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Graph Products of Open Distance Pattern Uniform Graphs

Bibin K. Jose

Department of Mathematics, S.D. College Alappuzha
University of Kerala, Kerala, India
E-mail: bibinkjose2002@yahoo.com

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Abstract

Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, each vertex u in G is associated with the set $f^o_M(u) = \{d(u, v) : v \in M, u \neq v\}$, called its open M-distance-pattern. The graph G is called open distance-pattern uniform (odpu-) graph if there exists a subset M of $V(G)$ such that $f^o_M(u) = f^o_M(v)$ for all $u, v \in V(G)$ and M is called an open distance-pattern uniform (odpu-) set of G. The minimum cardinality of an odpu-set in G, if it exists, is called the odpu-number of G and is denoted by $od(G)$. In this paper we characterize several odpu-graphs and constructed classes of odpu-graph products especially, join of two graphs, cartesian product, lexicographic Product and corona.

Keywords: *Graph products, Open distance-pattern uniform graphs, Open distance-pattern uniform (odpu-) set, Odpu-number.*

1 Introduction

All graphs considered in this paper are finite, simple, undirected and connected. For graph theoretic terminology we refer to Harary[7].

The concept of open distance-pattern and open distance-pattern uniform graphs were studied in [1, 2]. Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, the open M-distance-pattern of a vertex u in G is defined to be the set $f^o_M(u) = \{d(u, v) : v \in M, u \neq v\}$, where $d(x, y)$ denotes the distance between the vertices x and y in G. If there exists a non-empty set $M \subseteq V(G)$ such that $f^o_M(u)$ is independent of the choice of u, then G is called open distance-pattern uniform (odpu-) graph and the set M is called an open distance-pattern uniform (odpu-) set. The minimum cardinality of an odpu-set in G, if it exists, is the odpu-number of G and is denoted by $od(G)$. In this paper, we characterize several odpu-graphs which are formed by graph products especially, join of two graphs, cartesian product, lexicographic product and corona. We need the following definitions and previous results.

In paper [1], it is proved that, a graph G with radius $r(G)$ is an odpu graph if and only if the open distance pattern of any vertex u in G is $f^o_M(u) = \{1, 2, \cdots, r(G)\}$ and a graph is an odpu-graph if and only if its centre $Z(G)$ is an odpu-set, thereby characterizing odpu-graphs, which in fact suggests an easy method to check the existence of an odpu-set for a given graph.

Proposition 1. [1] *For any graph G, $od(G) = 2$ if and only if there exist at least two vertices $x, y \in V(G)$ such that $deg(x) = deg(y) = |V(G)| - 1$, where $deg(x)$ denote the degree of the vertex x in G.*

Proposition 2. [1] *There is no graph having odpu-number three.*

Proposition 3. [1] *A graph G is an odpu graph if and only if its centre $Z(G)$ is an odpu set and hence $|Z(G)| \geq 2$.*

Proposition 4. [1] *All self-centered graphs are odpu graphs.*

Theorem 1.1. [1] *The shadow graph of any complete graph K_n, $n \geq 3$ is an odpu-graph with odpu-number $n + 2$ (The shadow graph $S(G)$ of a graph G is obtained from G by adding for each vertex v of G a new vertex v', called the shadow vertex of v, and joining v' to all the neighbors of v in G).*

Theorem 1.2. [1] *Every odpu-graph G satisfies, $r(G) \leq d(G) \leq r(G) + 1$ where $r(G)$ and $d(G)$ denote the radius and diameter of G respectively.*

The join of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is denoted by $G_1 + G_2$ has the vertex set as $V = V_1 \cup V_2$ and the edge set E contains all the edges of G_1 and G_2 together with all edges joining the vertices of V_1 with the vertices of V_2.

The cartesian product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is denoted by $G_1 \times G_2$ has the vertex set $V = V_1 \times V_2$. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two vertices in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 is adjacent to $v_2]$ or $[u_2 = v_2$ and u_1 is adjacent to $v_1]$.

The composition (or lexicographic product) $G = G_1[G_2]$ also has the vertex set $V = V_1 \times V_2$ and the vertex $u = (u_1, u_2)$ is adjacent with the vertex $v = (v_1, v_2)$ whenever $[u_1$ is adjacent to $v_1]$ or $[u_1 = v_1$ and u_2 is adjacent to

$v_2]$. Obviously both compositions $G_1[G_2]$ and $G_2[G_1]$ are not isomorphic in general.

The Corona $G_1 \circ G_2$ of two graphs G_1 and G_2 was defined by Frucht and Harary [5] as the graph G obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. It follows from the definition of the corona that $G_1 \circ G_2$ has $p_1(1 + p_2)$ points and $q_1 + p_1q_2 + p_1p_2$ edges where G_1 and G_2 has q_1 and q_2 edges respectively.

Theorem 1.3. [8] *The cartesian product of two graphs is connected if and only if both factors are connected.*

Theorem 1.4. [8] *Let (u, v) and (x, y) be arbitrary vertices of the cartesian product $G \times H$. Then,*
$d_{G\times H}((u, v),(x, y)) = d_G(u, x) + d_H(v, y)$.

2 Main Results

First we characterize the odpu graphs which are obtained by taking join of two graphs. Recall that a universal vertex means a vertex which is adjacent to all other vertices of the graph.

Theorem 2.1. *Join of two graphs G_1 and G_2 is an odpu-graph if and only if exactly one of G_1 and G_2 does not have exactly one universal vertex.*

Proof. Assume that the join $G_1 + G_2$ is an odpu-graph. We shall prove that exactly one of G_1 and G_2 does not have exactly one universal vertex. If not, assume that exactly one of G_1 and G_2 (say G_1) has exactly one universal vertex $u \in V(G_1)$. Since u is a universal vertex in G_1, u is a universal vertex in G_1+G_2. Also, since u is the only universal vertex in G_1 and G_2 does not have a universal vertex, there is no vertex of degree $|V(G_1 + G_2)| - 1$ in $G_1 + G_2$ other than u. This implies $Z(G_1+G_2) = \{u\}$. Hence by Proposition 3, G_1+G_2 is not an odpu-graph, a contradiction to the assumption.
Conversely, assume that exactly one of G_1 and G_2 does not have exactly one universal vertex. Then, the following are the three possibilities.
(i) One of G_1 and G_2 has more than one universal vertices.
(ii) Both G_1 and G_2 have at least one universal vertex each.
(iii) None of G_1 and G_2 has a universal vertex.

Case:i One of G_1 and G_2 (say G_1) has at least two universal vertices u and v.
Since u and v are universal vertices of G_1, u and v are also universal vertices of $G_1 + G_2$. That is, there exist two universal vertices in $G_1 + G_2$ and hence

by Proposition 1, $G_1 + G_2$ is an odpu-graph.

Case:ii Both G_1 and G_2 have at least one universal vertex each.
Let $x \in V(G_1)$ and $y \in V(G_2)$ are universal vertices in G_1 and G_2 respectively. Then, x and y are universal vertices of $G_1 + G_2$ and hence by Proposiion 1, $G_1 + G_2$ is an odpu-graph.

Case:iii None of G_1 and G_2 has a universal vertex.
In this case the graph G_1+G_2 does not have a universal vertex. Let $u \in V(G_1)$. Then, for every vertices $v \in V(G_1)$ not adjacent to u, there exists a path uwv where $w \in V(G_2)$, in $G_1 + G_2$. This is true for all vertices of $V(G_1)$ and similarly in $V(G_2)$. Hence $G_1 + G_2$ is a self-centered graph of radius 2. Hence by Proposition 4, $G_1 + G_2$ is an odpu-graph. Hence the Theorem.

Corollary 2.2. *Join of two graphs G_1 and G_2 is an odpu-graph if and only if one of the following condition must hold.*
(i) One of G_1 and G_2 has more than one universal vertices.
(ii) Both G_1 and G_2 have at least one universal vertex.
(iii) None of G_1 and G_2 has a universal vertex.

Corollary 2.3. *$G_1 + G_2$ is an odpu graph if and only if $|Z(G_1 + G_2)| \geq 2$.*

Corollary 2.4. *G_1+G_2 is odpu if and only if $r(G_1+G_2) = r(Z(G_1+G_2))$.*

Theorem 2.5. *For any positive integer $n \neq 1, 3$, there exists an odpu-graph G with odpu-number n which is formed by join of two graphs G_1 and G_2.*

Proof. ***Case:i*** $n = 2$.
Consider the graph, $K_2 + H$, where H is any graph. Then the vertices of K_2 are universal vertices in K_2+H and hence by Corollary 2.2, it is an odpu-graph with odpu-number $n = 2$.

Case:ii $n = 4$.
Consider the graph, $\overline{K_2} + \overline{K_2}$, which isomorphic to $K_{2,2}$ and hence it is an odpu-graph with odpu-number 4.

Case:iii $n = 5$.
Consider the graph, $G = C_5 + \overline{K_2}$. Then $r(G) = 2$. Let M be a minimal odpu-set of G. Since $r(G) = 2$, $f^o_M(u) = \{1, 2\}$; $\forall\ u \in V(G)$. Since, for a vertex x in $\overline{K_2}$, the other vertex $y \in \overline{K_2}$ is the only vertex at a distance 2 in G, both the vertices of $\overline{K_2}$ must be in M. Now let $C_5 = (v_1, v_2, v_3, v_4, v_5, v_1)$.

Claim:1 M contains exactly three vertices from C_5.
Since, all the vertices of C_5 is adjacent to a vertex of $M \cap \overline{K_2}$, $1 \in f^o_M(v_i)$;

$(i = 1, 2, \ldots, 5)$. If none of the vertices of C_5 is in M, then $2 \notin f^o_M(v_i)\ \forall\ i = 1, 2, \ldots, 5$. So let $v_1 \in M$. But, for $2 \in f^o_M(v_1)$, either v_3 or v_4 must be in M. With out loss of generality assume that $v_3 \in M$. Then $v_2 \in N(v_1) \cap N(v_3)$. Since $2 \in f^o_M(v_2)$, either v_4 or v_5 must be in M. Let $v_5 \in M$. Hence M contains v_1, v_3 and v_5 and it is easy to see that $2 \in f^o_M(v_i)$; $i = 1, 2, \ldots, 5$ and hence three vertices of C_5 must be in M. Hence $M = \{v_1, v_3, v_5, u, v\}$ and $od(G) = 5$.

Case:iv $n = k$; where $k \geq 6$ and k is even.
Let $G = \overline{K_2} + K_{2,2,\ldots,2}$, where $K_{2,2,\ldots,2}$ be the complete t-partite graph of order $2t$. Then G is isomorphic to a complete $(t+1)$-partite graph with $2t + 2 = k$ vertices. Thus the graph G is self-centered with $r(G) = 2$ and hence it is an odpu-graph. Each partitions of G contains exactly two vertices u and v such that v is the only vertex of u at a distance 2 in G and conversely. Hence all the vertices of G must be in the minimum odpu set M. Hence $od(G) = k$, where $k \geq 6$ and k is even.

Case:v $n = k$; where $k \geq 7$ and k is odd.
Let $G = C_5 + K_{2,2,\ldots,2}$, where $K_{2,2,\ldots,2}$ be a complete t-partite graph of order $2t$. then, G is a self-centered graph of radius two. Each partitions of $K_{2,2,\ldots,2}$ contains exactly two vertices u and v such that v is the only vertex of u at a distance 2 in G and conversely. Hence all the vertices of $K_{2,2,\ldots,2}$ must be in the minimum odpu set M. By claim-1 in Case:(iii), exactly three vertices of C_5 must be in M. Hence M contains exactly $3 + 2t$ vertices of G and hence $od(G) = 3 + 2t$; $t \geq 2$. Thus $od(G) = k$, $k \geq 7$ and k is odd. Hence the theorem.

Corollary 2.6. *Let G be an odpu-graph formed by join of two graphs G_1 and G_2. Then, G has odpu-number two if and only if either*
(i) One of G_1 or G_2 has more than one universal vertices or
(ii) Both G_1 and G_2 have at least one universal vertex each.

Theorem 2.7. *Let G be an odpu-graph formed by join of two graphs G_1 and G_2. Then, $od(G) = 4$ if and only if there exist two non-adjacent vertices u_i and v_i in G_i $(i = 1, 2)$ such that $N(u_i) \cap N(v_i) = \phi$, for $i = 1, 2$.*

Proof. Let $G = G_1 + G_2$ and $od(G) = 4$. Then, there is no universal vertex in G and hence G is a self-centered graph of radius 2. Hence $f^o_M(u) = \{1, 2\}\ \forall\ u \in V(G)$. Since $2 \in f^o_M(u)\ \forall\ u \in V(G)$ exactly two nonadjacent vertices from G_1 and exactly two nonadjacent vertices from G_2 belongs to the minimum odpu set M. Let the vertices of M be u_1, v_1, u_2, v_2 where $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$. Now, if $N(u_1) \cap N(u_2) \neq \phi$, then let $w \in N(u_1) \cap N(v_1)$. Then $wu_1,\ wv_1,\ wu_2,\ wv_2 \in E(G)$ and hence $2 \notin f^o_M(w)$, a contradiction. Hence $N(u_1) \cap N(v_1) = \phi$. Similarly $N(u_2) \cap N(v_2) = \phi$. Hence there exist two non-adjacent vertices u_i and v_i in G_i $(i = 1, 2)$ such that $N(u_i) \cap N(v_i) = \phi$,

for $i = 1, 2$.
Conversely, assume that there exist two non-adjacent vertices u_i and v_i in G_i $(i = 1, 2)$ such that $N(u_i) \cap N(v_i) = \phi$, for $i = 1, 2$. Let $M = \{u_1, v_1, u_2, v_2\}$. Since $d(u_1, v_1) = 2$ and $d(u_1, u_2) = d(u_1, v_2) = 1$, $f^o_M(u_1) = \{1, 2\}$. Similarly, $f^o_M(v_1) = \{1, 2\}$. Since $d(u_2, v_2) = 2$ and $d(u_2, u_1) = d(u_2, v_1) = 1$, $f^o_M(u_2) = \{1, 2\}$. Similarly, $f^o_M(v_2) = \{1, 2\}$. Let $w \in V(G_1)$. Then, $d(w, u_2) = d(w, v_2) = 1$ and since $N(u_1) \cap N(v_1) = \phi$, at least $d(w, u_1) = 2$ or $d(w, v_1) = 2$ or both. Therefore, $f^o_M(w) = \{1, 2\}$ Similarly, $\forall\ z \in V(G_2)$, $f^o_M(z) = \{1, 2\}$. Hence 4 vertices of G form an odpu-set. By Proposition 1 and Proposition 2, we conclude that $od(G) = 4$. Hence the theorem.

The next Lemma help us to characterize the odpu graphs which is formed by the cartesian product of two graphs.

Lemma 2.8. *Let (u, v) be a vertex of the cartesian product $G \times H$. Then, $e_{G\times H}(u, v) = e_G(u) + e_H(v)$ where $e_G(u)$ denote the eccentricity of u in G.*

Proof. Let $e_{G\times H}(u, v) = k$. Then, there exists a vertex $(x, y) \in G \times H$ such that $d((u, v), (x, y)) = k$ and there is no vertex (a, b) such that $d(u, a)+d(v, b) > k$. Thus, $d(u, x)$ and $d(v, y)$ are maximum with respect to u and v respectively. Hence, $d(u, x) = e_G(u)$ and $d(v, y) = e_H(v)$. Hence by the Theorem 1.4, $e_{G\times H}(u, v) = e_G(u) + e_H(v)$. Hence the lemma.

Corollary 2.9. *$r(G \times H) = r(G) + r(H)$ and $d(G \times H) = d(G) + d(H)$.*

Proof. $r(G \times H) = \min \{e(u, v) : (u, v) \in G \times H\}$.
$= \min \{e(u) + e(v) : u \in G, v \in H\}$, by the Theorem 1.4.
$= \min \{e(u) : u \in G\} + \min \{e(v) : v \in H\}$.
$= r(G) + r(H)$.
Similarly, $d(G \times H) = d(G) + d(H)$.

Theorem 2.10. *The cartesian product $G_1 \times G_2$ is an odpu-graph if and only if one of G_1 and G_2 is self-centered and the other graph is an odpu-graph.*

Proof. For notational convenience let $r(G_1 \times G_2) = r$, $r(G_1) = r_1$ and $r(G_2) = r_2$. Similarly, $d(G_1 \times G_2) = d$, $d(G_1) = d_1$ and $d(G_2) = d_2$. Hence by Corollary 2.9 $r = r_1 + r_2$ and $d = d_1 + d_2$.

First we assume that $H = G_1 \times G_2$ is an odpu graph. We shall prove, one of G_1 and G_2 (say G_1) is self-centered and the other is an odpu graph. If not, there are two possibilities.

Case:1 None of G_1 and G_2 are self-centered.
Then, $r_1 \neq d_1$ and $r_2 \neq d_2$ or in particular, $r_1 + 1 \leq d_1$ and $r_2 + 1 \leq d_2$. Thus $r_1 + r_2 + 2 \leq d_1 + d_2$. Hence by Corollary 2.9, $r + 2 \leq d$. Which is a

contradiction to Theorem 1.2.

Case:2 G_1 is self-centered; but G_2 is not an odpu-graph.
Since H is an odpu graph, $f^o_M(u,v) = \{1,2,\ldots,r\}\ \forall\ (u,v) \in V(H)$ where $r = r_1+r_2$. Now, $Z(H) = \{(u,v) : u \in Z(G_1),\ v \in Z(G_2)\} = \{(u,v) : v \in Z(G_2)\}$, since G_1 is self-centered. Since G_2 is not odpu, there exist a vertex $v \in V(G_2)$ such that $f^o_M(v) \neq \{1,2,\ldots,r_2\}$. Thus, there are two possibilities.

Subcase:1 there exists a number $k > r_2$ such that $k \in f^o_M(v)$.
Then there exists a vertex $x \in Z(G_2)$ such that $d(v,x) = k$. Now let $w,z \in V(G_1)$ such that $d(w,z) = r_1$. Since $x \in Z(G_2)$, the vertices $(x,z),(x,w) \in Z(G_1 \times G_2)$. By Theorem 1.4, $d((v,w),(x,z)) = d(v,x) + d(w,z) = k + r_1 > r_1 + r_2 = r$. Therefore, $r_1 + k \in f^o_M(v,w)$ is a contradiction that H is an odpu graph.

Subcase:2 There exists a number $k;\ 1 \leq k \leq r_2$ such that $k \notin f^o_M(v)$.
Thus there does not exist a vertex $x \in Z(G_2)$ such that $d(v,x) = k$. Correspondingly, $k + r_1 \notin f^o_M(v,w)$ for $(v,w) \in V(G_1 \times G_2)$. A contradiction that H is an odpu graph. Hence, one of G_1 and G_2 is self-centered and the other is an odpu graph.

Conversely, assume that one of G_1 and G_2 (say G_1) is self-centered and the other is an odpu graph. Since G_1 is self-centered, $Z(G_1 \times G_2) = \{(u,v) : v \in Z(G_2)\}$. Let $(u,v) \in V(H)$. Since G_1 is self-centered with radius r_1 and G_2 is an odpu-graph with radius r_2, $f^o_M(u) = \{1,2,\ldots,r_1\}$ and $f^o_M(v) = \{1,2,\ldots,r_2\}$. That is, there exist vertices $x_i \in Z(G_1)$ and $y_j \in Z(G_2)$ such that $d_{G_1}(u,x_i) = i;\ 1 \leq i \leq r_1$ and $d_{G_2}(v,y_j) = j;\ 1 \leq j \leq r_2$. Therefore, $d(u,x_i) + d(v,y_j) = d((u,v),(x_i,y_j)) = i + j,\ \forall\ 2 \leq i + j \leq r_1 + r_2 = r$. Also, $d(u,v),(u,y_1) = 1$, $1 \in f^o_M(u,v)$. Hence, $f^o_M(u,v) = \{1,2,\ldots,r\}$. Since (u,v) is arbitrary, $G_1 \times G_2$ is an odpu-graph. Hence the theorem.

Theorem 2.11. *Lexicographic product $H = G_1[G_2]$ of two graphs G_1 and G_2 is an odpu-graph if and only if either*
(i) G_1 is an odpu-graph. Or
(ii) $G_1 = K_1$ and G_2 is an odpu-graph. Or
(iii) $G_1 \neq K_1$ has exactly one universal vertex and G_2 does not have exactly one universal vertex.

Proof. First assume that lexicographic product $H = G_1[G_2]$ is an odpu-graph. Suppose G_1 and G_2 does not satisfy any of the three conditions. Then the possible cases are discussed as below.

Case:(i) $G_1 = K_1$ and G_2 is not an odpu-graph.

Then, $H = G_1[G_2] = G_2$ is not an odpu-graph.

Case:(ii) $G_1 \neq K_1$, G_1 is not an odpu-graph and G_1 has no universal vertex.
That is, $G_1 \neq K_1$, G_1 is not an odpu-graph with $r(G_1) \geq 2$. Then H has the same radius and diameter as in G_1 and hence $f^o_M(u) = f^o_M(u, x)$, $\forall\ u \in V(G_1)$, $x \in V(G_2)$. Since G_1 is not an odpu-graph, there exist two vertices u and v such that $f^o_M(u) \neq f^o_M(v)$. Correspondingly $f^o_M(u, x) \neq f^o_M(v, y)$, $x, y \in V(G_2)$. Hence, $H = G_1[G_2]$ is not an odpu-graph.

Case:(iii) $G_1 \neq K_1$ and G_1 and G_2 has exactly one universal vertices each (say u and v respectively).
Then $Z(G_1[G_2])$ has exactly one vertex (u, v) and hence H is not an odpu-graph.
Hence, all the above cases we arrived a contradiction that $H = G_1[G_2]$ is not an odpu-graph.

Conversely, assume that G_1 and G_2 satisfy any one of the given three conditions.
Case:(i) G_1 is an odpu-graph.
Then $f^o_M(u) = f^o_M(v)$, $\forall\ u, v \in V(G_1)$. Hence, $f^o_M(u, x) = f^o_M(v, y)$ $\forall\ x, y \in V(G_2)$ and $(u, x), (v, y) \in V(H)$. Hence $H = G_1[G_2]$ is an odpu-graph. Note that, in the case of G_1 have two or more universal vertices make G_1, an odpu graph. Hence this case is already discussed here.

Case:(ii) $G_1 = K_1$ and G_2 is an odpu-graph.
Then, $H = G_1[G_2] = G_2$ is an odpu-graph.

Case:(iii) $G_1 \neq K_1$, G_1 has a universal vertex u and G_2 does not have exactly one universal vertex.

Subcase:(i) $G_1 \neq K_1$, G_1 has a universal vertex u and G_2 has more than one universal vertex (say x and y).
Then $H = G_1[G_2]$ has at least two universal vertices (u, x) and (v, y) and hence by proposition 1, H is an odpu-graph.

Subcase:(ii) $G_1 \neq K_1$, G_1 has a universal vertex u and none of the vertices of G_2 are universal vertices.
Then $\forall\ x \in V(G_2)$, there exists a vertex $y \in V(G_2)$ such that $x, y \notin E(G_2)$. Hence, $d((u, x), (u, y)) = 2$ and $d((u, x), (w, y)) = 1$ $\forall\ u \neq w$ consequently $e_H(u, x) = 2$. Also, $\forall\, w, v \neq u$, $d((w, x), (v, y)) = 2$, since, $((w, x), (u, x), (v, y))$ is a path of length 2 in H. Hence, $e_H(w, x) = 2$. Thus, H is self-centered and

hence it is an odpu-graph.

Lemma 2.12. *If G is an odpu-graph then, G has no cut vertices.*

Proof. Suppose, G has a cut vertex u, then the graph G has at least two blocks (say B_1 and B_2) such that $V(B_1) \cap V(B_2) = \{u\}$ and $E(B_1) \cap E(B_2) = \phi$. Since, the center of a graph lies in a block, with out loss of generality, assume that the center $Z(G)$ lies in the block B_1. Let $v \in V_2$ such that $uv \in E(G)$. If G is an odpu-graph then, there exists an odpu-set $M \subseteq Z(G)$ and $f_M^o(u) = \{1, 2, \ldots, r\}$ $\forall$ $u \in V(G)$. Then there exists a vertex $w \in M$ such that $d(u, w) = r$, and $d(v, w) = r + 1$. Then $r + 1 \in f_M^o(u)$, which is a contradiction. Hence an odpu-graph G cannot have cut vertices.

The *Corona* $G \circ H$ was defined by Frucht and Harary[5] as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertices in the i^{th} copy of G_2.

Theorem 2.13. *Corona $G \circ H$ is an odpu- graph if and only if the graph $G \approx K_1$ and the graph H has at least one universal vertex.*

Proof. Assume that $G \circ H$ is an odpu-graph. We shall prove, $G_1 \approx K_1$ and G_2 has at least one universal vertex. If not, there are two possibilities.

Case:(i) G is not isomorphic to K_1.
Then, G has at least two vertices. Let the vertices of G be $v_1, v_2, \ldots, v_{p_1}$. Then $G \circ H - \{v_1, v_2, \ldots, v_{p_1}\}$ is p_1 disconnected copies of H. Hence, each vertices of G in $G \circ H$ are cut vertices, a contradiction to Lemma 2.12.

Case:(ii) $G = K_1$, but H has no universal vertices.
Then $G \circ H$ has exactly one universal vertex and hence it is not an odpu-graph, a contradiction to the assumption.
Conversely, assume that $G = K_1$ and H has at least one universal vertex. Then the corona $G \circ H$ has at least two universal vertices and hence by Proposition 1, it is an odpu-graph with $od(G \circ H) = 2$.

Corollary 2.14. *If corona $G \circ H$ is an odpu-graph then, $od(G \circ H) = 2$.*

The next Corollary gives a necessary condition for a graph to be an odpu-graph.

Corollary 2.15. *A graph G is odpu then, it cannot be represented as corona of two non-trivial graphs G and H.*

Corollary 2.16. *Any odpu-graph H with $od(H) = 2$ can be represented as corona $H = K_1 \circ G$, where $K_1 = \{v\}$ and $G = \langle H - v \rangle$.*

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