^n\right]=nL_n^{\alpha}(\mathrm{x},\mathrm{y})\mathrm{y}^{-1}z^n$$
$$A_{13}\left[L_n^{\alpha}(\mathrm{x},\mathrm{y})z^n\right]=(\frac{n}{x}-1)yL_n^{\alpha}(\mathrm{x},\mathrm{y})z^n-\frac{(\alpha+n)}{x}L_{n-1}^{\alpha}(\mathrm{x},\mathrm{y})z^n$$
$$A_{21}\left[L_n^{\alpha}(\mathrm{x},\mathrm{y})z^n\right]=nL_n^{\alpha}(\mathrm{x},\mathrm{y})z^n$$
$$A_{22}\left[L_n^{\alpha}(\mathrm{x},\mathrm{y})z^n\right]=-2(\alpha+n)L_{n-1}^{\alpha}(\mathrm{x},\mathrm{y})z^{n-1}$$
$$A_{23}\left[L_n^{\alpha}(\mathrm{x},\mathrm{y})z^n\right]=\left[\frac{n}{x}L_n^{\alpha}(\mathrm{x},\mathrm{y})-\frac{(n-\alpha)}{x}yL_{n-1}^{\alpha}(\mathrm{x},\mathrm{y})\right]yz^{n-1},$$

3 Lie Algebra

Now we shall find the commutator relations by using the commutator notation with

$[A,B]u=(AB-BA)u$ we have

$[A_{22},A_{21}]=-A_{22}$, $\quad [A_{23},A_{21}]=A_{23}$

$[A_{22},A_{22}]=0$, $\quad [A_{23},A_{22}]=-1$

$[A_{22},A_{23}]=1$, $\quad [A_{23},A_{23}]=0$

$[A_{11},A_{21}]=0$; $[A_{11},A_{12}]=-A_{12}$; $[A_{11},A_{22}]=-A_{22}$

$[A_{11},A_{23}]=-A_{23}$; $[A_{12},A_{13}]=-1$; $[A_{21},A_{22}]=A_{22}$; $[A_{21},A_{23}]=-A_{23}$

$[A_{11},A_{13}]=A_{13}$;
$[A_{12},A_{21}]=[A_{12},A_{22}]=[A_{12},A_{23}]=0$
$[A_{13},A_{21}]=[A_{13},A_{23}]=0$

So we see from the above commutator relations that the set of operators $\{1, A_{ij}\ i=1,2;\ j=1,2,3\}$ generates a Lie algebra.
Also the partial differential operator L given

$$L=x\frac{\partial^2}{\partial x^2}+(1-x)\frac{\partial}{\partial x}+y\frac{\partial^2}{\partial y\partial x}+z\frac{\partial}{\partial z}\ ,$$

which can be expressed in the following forms:

$L_1=A_{13}A_{12}+A_{11}+A_{21}$
and $L_2=A_{22}A_{23}+A_{21}$
commutes with $A_{ij}\ (i=1,2;\ j=1,2,3)$

$$[L_k,A_{ij}]=0 \quad k=i=1,2;\ j=1,2,3\ , \tag{3.1}$$

The extended form of the groups generated by $A_{ij}\ (i=1,2;\ j=1,2,3)$ are given by

$$e^{a_{11}A_{11}}u(x,y,z)=u(x,e^{a_{11}}y,z)\ , \tag{3.2}$$

$$e^{a_{21}A_{21}}u(x,y,z)=u(x,y,e^{a_{21}}z) \tag{3.3}$$

$$e^{a_{12}A_{12}}u(x,y,z)=u(\frac{x}{y}(a_{12}+y),a_{12}+y,z) \tag{3.4}$$

$$e^{a_{22}A_{22}}u(x,y,z)=e^{-a_{22}\frac{xz}{y}}u(x+a_{22}\frac{xz}{y},y+a_{22}z,z), \tag{3.5}$$

$$e^{a_{13}A_{13}}u(x,y,z)=e^{-a_{13}y}u(x+a_{13}y,y,z), \tag{3.6}$$

$$e^{a_{23}A_{23}}u(x,y,z)=u(x,a_{23}\frac{y}{z},y,z), \tag{3.7}$$

Therefore, we get

$$\begin{aligned}&e^{a_{23}A_{23}}e^{a_{13}A_{13}}e^{a_{22}A_{22}}e^{a_{12}A_{12}}e^{a_{21}A_{21}}e^{a_{11}A_{11}}f(x,y,z)=\\&\exp\left[-a_{13}y-\frac{a_{22}}{y}\{a_{23}y+(x+a_{13}y)z\}\right]f(\zeta,\eta,\varsigma)\end{aligned}, \tag{3.8}$$

Where

$$\zeta=\frac{1}{yz}[a_{23}y+(x+a_{13}y)z](a_{12}+y+a_{22}z)$$
$$\eta=e^{a_{11}}(a_{12}+y+a_{22}z)$$
$$\varsigma=e^{a_{21}}z$$

4 Generating Functions

From (2,1) $u(x,y,z)=L_n^{\alpha}(x,y)z^n$ is a solution of the system

$$\begin{cases}L_1u=0\\(A_{11}-\alpha-xy^{-1}+x)u=0\end{cases};\qquad\begin{cases}L_1u=0\\(A_{21}-y\ \ n)u=0\end{cases}$$

From (3,1) we get
$SL_i(L_n^{\alpha}(x,y)z^n)=L_iS(L_n^{\alpha}(x,y)z^n)=0,\ i=1,2$
where $S=e^{a_{23}A_{23}}e^{a_{13}A_{13}}e^{a_{22}A_{22}}e^{a_{12}A_{12}}e^{a_{21}A_{21}}e^{a_{11}A_{11}}$,

Therefore, the transformation $S(L_n^{\alpha}(x,y)z^n)$ is also annulled by $L_i,\ i=1,2$.
By setting $a_{11}=a_{21}=0$ in (3.8) we get

$$e^{a_{23}A_{23}}e^{a_{13}A_{13}}e^{a_{22}A_{22}}e^{a_{12}A_{12}}\left[L_n^{\alpha}(x,y)z^n\right]$$
$$=\exp\left[-a_{13}y-\frac{a_{22}}{y}\{a_{23}y+(x+a_{13}y)z\}\right]z^n$$
$$L_n^{\alpha}\left[\frac{1}{yz}[a_{23}y+(x+a_{13}y)z](a_{12}+y+a_{22}z),a_{12}+y+a_{22}z\right] \tag{4.1}$$

$$=\sum_{p=0}^{\infty}\frac{(a_{23})^p}{p!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k!}(-1)^{m+p}$$
$$(n-k+l)_m(n+1)_l L_{n+l-p}^{-k-l+m+p}[x,a_{12}+y+a_{22}z]y^{-k-1+m+p}z^{l-p+n}$$

From (4.1) several generating relations have been derived in this section by attributing different values to a_{ij} 's.

Now we shall consider the following different cases:

writing $-y=t_1,-z=t_2$ in (4.1) and putting

Case 1:

$a_{12}=a_{22}=\frac{-1}{w},a_{13}=a_{23}=1$ we get

$$\exp\left[t_1+\frac{1}{wt_1}\{t_1+t_2(x-t_1)\}\right](-t_2)^n L_n^{\alpha}\left[\frac{1}{t_1t_2}[t_1+t_2(x-t_1)](t_1+\frac{1}{w}(1-t_2),-t_1-\frac{1}{w}(1-t_2)\right]$$

$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{w^{-1}}{l!}\sum_{k=0}^{\infty}\frac{w^{-k}}{k!}(-1)^{p+l}(n-k+l)_k(n+l)_l \quad , \tag{4.2}$$
$$L_{n+l-p}^{-k-l+m+p}[x,-t_1-\frac{1}{w}(1-t_2)]t_1^{-k-1+m+p}t_2^{l-p+n}$$

Case 2:

$a_{12}=a_{22}=\frac{-1}{w},a_{13}=1,a_{23}=0$ we get

$$\exp\left[t_1+\frac{t_2}{wt_1}(x-t_1)\right](-t_2)^n L_n^{\alpha}\left[\frac{1}{t_1}(x-t_1)(t_1+\frac{1}{w}(1-t_2),(-t_1-\frac{1}{w}(1-t_2)\right]$$

$$=\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{w^{-1}}{l!}\sum_{k=0}^{\infty}\frac{w^{-k}}{k!}(-1)^l(n-k+l)_k(n+l)_l \quad , \tag{4.3}$$
$$L_{n+l}^{-k-l+m}[x,-t_1-\frac{1}{w}(1-t_2)]t_1^{-k-l+m}t_2^{l+n}$$

Case 3:

$a_{12} = a_{22} = \frac{-1}{w}, a_{13} = 0, a_{23} = 1$ we get

$$= \exp\left[\frac{1}{wt_1}\{t_1 + xt_2)\}\right](-t_2)^n L_n^{\alpha}\left[\frac{1}{t_1t_2}[t_1 + xt_2\{t_1 + \frac{1}{w}(1-t_2)\}, t_1 + \frac{1}{w}(1-t_2)\right]$$

$$= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{l=0}^{\infty}\frac{w^{-1}}{l!}\sum_{k=0}^{\infty}\frac{w^{-k}}{k!}(-1)^{p+l}(n-k+l)_k \quad , \qquad (4.4)$$

$$(n+l)_l L_{n-p}^{-k-l+p}[x, t_1 + \frac{1}{w}(1-t_2)]t_1^{-k-1+p}t_2^{l-p+n}$$

Case 4:

$a_{12} = \frac{-1}{w}, a_{22} = 0, a_{13} = a_{23} = 1$ we get

$$\exp(t_1)(-t_2)^n L_n^{\alpha}\left[\frac{1}{t_1t_2}[t_1 + t_2(x-t_1)](t_1 + \frac{1}{w}), -t_1 - \frac{1}{w}\right] \qquad (4.5)$$

$$= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{\infty}\frac{w^{-k}}{k!}(-1)^{p+l}(n-k+l)_m \quad ,$$

$$L_{n+p}^{-k+m+p}[x, -t_1 - \frac{1}{w}]t_1^{-k+m+p}t_2^{-p+n}$$

Case 5:

$a_{12} = 0, a_{22} = \frac{-1}{w}, a_{13} = a_{23} = 1$ we get

$$\exp\left[t_1 + \frac{1}{wt_1}\{t_1 + t_2(x-t_1)\}\right](-t_2)^n L_n^{\alpha}\left[-\frac{1}{t_1t_2}[t_1 + t_2(x-t_1)](-t_1 + \frac{t_2}{w}), -t_1 + \frac{t_2}{w}\right] \qquad (4.6)$$

$$= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{w^{-1}}{l!}(-1)^{p+l}(n+l)_l$$

$$L_{n+l-p}^{-l+m+p}[x, -t_1 - \frac{t_2}{w}]t_1^{-1+m+p}t_2^{l-p+n}$$

,

Let $y = -t_1, z = -t_2$ in (4.1) and putting

Case 6:

$a_{12} = a_{22} = 0, a_{13} = a_{23} = 1$ we get

$$e^{t_1}(-t_2)^n L_n^{\alpha}\left[-\frac{1}{t_1 t_2}[t_1+(x-t_1)t_2],-t_1\right]$$
, (4.7)
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}(-1)^p L_{n-p}^{m+p}[x,-t_1]t_1^{m+p}t_2^{n-p}$$

Case 7:

$a_{12}=0, a_{22}=\frac{-1}{w}, a_{13}=0, a_{23}=1$ we get

$$\exp\left[\frac{1}{w}+\frac{xt_2}{wt_1}\right](-t_2)^n L_n^{\alpha}\left[\frac{1}{t_1t_2}[t_1+xt_2)](t_1-\frac{t_2}{w}),(-t_1+\frac{t_2}{w})\right]$$
, (4.8)
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{l=0}^{\infty}\frac{w^{-1}}{l!}(-1)^{l+p}(n+l)_l L_{n+l-p}^{-l+p}[x,-t_1+\frac{t_2}{w}]t_1^{-1+p}t_2^{l-p+n}$$

Case 8:

$a_{12}=a_{22}=a_{13}=0, a_{23}=1$ we get

$$(-t_2)^n L_n^{\alpha}\left[\frac{1}{t_1}(t_1+t_2x),-t_1\right]$$
, (4.9)
$$=\sum_{p=0}^{\infty}\frac{1}{p!}(-1)^p L_{n-p}^{p}[x,-t_1]t_1^p t_2^{-p+n}$$

writing $\frac{1}{y}=t_1, \frac{1}{z}=t_2$ in (4.1) and putting

Case 9:

$a_{12}=a_{22}=\frac{-1}{w}, a_{13}=a_{23}=0$ we get

$$\exp\left[\frac{xt_1}{wt_2}\right]\left(\frac{1}{t_2}\right)^n L_n^{\alpha}\left[xt_1\left(\frac{1}{t_1}-\frac{1}{w}(1+\frac{1}{t_2})\right),\frac{1}{t_1}-\frac{1}{w}(1+\frac{1}{t_2})\right]$$
(4.10)
$$=\sum_{l=0}^{\infty}\frac{w^{-1}}{l!}\sum_{k=0}^{\infty}\frac{w^{-k}}{k!}(-1)^{l+k}(n-k+l)_k$$,
$$(n+l)_l L_{n+l}^{-k-l}[x,\frac{1}{t_1}-\frac{1}{w}(1+\frac{1}{t_2})]t_1^{k+1}t_2^{-l+n}$$

Case 10:

$a_{22}=1, a_{12}=a_{13}=a_{23}=0$ we get

$$\exp\left[\frac{xt_1}{t_2}\right]\left(\frac{1}{t_2}\right)^n L_n^{\alpha}\left[\frac{x}{t_2}(t_1+t_2), \frac{t_1+t_2}{t_1t_2}\right]$$

$$= \sum_{l=0}^{\infty}\frac{1}{l!}(n+l)_l L_{n+l}^{-l}[x, \frac{t_1+t_2}{t_1t_2}]t_1^l t_2^{n-l} \qquad (4.11)$$

writing $y^{-1}=t_1, -z=t_2$ in (4.1) and putting

Case 11:

$a_{12}=\frac{-1}{w}, a_{22}=a_{13}=0, a_{23}=1$ we get

$$(-t_2)^n L_n^{\alpha}\left[-\frac{t_1}{t_2}\left(\frac{1}{t_1}-xt_2\right)\left(\frac{1}{t_1}-\frac{1}{w}\right), \frac{1}{t_1}-\frac{1}{w}\right]$$

$$= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{k=0}^{\infty}\frac{w^{-k}}{k!}(-1)^k(n-k+l)_k L_{n-p}^{-k+p}[x, \frac{1}{t_1}-\frac{1}{w}]t_1^{k-p}t_2^{n-p} \qquad (4.12)$$

writing $-y=t_1, z^{-1}=t_2$ in (4.1) and putting

Case 12:

$a_{22}=\frac{-1}{w}, a_{12}=0, a_{13}=1, a_{23}=0$ we get

$$\exp\left[t_1-\frac{1}{t_1t_2w}(x-t_1)\right]\left(\frac{1}{t_2}\right)^n L_n^{\alpha}\left[\frac{1}{t_1}(x-t_1)\left(t_1+\frac{1}{wt_2}\right), -t_1-\frac{1}{wt_2}\right]$$

$$= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{w^{-1}}{l!}(n+l)_l L_{n+l}^{-l+m}\left(x, -t_1-\frac{1}{wt_2}\right)t_1^{-1+m}t_2^{n-l} \qquad (4.13)$$

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