Gen. Math. Notes, Vol. 29, No. 2, August 2015, pp. 48-54
ISSN 2219-7184; Copyright © ICSRS Publication, 2015
www.i-csrs.org
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Y-Supplement Extending Modules

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(Received: 8-6-15 / Accepted: 23-7-15)

Abstract

Let R be a commutative ring with unitary and let M be any unitary R-module. In this work we present Y-supplement extending module concept as a generalization of supplement extending module. Also we generalize some properties of cls-module to Y-supplement extending module. And we study the relation between supplement extending and Y-supplement extending module.

Keywords: *Extending module, Supplement submodule, Y-closed submodule.*

1 Introduction and Preliminaries

Throughout this paper *R* will be a commutative ring with identity and all modules will be unitary left *R*-module. A submodule *N* of *M* is called an essential in *M* if for every nonzero submodule *K* of *M* then $N \cap K \neq 0$ [1]. A submodule *N* of *M* is called small in *M* if for any proper submodule *K* of *M* then $N+K \neq M$ [1]. A submodule *N* of *M* is called supplement in *M* if there exists a submodule *K* of such that $N+K=M$ and *N* is minimal with this property. Equivalently, if $N+K=M$ and $N \cap K \ll N$ [1]. A submodule *N* of *M* is called closed in *M* if it has no proper essential extension in *M* [1]. Recall that a submodule *Z (M) ={x*∈*M* such that *ann(x)* is essential in *R*} see [1], if *Z (M)* =0 then *M* is called a nonsingular and called singular if *Z (M)* = *M* [1]. A submodule *N* of *M* is called *Y*-closed submodule in *M* provided $\frac{M}{N}$ is nonsingular see [2]. A module *M* is called cls-module provided every closed submodule in *M* is a direct summand of *M* see [3]. Recall that an *R*-module *M* is called extending (cs) if every closed submodule is a direct summand of *M* see [4]. An *R*-module *M* is called supplement extending if every submodule of *M* is essential in a supplement submodule in *M*. Equvalently, if and only if every closed submodule in *M* is supplement submodule in *M* see [10].

Remarks and Examples 1.1[10]:

1. It is clear that every extending module is supplement extending then Z, Q, M=$Z_2 \oplus Z_4$as Z-module are supplement extending.
2. Every semisimpleR-module is supplement extending.
3. A Z-module M=$Z_4 \oplus Z_4$ is not supplement extending since $\{(\bar{0},\bar{2})\}$ is closed submodule in M which is not supplement submodule.

Proposition 1.2[2]: *Let M be a singular R-module. Then M is the only Y-closed submodule in M.*

Remark 1.3[2]: *Let A and B be submodules of an R-module M if A is a Y-closed submodule in B and B is a Y-closed submodule in M, then A is a Y-closed submodule in M.*

Remark 1.4[2]: *Let M be an R-module and let A, B be submodules of M such that A⊆B, then*

1. *If A is a Y-closed submodule in M, then A is Y-closed submodule in B.*
2. *B is Y-closed submodule in M if and only if* $\frac{B}{A}$ *is a Y-closed submodule in* $\frac{M}{A}$.

Remark 1.5[2]: *Let M be an R-module and N be a Y-closed submodule in M, then [N:M] is a Y-closed ideal in R.*

Remark 1.6[2]: *Let M be an R-module and let* $\{B_i, i\in I\}$ *be an independent family of submodules of M. If* $\{A_i, i\in I\}$ *is a family of submodules of M such that* $A_i\subseteq B_i$, $\forall\ i \in I$. *Then* $\oplus_{i\in I} A_i$*is a Y-closed submodule in* $\oplus_{i\in I} B_i$*if and only if* A_i *is a Y-closed submodule in* B_i, $\forall\ i \in I$.

Proposition 1.7[2,Prop.1.3,P.17]: *Let M be an R-module and A be a submodule of M. If B is any relative complement for A in M, then* $A\oplus B \subseteq_e M$.

Remark 1.8[2]: *For any R-module M*

1. *M is Y-closed in M.*
2. *Every Y-closed submodule in M is closed but the converse is true when M is nonsingular.*

2 *Y*-Supplement Extending Modules

In this section we introduction a generalization for supplement extending module namely *Y*-supplement extending.

Definition 2.1: *Let M be an R-module, then M is called Y-supplement extending if every Y-closed submodule in M is supplement submodule.*

Examples and Remarks 2.2:

1. *Z* as *Z*-module is *Y*-supplement extending, since the only *Y*-closed submodule in *Z* is *Z* and 0 which are supplement submodules in *Z*.
2. Z_6 as Z_6-module is a *Y*-supplement extending, since the only *Y*-closed submodule are Z_6 and $\{\bar{0}\}$ which are supplement submodules in Z_6.
3. Every singular *R*-module *M* is *Y*-supplement extending. In particular, every torsion module over an integral domain is a *Y*-supplement extending.

Proof: Let *M* be a singular *R*-module, then *M* is the only *Y*-closed submodule in *M*, by Prop. 1.2, but *M* is supplement then *M* is a *Y*-supplement extending module.

4. Clear that every supplement extending module *M* is *Y*-supplement extending. But the converse in not true in general, for example.

5. $Z_4 \oplus Z_4$as *Z*-module is singular since Z_4 as *Z*-module is singular and every direct sum of singular is also singular by [2] and hence by Prop. 1.2, $Z_4 \oplus Z_4$is the only *Y*-closed submodule but $Z_4 \oplus Z_4$is supplement submodule . So, $Z_4 \oplus Z_4$as *Z*-module is *Y*-supplement extending. But by Remarks and Examples 1.1(3) is not supplement extending module.

Proposition 2.3: *Every nonsingular Y-supplement extending R-module is supplement extending. In particular, every torsion free module over an integral domain is Y-supplement extending module.*

Proof: Let *M* be a nonsingular module and *A* be a closed submodule in *M*. Since *M* is nonsingular then by Remark 1.8(2), *A* is a *Y*-closed submodule in *M* but *M* is *Y*-supplement extending then *A* is supplement submodule and hence *M* is supplement extending.

Proposition 2.4: *Let M be an R-module such that for every submodule X of M, there exists a Y-closed submodule A in M such that X is essential in A. Then M is Y-supplement extending if and only if M is supplement extending.*

Proof: Let X be a submodule of M and M be a Y-supplement extending, then by assumption, there is a Y-closed submodule A in M such that X is essential in A but M is a Y-supplement extending then A is supplement submodule in M. Now X is essential in A where A is supplement in M. Hence M is supplement extending.
The converse is true from Examples and Remarks 2.2(4).

Proposition 2.5: *Any direct summand of Y-supplement extending module is Y-supplement extending.*

Proof: Let $M=A\oplus B$ be a Y-supplement extending module. To show that A is a Y-supplement extending, let K be a Y-closed submodule in A, by the third and second isomorphism theorems we have $\frac{M}{K\oplus B}=\frac{A\oplus B}{K\oplus B}\cong\frac{\frac{A\oplus B}{B}}{\frac{K\oplus B}{B}}\cong\frac{\frac{A}{A\cap B}}{\frac{K}{K\cap B}}=\frac{A}{K}$ but K is a Y-closed submodule in A then $\frac{A}{K}$ is nonsingular and hence $K\oplus B$ is a Y-closed in M but M is a Y-supplement extending then $K\oplus B$ is supplement in $M= A\oplus B$. So, by [7] K is supplement in A and hence A is Y-supplement extending.

Proposition 2.6: *Every Y-closed submodule in Y-supplement extending module is again Y-supplement extending.*

Proof: Let A be a Y-closed submodule in M to show that A is Y-supplement extending. Let K be a Y-closed submodule in A by Prop. 1.3 K is a Y-closed in M but M is Y-supplement extending then K is supplement in M and hence K is supplement in A by [9, lemma 1.5]. So A is Y-supplement extending.

Proposition 2.7: *Let A and B be submodules of an R-module M. If B is a Y-supplement extending and A is Y-closed submodule in M then A∩B is supplement submodule in B.*

Proof: Assume that B is Y-supplement extending and A is Y-closed in M by the second isomorphism theorem $\frac{B}{B\cap A}\cong\frac{A+B}{A}$. Since $\frac{A+B}{A}\subseteq\frac{M}{A}$ and A is Y-closed in M then $\frac{M}{A}$ is nonsingular. So $\frac{A+B}{A}$ is nonsingular and hence $A\cap B$ is Y –closed in B but B is Y-supplement extending then $A\cap B$ is supplement in B.

Proposition 2.8: *Let A be a submodule of an R-module M. If M is a Y-supplement extending then* $\frac{M}{A}$ *is Y-supplement extending.*

Proof: Let $\frac{B}{A}$ be a Y-closed submodule in $\frac{M}{A}$, Then by [Prop. 1.4 (2)] B is Y-closed submodule in M. But M is Y-supplement extending, therefore B is supplement in M. Thus $M=B+K$, where K is a submodule of M and $B\cap K\ll B$ but $A\subseteq B$, then one can easily show that $\frac{M}{A}=\frac{B}{A}+\frac{K+A}{A}$ to show $\frac{B}{A}\cap\frac{K+A}{A}\ll\frac{B}{A}$ i.e. $\frac{B\cap(K+A)}{A}\ll\frac{B}{A}$ i.e. $\frac{A+(K\cap B)}{A}\ll\frac{B}{A}$ [by modular law]. Let $\frac{L}{A}+\frac{A+(K\cap B)}{A}=\frac{L+A+(B\cap K)}{A}=\frac{B}{A}$ then

$L+A+(B\cap K)=B$ but $B\cap K\ll B$. Hence $L+A=B$ but $A\subseteq L$ then $L=B$ and $\frac{L}{A}=\frac{B}{A}$. Hence $\frac{A+(B\cap K)}{A}\ll\frac{B}{A}$. Thus $\frac{B}{A}$ is supplement in $\frac{M}{A}$ and hence $\frac{M}{A}$ is Y-supplement extending.

Proposition 2.9: *Let M be a faithful multiplication R-module. If R is a Y-supplement extending ring then M is Y-supplement extending module.*

Proof: Let A be a Y-closed submodule in M, but M is multiplication, so $[A:M]M=A$. But A is a Y-closed in M then by Prop. 1.5, $[A:M]$ is Y-closed ideal in R. But R is Y-supplement extending ring then $[A:M]$ is supplement ideal in R i.e. there exists an ideal J of R such that $[A:M]+J=R$ and $[A:M]\cap J\ll[A:M]$. Now, $M=RM=([A:M]+J)M=[A:M]M+JM=A+JM$ and we have $A\cap JM=[A:M]M\cap JM$ since M is faithful then $[A:M]M\cap JM=([A:M]\cap J)M$. Now, to show that $([A:M]\cap J)M\ll[A:M]M=A$. Let $([A:M]\cap J)M+KM=[A:M]M$ then $(([A:M]\cap J)+K)M=[A:M]M$ but M is multiplication then $([A:M]\cap J)+K=[A:M]$ but $[A:M]\cap J\ll[A:M]$. So, $K=[A:M]$ and $KM=[A:M]M$ and hence $[A:M]\cap J)M\ll[A:M]M=A$, then A is supplement in M and M is Y-supplement extending.

Let M be an R-module. M is called Y-extending if for any submodule A of M there exists a direct summand K of M such that $A\cap K$ is essential in A and $A\cap K$ is essential in K see [5].

Proposition 2.10: *Let M be a Y-extending R-module then M is Y-supplement extending.*

Proof: See [3] because every cls-module is Y-supplement extending.

Now, we take the following remark.

Remark 2.11 [2, p.49]: *Let A be a submodule of an R-module M. By Zorn's Lemma, there is a smallest Y-closed submodule H of M containing A called the Y-closure of A in M {we denote it by A^{-y}}.*

Proposition 2.12: *An R- module M is Y-supplement extending if and only if A^{-y} is supplement in M, for every submoduleA of M.*

Proof: Let M be a Y-supplement extending module and let A be a submodule of M. Since $A^{-y}$is a Y-closed submodule in M, then A^{-y} is supplement submodule in M. The converse, let A be a Y-closed submodule in M, then $A^{-y}= A$. Thus A is supplement in M.

3 Direct Sum of *Y*-Supplement Extending Module

In this section, direct sums of Y-supplement extending are studied. It is shown that if $M=M_1\oplus M_2$when $M_1$and M_2 are Y-supplement extending modules.

Proposition 3.1: *Let $M_1$and M_2 be Y-supplement extending modules such that $annM_1+annM_2=R$, then $M_1\oplus M_2$ is Y-supplement extending module.*

Proof: Let *A* be a *Y*-closed submodule in $M_1\oplus M_2$. Since $annM_1+annM_2=R$ then by the same way of the proof [6, prop 4.2, ch.1], $A=C\oplus D$ where *C* and *D* are submodules of $M_1$and M_2 respectively. Since $A=C\oplus D$ is a *Y*-closed in $M_1\oplus M_2$ then *C* and *D* are *Y*-closed in M_1 and M_2respectively by [prop 1.5]. But $M_1$and M_2 are *Y*-supplement extending modules then *C* and *D* are supplement submodules in M_1 and M_2 respectively. So, $A=C\oplus D$ is supplement in $M_1\oplus M_2$ by [7]. Hence $M_1\oplus M_2$ is *Y*-supplement extending module.

Definition 3.2: *Let M be an R-module. We say that M is distributive module if $A\cap(B+C)=(A\cap B)+(A\cap C)$, for all submodules A,B and C of M, see [8].*

Proposition 3.3: *Let $M=M_1\oplus M_2$ be a distributive module. Then M is a Y-supplement extending if and only if $M_1$and M_2 are Y-supplement extending modules.*

Proof: → Clear by [Prop.2.5].

Conversely, let *K* be a *Y*-closed submodule in *M*. But $M=M_1\oplus M_2$ then $K=K\cap(M_1\oplus M_2)$. Since *M* is distributive module then $K=(K\cap M_1)\oplus(K\cap M_2)$ is a Y-closed submodule in $M=M_1\oplus M_2$ by Prop. 1.6, $K\cap M_1$ and $K\cap M_2$ are *Y*-closed submodules in M_1 and M_2 respectively. But M_1 and M_2 are *Y*-supplement extending modules, then $K\cap M_1$ and $K\cap M_2$ are supplement submodules in M_1 and M_2 respectively, by [7], $K=(K\cap M_1)\oplus(K\cap M_2)$ is supplement in $M_1\oplus M_2=M$. Hence *M* is *Y*-supplement extending.

Proposition 3.4: *Let $M=\oplus_i M_i$ where $i\in I$ be an R-module such that every Y-closed submodule in M is fully invariant then M is Y-supplement extending if and only if M_i are Y-supplement extending for all $i\in I$.*

Proof: Let *A* be an *Y*-closed submodule in *M*. For each $i\in I$, if $f_i: M\to M_i$ is the projection map. Now, let $x\in A$ then $x=\sum_{i\in I} m_I$, $m_i\in M_i$ and $m_i=0$ for all except a finite number of $i\in I$. Clearly that $f_i(x)=m_i$, for all $i\in I$. Since *A* is a *Y*-closed submodule in *M*, then by assumption, *A* is fully invariant and hence $f_i(x)=m_i\in A\cap M_i$. So, $x\in\oplus(A\cap M_i)$, thus $A\subseteq\oplus(A\cap M_i)$, but $\oplus(A\cap M_i)\subseteq A$ thus $\oplus(A\cap M_i)=A$. Since *A* is a *Y*-closed submodule in *M* then $(A\cap M_i)$ is a *Y*-closed in M_i for all $i\in I$, by Remark 1.6. But M_i is *Y*-supplement extending modules for all $i\in I$ then $(A\cap M_i)$ is supplement in M_i for all $i\in I$. But by [7] $A=\oplus(A\cap M_i)$ is supplement in $\oplus_i M_i=M$. Hence *M* is *Y*-supplement extending.

Proposition 3.5: *An R-module M is Y-supplement extending module if and only if for every direct summand A of the injective hull E(M) of M such that $A\cap M$ is a Y-closed submodule in M, then $A\cap M$ is a supplement submodule in M.*

Proof:→ Clear.

Conversely, let A be a Y-closed submodule in M and let B be a relative complement of Ain M, then by Prop.1.9, $A\oplus B$ is essential in M. Since M is essential in $E(M)$, then $A\oplus B$ is essential in $E(M)$. Thus $E(M) = E(A\oplus B) = E(A) \oplus E(B)$. Since $E(A)$ is direct summand and $A = A\cap M$ is essential in $E(A)\cap M$. So, $\frac{E(A)\cap M}{A}$ is singular. Now, $\frac{E(A)\cap M}{A} \subseteq \frac{M}{A}$ which is nonsingular then $\frac{E(A)\cap M}{A}$ is nonsingular. So, $\frac{E(A)\cap M}{A} = 0$ i.e. $E(A)\cap M = A$ is Y-closed and by assumption $E(A)\cap M = A$ is supplement in M.

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