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2- Absorbing Sub Semi Modules of Partial Semi Modules

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Abstract

The partial functions under disjoint-domain sums and functional composition do not form a field, and thus conventional linear algebra is not applicable. However they can be regarded as a so-ring, an algebraic structure possessing a natural partial ordering, an infinitary partial addition and a binary multiplication, subject to a set of axioms. In this paper, we introduce the notions of 2-absorbing ideal of so-rings and 2-absorbing subsemimodule of partial semimodules and study their characteristics.

Keywords: *So-ring, Ideal, Prime ideal, 2-absorbing ideal, Partial semimodule, Subsemimodule, Multiplication partial semimodule, 2-absorbing subsemimodule.*

Introduction:

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming language semantics. The general cardinal algebras studied by Tarski in 1949, ∑- structures studied by Higgs in 1980, Housdorff topoligical commutative groups studied by Bourbaki in 1966, sum-ordered partial monoids and sum-ordered partial semirings studied by Arbib, Manes, Benson and Streenstrup are some of the algebraic structures of the above type.

Motivated by the work done in partially-additive semantics by Arbib, Manes [2] and in the development of matrix theory of so-rings by Martha E. Streenstrup [6], G. V. S. Acharyulu [1] in 1992 studied the conditions under which an arbitrary so-ring becomes a $pfn(D,D)$, $Mfn(D,D)$ and $Mset(D,D)$. Continuing this study, P.V. Srinivasa Rao [8] in 2011 developed the ideal theory for so-rings and partial semimodules over partial semirings. In this paper, we generalise the concept of prime ideals in a different way as 2-absorbing ideals. In addition to it we introduce the notion of 2-absorbing subsemimodule of partial semimodules and characterize 2-absorbing subsemodules interms of 2-absorbing partial ideals of a partial semiring *R*.

1 Preliminaries

In this section we collect important definitions, results and examples which were already proved for our use in the next sections.

Definition 1.1 [5]: *A partial monoid is a pair* (M,Σ) *where* M *is a non empty set and* $\sum$ *is a partial addition defined on some, but not necessarily all families* $(x_i : i\in I)$ *in* M *subject to the following axioms:*

(i) Unary Sum Axiom: If $(x_i : i\in I)$ is a one element family in M and $I = \{ j \}$, then $\sum(x_i : i\in I)$ is defined and equals x_j.

(ii) Partition-Associatively Axiom: If $(x_i : i\in I)$ is a family in M and If $(I_j : j\in J)$ is a partition of I, then $(x_i : i\in I)$ is summable if and only if $(x_i : i\in I_j)$ is summable for every *j* in *J* and $(\sum(x_i : i\in I_j) : j\in J)$ is summable.

We write $\sum(x_i : i\in I) = \sum(\sum(x_i : i\in I_j) : j\in J)$.

Definition 1.2 [5]: *The sum ordering* $\leq$ *on a partial monoid* (M,Σ) *is the binary relation* $\leq$ *such that* $x \leq y$ *if and only if there exists a h in* M *such that* $y = x + h$, *for* $x, y\in M$.

Definition 1.3 [5]: *A partial semiring is a quadruple* $(R,\Sigma,\cdot,1)$, *Where* (R,Σ) *is a partial monoid with partial addition* $\sum$, $(R,\cdot,1)$ *is a monoid with multiplicative operation '$\cdot$' and unit '1', and the additive and multiplicative structures obey the following distributive laws:*

If $\sum(x_i : i\in I)$ is defined in *R*, then for all *y* in *R*, $\sum(y\cdot x_i : i\in I)$ and $\sum(x_i \cdot y : i\in I)$ are defined and $y\cdot[\sum_i x_i] = \sum_i (y\cdot x_i),[\sum_i x_i]\cdot y = \sum_i (x_i \cdot y)$.

Definition 1.4 [5]: *A sum-ordered partial semiring (or so-ring for short), is a partial semiring in which the sum ordering is a partial ordering.*

Definition 1.5 [1]: *Let R be so-ring. A subset N of R is said to be an ideal of R if the following are satisfied:*

(I_1) if $(x_i : i\in I)$ is a summable family in *R* and $x_i\in N$ for every $i\in I$ then $\sum x_i\in N$,
(I_2) if $x \le y$ and $y\in N$ then $x\in N$, and
(I_3) if $x\in N$ and $r\in R$ then $xr, rx\in N$.

Theorem 1.6 [6]: *An ideal of P of a complete so-ring R is prime if and only if for any* $a, b\in R$, $ab\in P$ *implies* $a\in P$ *or* $b\in P$.

Definition 1.7 [7]: *Let* $(R,\Sigma,\cdot,1)$ *be a partial semiring and* $(M,\overline{\Sigma})$ *be a partial monoid. Then M is said to be a left partial semimodule over R if there exists a function* $*: R\times M \to M : (r,x)\mapsto r*x$ *which satisfies the following axioms for* x, $(x_i : i\in I)$ *in M and* $r_1, r_2, (r_j : j\in J)$ *in R*

(i) if $\overline{\Sigma}_i x_i$ exists then $r*(\overline{\Sigma}_i x_i) = \overline{\Sigma}_i (r*x_i)$,
(ii) if $\sum_j r_j$ exists then $(\sum_j r_j)*x = \overline{\Sigma}_j (r_j * x)$,
(iii) $r_1*(r_2*x) = (r_1\cdot r_2)*x$, and
(iv) $1_R * x = x$.

Definition 1.8 [7]: *Let* $(M,\overline{\Sigma})$ *be a left partial semimodule over a partial semiring R. Then a nonempty subset N of M is said to be a subsemimodule of M if N is closed under* $\overline{\Sigma}$ *and* $*$.

Remark 1.9 [7]: If *N* is a proper subsemimoule of a partial semimodule *M* over *R* then $(N : M) = \{r\in R \mid rM \subseteq N\}$.

Definition 1.10 [7]: *Let M be a partial semimodule over R. Then M is said to be multiplication partial semimodule if for all subsemimodules N of M there exists a partial ideal I of R such that N = IM.*

Theorem 1.11 [7]: *A partial semimodule M over R is a multiplication partial semimodule if and only if there exists a partial ideal I of R such that Rm=IM for each* $m\in M$.

Definition 1.12 [7]: *Let M be a multiplication partial semimodule over R and N, K be subsemimodules of M such that N =IM and K=JM for some partial ideals I, J of R. Then the multiplication of N and K is defined as NK = (IM) (JM) = (IJ)M.*

Definition 1.13 [7]: *Let M be a multiplication partial semimodule over R and* m_1, $m_2\in M$ *such that* $R\,m_1= IM$ *and* $R\,m_2= JM$ *for some partial ideals I, J of R. Then the multiplication of* m_1 *and* m_2 *is defined as* $m_1\,m_2= (IM)(JM)= (IJ)M$.

2 2- Absorbing Ideals

Throughout this section *R* denotes commutative so-ring. In this section we introduce the notion of 2-absorbing ideal and prove that radical of *I* is 2-absorbing ideal of so-ring.

Definition 2.1: *A proper ideal of a so-ring R is called 2- absorbing if for any a, b,* $c\in R$, $abc\in I$ *implies* $ab\in I$ *or* $ac\in I$ *or* $bc\in I$.

Remark 2.2: Every prime ideal of a so-ring *R* is 2-absorbing.

Proof: Suppose *P* is a prime ideal of *R*.

Let $a, b, c\in R \ni abc \in P$. Since *P* is prime,

Case-1: $a\in P$ or $bc\in P$.
$\Rightarrow ab\in P$ or $bc\in P$.

Case-2: $ab\in P$ or $c\in P$.
$\Rightarrow ab\in P$ or $ac\in P$.

From case-1 and case-2, $ab\in P$ or $bc\in P$ or $ac\in P$. Hence *P* is a 2-absorbing ideal of *R*.

Note that the converse need not be true.

Example 2.3: Consider the so-ring $R = \{0, u, v, x, y, 1\}$ with $\sum$ defined on *R* by

$$\sum_i x_i = \begin{cases} x_j & \text{if } x_i = 0 \quad \forall i \neq j, \quad \text{for some} \quad j, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

And ‘ · ’ defined by the following table:

·	0	u	v	x	y	1
0	0	0	0	0	0	0
u	0	u	0	0	0	u
v	0	0	v	0	0	v
x	0	0	0	0	0	x
y	0	0	0	0	0	y
1	0	u	v	x	y	1

Then the ideal $I = \{0, u, x\}$ is a 2-absorbing ideal. Since $v.y = 0 \in I$, but $v \notin I$ and $y \notin I$, I is not prime.

Theorem 2.4: *If I and J are prime ideals of a so-ring R, then $I \cap J$ is 2-absorbing.*

Proof: Suppose I and J are prime ideals of R. Let $a, b, c \in R \ni abc \in I \cap J$. Then $abc \in I$ and $abc \in J$. $\Rightarrow a \in I$ or $bc \in I$ and $a \in J$ or $bc \in J$. $\Rightarrow a \in I$ or $b \in I$ or $c \in I$ and $a \in J$ or $b \in J$ or $c \in J$. $\Rightarrow ab \in I \cap J$ or $bc \in I \cap J$ or $ac \in I \cap J$. Hence $I \cap J$ is a 2-absorbing ideal of R.

Remark 2.5 [8]: If I is an ideal of a so-ring R then the radical of I is $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$.

Theorem 2.6: *If I is a 2-absorbing ideal of so-ring R, then $\sqrt{I}$ is a 2-absorbing ideal of a so-ring R.*

Proof: Let I be a 2-absorbing ideal of so-ring R. Let $a, b, c \in R \ni abc \in \sqrt{I}$. $\Rightarrow (abc)^n \in I$ for some $n \in \mathbb{N}$. $\Rightarrow a^n b^n c^n \in I$ for some $n \in \mathbb{N}$. Since I is 2-absorbing, $a^n b^n \in I$ or $b^n c^n \in I$ or $a^n c^n \in I$ for some $n \in \mathbb{N}$. $\Rightarrow (ab)^n \in I$ or $(bc)^n \in I$ or $(ac)^n \in I$ for some $n \in \mathbb{N}$. $\Rightarrow ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Hence $\sqrt{I}$ is a 2-absorbing ideal of R.

3 2-Absorbing Subsemimodules

Throughout this section R denotes a partial semiring.

In this section we introduce the notions of 2-absorbing subsemimodules of partial semimodules and characterize 2-absorbing subsemodules interms of 2-absorbing partial ideals of a partial semiring R.

Remark 3.1: Let R be a partial semiring. Then a partial ideal I is *2-absorbing* iff for any partial ideals A, B and C of R, $ABC \subseteq I$ implies $AB \subseteq I$ or $BC \subseteq I$ or $AC \subseteq I$.

Definition 3.2: *Let M be a partial semimodule over R and N be a proper subsemimodule of M. Then N is said to be a 2-absorbing subsemimodule of M if for any a, b*$\in R$ *and* $m \in M$, $ab * m \in N$ *implies* $ab \in (N : M)$ *or* $a * m \in N$ *or* $b * m \in N$.

Theorem 3.3: *Let M be a partial semimodule over R and K be a proper subsemimodule of M. If K is a 2-absorbing subsemimodule of M then its associated partial ideal* $(K : M)$ *is a 2-absorbing partial ideal of R.*

Proof: Suppose K is a 2-absorbing subsemimodule of M. Let a, b, $c \in R \ni abc \in (K : M)$. Then $(abc)M \subseteq K$. $\Rightarrow ab(cM) \subseteq K$. $\Rightarrow ab * (c * m) \in K \ \forall m \in M$. $\Rightarrow ab \in (K : M)$ or $a * (c * m) \in K$ or $b * (c * m) \in K \ \forall m \in M$. $\Rightarrow ab \in (K : M)$ or $a(cM) \subseteq K$ or $b(cM) \subseteq K$. $\Rightarrow ab \in (K : M)$ or $ac \in (K : M)$ or $bc \in (K : M)$.

Hence $(K : M)$ is a 2-absorbing partial ideal of R.

Example 3.4: Let R be the partial semiring $\mathbb{N}$ with finite support addition and usual multiplication. Then $M = \mathbb{N}\text{x}\mathbb{N}$ is a left partial semimodule over R by the scalar multiplication $* : (x, (a, b)) \mapsto (xa, xb)$ and $K = 0\text{x}4\mathbb{N}$ is a subsemimodule of M. Here $(K : M) = \{0\}$ which is a prime partial ideal of R. Hence it is 2-absorbing. Since $2 \cdot 2 * (0,1) \in K$, but $2 \cdot 2 = 4 \notin (K : M)$, $2 * (0,1) \notin K$ and hence K is not a 2-absorbing partial ideal of R.

Theorem 3.5: *Let M be a multiplication partial semimodule over R and N be a subsemimodule of M. Then N is 2- absorbing subsemimodule of M if and only if* $(N : M)$ *is a 2-absorbing partial ideal of R.*

Proof: By the theorem.3.3, we get the necessary part. For the sufficient part, suppose $(N : M)$ is a 2-absorbing partial ideal of R. Let I, J be a partial ideals of R and K be a subsemimodule of $M \ni (IJ)K \subseteq N$. Since M is multiplication partial semimodule, $\exists$ a partial ideal L of $R \ni K = LM$. $\Rightarrow N \supseteq (IJ)(LM) = (IJL)M$. $\Rightarrow IJL \subseteq (N : M)$. Since $(N : M)$ is a 2-absorbing partial ideal of R, $IJ \subseteq (N : M)$ or $JL \subseteq (N : M)$ or $IL \subseteq (N : M)$. $\Rightarrow IJ \subseteq (N : M)$ or $JLM \subseteq N$ or $ILM \subseteq N$.

$\Rightarrow IJ \subseteq (N:M)$ or $JK \subseteq N$ or $IK \subseteq N$. Hence N is a 2-absorbing subsemimodule of M.

Theorem 3.6: *Let M be a multiplication partial semimodule over R and N be a subsemimodule of M. Then the following conditions are equivalent:*

(i) N is a 2-absorbing subsemimodule of M.

(ii) For any subsemimodules U, V and W of M, $UVW \subseteq N$ implies $UV \subseteq N$ or $VW \subseteq N$ or $UW \subseteq N$.

(iii) For any m_1, m_2, $m_3 \in M$, $m_1m_2m_3 \subseteq N$ implies $m_1m_2 \in N$ or $m_2m_3 \in N$ or $m_1m_3 \in N$.

Proof: (i) $\Rightarrow$(ii): Suppose N is a 2-absorbing subsemimodule of M. Let U, V and W be the subsemimodules of $M \ni UVW \subseteq N$. Since M is a multiplication partial semimodule, $\exists$partial ideals I, J, K of $R \ni U = IM$, $V = JM$ and $W = KM$. $\Rightarrow UVW = (IJK)M \subseteq N$. $\Rightarrow IJK \subseteq (N:M)$. Since by the theorem.3.5, $(N:M)$ is a 2-absorbing partial ideal of R, and so, $IJ \subseteq (N:M)$ or $JK \subseteq (N:M)$ or $IK \subseteq (N:M)$. $\Rightarrow (IJ)M \subseteq N$ or $(JK)M \subseteq N$ or $(IK)M \subseteq N$. $\Rightarrow UV \subseteq N$ or $VW \subseteq N$ or $UW \subseteq N$.

(ii) $\Rightarrow$(iii): Suppose for any subsemimodules U, V and W of M, $UVW \subseteq N \Rightarrow UV \subseteq N$ or $VW \subseteq N$ or $UW \subseteq N$. Let m_1, m_2, $m_3 \in M$, $m_1m_2m_3 \subseteq N$. Since M is a multiplication partial semimodule, $\exists$partial ideals I, J, K of $R \ni Rm_1=IM$, $Rm_2=JM$, $Rm_3=KM$. $\Rightarrow m_1m_2m_3 = (Rm_1)(Rm_2)(Rm_3) = (IJK)M \subseteq N$. $\Rightarrow (Rm_1)(Rm_2)(Rm_3) = (IJK)M \subseteq N$. $\Rightarrow (Rm_1)(Rm_2)(Rm_3) \subseteq N$. $\Rightarrow (Rm_1)(Rm_2) \subseteq N$ or $(Rm_2)(Rm_3) \subseteq N$ or $(Rm_1)(Rm_3) \subseteq N$. $\Rightarrow m_1m_2 \in N$ or $m_2m_3 \in N$ or $m_1m_3 \in N$.

(iii) $\Rightarrow$(i): Suppose for any m_1, m_2, $m_3 \in M$, $m_1m_2m_3 \subseteq N \Rightarrow m_1m_2 \in N$ or $m_2m_3 \in N$ or $m_1m_3 \in N$. Now we prove $(N:M)$ is a 2-absorbing partial ideal of R. Let I, J and K be the partial ideals of $R \ni IJK \subseteq (N:M)$. Then$(IJK)M \subseteq N$. Suppose $IJ \not\subset (N:M)$, $JK \not\subset (N:M)$ and $IK \not\subset (N:M)$. $\Rightarrow (IJ)M \not\subset N$, $(JK)M \not\subset N$ and $(IK)M \not\subset N$. $\Rightarrow \exists i \in I$, $j \in J$ and $k \in K$, m_1, m_2, $m_3 \in M \ni (ij)*m_1 \in (IJ)M \setminus N$, $(jk)*m_2 \in (JK)M \setminus N$ and $(ik)*m_3 \in (IK)M \setminus N$.

$\Rightarrow [(ij)*m_1][(jk)*m_2][(ik)*m_3] = [(IJ)M][(JK)M][(IK)M] = (IJK)M \subseteq N$.

$\Rightarrow (ij)*m_1 \in N$ or $(jk)*m_2 \in N$ or $(ik)*m_3 \in N$, a contradiction.

Hence $(N:M)$ is a 2-absorbing partial ideal of R. Hence by the theorem.3.5, N is a 2-absorbing subsemimodule of M.

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