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The Relationship between Weak almost Dunford-Pettis and Dunford-Pettis Operators

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Abstract

We give some necessary and sufficient conditions for which the class of weak almost Dunford-Pettis operators coincide with that of Dunford-Pettis. Next, we characterize Banach space X and Banach lattice F' topological dual of Banach lattice F for which each weak almost Dunford-Pettis operator $T : X \to F'$ is Dunford-Pettis, and we derive some consequences.

Keywords: *Weak almost Dunford-Pettis operator, Dunford-Pettis operator, Schur property, KB-space, AM-compactness property, Order continuous norm.*

1 Introduction and Notation

An operator T from a Banach space X into another Y is called Dunford-Pettis if $\|T(x_n)\| \to 0$ for every weakly null sequence (x_n) in E [1]. A norm bounded subset A of a Banach lattice E is said to be almost Dunford-Pettis set , if every disjoint weakly null sequence (f_n) in E' converges uniformly to zero on A, that is, $\lim_{n\to\infty} \sup_{x\in A} f_n(x) = 0$. Recall from [5] an operator $T : X \to F$ from a Banach space X into a Banach lattice F is called weak almost Dunford-Pettis if T carries each relatively weakly compact set in X to an almost Dunford-Pettis set in F, equivalently, whenever $f_n(T(x_n)) \to 0$ for every weakly null

sequence (x_n) in X and every disjoint weakly null sequence (f_n) in F'.
A Banach space X has

- the Schur property, if $\|x_n\| \to 0$ for every weakly null sequence $(x_n) \subset E$.
- the Dunford-Pettis property (DP property for short), if $x_n \xrightarrow{w} 0$ in X and $f_n \xrightarrow{w} 0$ in X' imply $f_n(x_n) \to 0$.

A Banach lattice E has the weak Dunford-Pettis property (wDP property for short), if every relatively weakly compact set in E is almost Dunford-Pettis, equivalently, whenever $f_n(x_n) \to 0$ for every weakly null sequence (x_n) in E and for every disjoint weakly null sequence (f_n) in E' (see Corollary 2.6 of [5]).

A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent [1]. For exemple, each reflexive Banach lattice is a KB-space, but ℓ^∞ is not a KB-space.

It is clair that each KB-space has an order continuous norm, but a Banach lattice with order continuous norm is not necessary a KB-space. In fact, the Banach lattice c_0 has an order continuous norm but it is not a KB-space. However, for each Banach lattice E, its topological dual E' is a KB-space if and only if its norm is order continuous (see Theorem 4.59 of [1]).

It follows from Proposition 3.1 of [3] that a Banach lattice E has the AM-compactness property if and only if for every weakly null sequence (f_n) of E', we have $|f_n| \xrightarrow{w^\star} 0$. For exemple, the Banach lattice ℓ^1 has the AM-compactness property, but ℓ^∞ does not have this property.

A linear mapping T from a vector lattice E into a vector lattice F is called a lattice homomorphism, if $x \wedge y = 0$ in E implies $T(x) \wedge T(y) = 0$ in F. An operator $T : E \to F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. If $T : E \to F$ is a positive operator between two Banach lattices, then its adjoint $T' : F' \to E'$, defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$, is also positive. For the theory of Banach lattices and positive operators, we refer the reader to monographs [1, 7].

Note that every Dunford-Pettis operator $T : X \to F$ is weak almost Dunford-Pettis, but the converse is not always true. In fact, the identity operator of the Banach lattice ℓ^∞ is weak almost Dunford-Pettis (because ℓ^∞ has the weak Dunford-Property) but it is not Dunford-Pettis (because ℓ^∞ does not have the Schur property).

In this paper, we establish a necessary and sufficient conditions for which each weak almost Dunford-Pettis operator is Dunford-Pettis (Theorem 2.2, Theorem 2.5 and Theorem 2.8). Also, we deduce that if X be a Banach space and F be a Banach lattice such that F has the AM-compactness property, then each weak almost Dunford-Pettis operator $T : X \to F'$ is Dunford-Pettis if and only if X has the Schur property or F' is a KB-space (Corollary 2.9). As consequences, we derive some interesting results (Corollaries 2.3, 2.4, 2.6, 2.10 and 2.11).

2 Main Results

The proof of the next Theorem is based on the following Proposition.

Proposition 2.1 *Let X be a Banach space and F be a Banach lattice. Then, each operator $T : X \to F$ that admits a factorization through the Banach lattice ℓ^∞, is weak almost Dunford-Pettis.*

Let $P : X \to \ell^\infty$ and $Q : \ell^\infty \to F$ be two operators such that $T = Q \circ P$. Let (x_n) be a weakly null sequence in X and let (f_n) be a disjoint weakly null sequence in F'. It is clear that $P(x_n) \xrightarrow{w} 0$ in ℓ^∞ and $Q'(f_n) \xrightarrow{w} 0$ in $(\ell^\infty)'$. Since ℓ^∞ has the Dunford-Pettis property, then

$$f_n(Tx_n) = f_n(Q \circ P(x_n)) = (Q'f_n)(P(x_n)) \to 0.$$

This prove that, T is weak almost Dunford-Pettis.

The following Theorem gives some necessary conditions of a Banach lattices E and F under which each positive weak almost Dunford-Pettis operator from E into F is Dunford-Pettis.

Theorem 2.2 *Let E and F be two Banach lattices such that F is Dedekind σ-complete. If each positive weak almost Dunford-Pettis operator $T : E \to F$ is Dunford-Pettis then one of the following assertions is valid:*

1. *E has the Schur property.*

2. *The norm of F is order continuous.*

Assume by way of contradiction that E does not have the Schur property and F does not have the order continuous norm. We have to construct a positive weak almost Dunford-Pettis operator which is not Dunford-Pettis. As E does not have the Schur property, then there exists a weakly null sequence (x_n) in E which is not norm convergent to 0. As $\|x_n\| = \sup\{|f(x_n)| : f \in (E')^+, \|f\| = 1\}$, there exist a sequence (f_n) in $(E')^+$ with $\|f_n\| = 1$, some $\epsilon > 0$ and a subsequence (y_n) of (x_n) such that $|f_n(y_n)| \geq \epsilon$ for all n.

Now, consider the operator $P : E \to \ell^\infty$ defined by

$$P(x) = (f_k(x))_{k=1}^\infty$$

Clearly, that P is positive. Also, since the norm of the Dedekind σ-complete Banach lattice F is not order continuous, it follows from Theorem 4.51 of [1] that ℓ^∞ is lattice embeddable in F. Let $Q : \ell^\infty \to F$ be a lattice embedding, then there exists $m > 0$ and $M > 0$ such that

$$m.\,\|((\lambda)_{k=1}^\infty)\|_\infty \leq \|Q((\lambda)_{k=1}^\infty)\| \leq M.\,\|((\lambda)_{k=1}^\infty)\|_\infty$$

for all $((\lambda)_{k=1}^{\infty}) \in \ell^{\infty}$.

Let $T = Q \circ P : E \to \ell^{\infty} \to F$. It follows From Proposition 2.1 that T be a positive weak almost Dunford-Pettis but is not Dunford-Pettis. In fact, note that (y_n) is a weakly null sequence in E and for every n we have

$$\|T(y_n)\| = \|Q((f_k(y_n))_{k=1}^{\infty})\|_{\infty} \geq m.\,\|(f_k(y_n))_{k=1}^{\infty}\|_{\infty} \geq m.\,|f_n(y_n)| \geq m.\epsilon$$

This show that T is not Dunford-Pettis.

If we put $E = F$ in Theorem 2.2, we give a condition sufficient for which a Banach lattice Dedekind σ-complete E has a order continuous norm.

Corollary 2.3 *Let E a Banach lattice Dedekind σ-complete. If each positive weak almost Dunford-Pettis operator $T : E \to E$ is Dunford-Pettis then the norm of E is order continuous.*

As a consequence of Theorem 2.2, we obtain an operator characterization of the Schur property of a Banach lattice.

Corollary 2.4 *Let E be a Banach lattice. Then the following assertions are equivalent:*

1. *Every operator $T : E \to \ell^{\infty}$ is Dunford-Pettis.*
2. *Every positive operator $T : E \to \ell^{\infty}$ is Dunford-Pettis.*
3. *Every positive weak almost Dunford-Pettis operator $T : E \to \ell^{\infty}$ is Dunford-Pettis.*
4. *E has the Schur property.*

(1) $\Rightarrow$ (2) Obvious.
(2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (4) It follows from Theorem 2.5 by noting that ℓ^{∞} is Dedekind σ-complete and its norm is not order continuous.
(4) $\Rightarrow$ (1) Obvious.

By a similar proof as the previous Theorem, we obtain the following result.

Theorem 2.5 *Let X be a Banach space and F be a Dedekind σ-complete Banach lattice. If each weak almost Dunford-Pettis operator $T : X \to F$ is Dunford-Pettis then one of the following assertions is valid:*

1. *X has the Schur property.*
2. *The norm of F is order continuous.*

Remark 1 *The assumption "F Dedekind σ-complete" is essential in Theorem 2.2 (resp, Theorem 2.5). In fact, if we consider $E = \ell^\infty$ (resp, $X = \ell^\infty$) and $F = c$ the Banach lattice of all convergent sequences, it is clear that $F = c$ is not Dedekind σ-complete, and it follows from the proof of Proposition 1 of [10] and Theorem 5.99 of [1] that each operator from ℓ^∞ into c is Dunford-Pettis. But ℓ^∞ does not have the Schur property and the norm of c is not order continuous.*

Remark 2 *The second necessary condition of Theorem 2.2 (resp, Theorem 2.5) is not sufficient. In fact, the identity operator $I_{c_0} : c_0 \to c_0$ is positive weak almost Dunford-Pettis (resp, weak almost Dunford-Pettis) (because c_0 has the weak Dunford-Pettis property) but is not Dunford-Pettis (because c_0 does not have the Schur property). However the norm of c_0 is order continuous.*

As a consequence of Theorem 2.5, we obtain an operator characterization of the Schur property of a Banach space.

Corollary 2.6 *Let X be a Banach space. Then the following assertions are equivalent:*

1. *Every operator $T : X \to \ell^\infty$ is Dunford-Pettis.*
2. *Every weak almost Dunford-Pettis operator $T : X \to \ell^\infty$ is Dunford-Pettis.*
3. *X has the Schur property.*

(1) $\Rightarrow$ (2) Obvious.
(2) $\Rightarrow$ (3) It follows from Theorem 2.5 by noting that ℓ^∞ is Dedekind σ-complete and its norm is not order continuous .
(3) $\Rightarrow$ (1) Obvious.

For proof of the next Theorem, we need the following Lemma which is just Corollary 2.7 of Dodds and Fremlin in [4]

Lemma 2.7 *Let E be a Banach lattice and let (f_n) be a sequence of E'. Then the following assertions are equivalent:*

1. *$\|f_n\| \to 0$.*
2. *$|f_n| \overset{w^\star}{\to} 0$ and $f_n(x_n) \to 0$ for every norm bounded disjoint sequence (x_n) in E^+.*

Now, we give some sufficient conditions for which every weak almost Dunford-Pettis operator T from a Banach space X into a dual topological F' of a Banach lattice F is Dunford-Pettis.

Theorem 2.8 *Let X be a Banach space and F be a Banach lattice. Then every weak almost Dunford-Pettis operator $T : X \to F'$ is Dunford-Pettis if one of the following assertions is valid:*

1. *X has the Schur property.*
2. *F' has the Schur property.*
3. *F' is a KB-space and F has the AM-compactness property.*

(1) Obvious.
(2) Obvious.
(3) Let (x_n) be a sequence in X such that $x_n \xrightarrow{w} 0$ in X. We show that $\|T(x_n)\| \to 0$. By Lemma 2.7, it suffices to prove that $|T(x_n)| \xrightarrow{w^\star} 0$ in F' and $T(x_n)(y_n) \to 0$ for each norm bounded disjoint sequence (y_n) in F^+. It is clear that $(T(x_n))$ be a weakly null sequence in F', as F has the AM-compactness property then $|T(x_n)| \xrightarrow{w^\star} 0$ in F'.

On the other hand, let (y_n) be a norm bounded disjoint sequence in F^+, Since F' is a KB-space then its norm is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [4] that $(y_n) \xrightarrow{w} 0$ in F. Now, we have the canonical injection $\tau : F \to F''$ is a lattice homomorphism, we obtain that $\tau(y_n)$ is a disjoint weakly null sequence in F''. Finally, as T is weak almost Dunford-Pettis, then $\tau(y_n)(T(x_n)) \to 0$. Also by the equality

$$\tau(y_n)(T(x_n)) = T(x_n)(y_n)$$

for each n, we deduce that $T(x_n)(y_n) \to 0$, and this complete the proof. Our major result is given by the following characterization.

Corollary 2.9 *Let X be a Banach space and F be a Banach lattice such that F has the AM-compactness property. Then the following assertions are equivalent:*
(1) *Every weak almost Dunford-Pettis operator $T : X \to F'$ is Dunford-Pettis,*
(2) *One of the following assertions is valid:*
(*a*) *X has the Schur property,*
(*b*) *F' is a KB-space.*

(1) $\Rightarrow$ (2) Immediately from Theorem 2.5 by noting that if the norm of F' is order continuous then F' is a KB-space (see Theorem 2.4.14 of [7]).
(2) $\Rightarrow$ (1) It follows from Theorem 2.8.

Remark 3 *The assertion "Each weak almost Dunford-Pettis operator $T : X \to F$ is Dunford-Pettis" is not equivalent to the assertion "X has the Schur property or F is a KB-space". In fact, if we put $X = \ell^\infty$ and $F = c_0$, then by Proposition 2.1 every operator from ℓ^∞ into c_0 is weak almost Dunford-Pettis. But ℓ^∞ does not have the Schur property and c_0 is not a KB-space.*

As a consequences of Theorem 2.2 and Corollary 2.9, we obtain

Corollary 2.10 *Let E and F be two Banach lattices such that F has the AM-compactness property. Then the following assertions are equivalent:*
(1) *Every positive weak almost Dunford-Pettis operator $T : E \to F'$ is Dunford-Pettis,*
(2) *One of the following assertions is valid:*
  (a) *E has the Schur property,*
  (b) *F' is a KB-space.*

Another consequence of Corollary 2.9 is the following result.

Corollary 2.11 *Let F be a Banach lattice such that F has the AM-compactness property.*
F' is a KB-space if and only if every weak almost Dunford-Pettis operator $T : \ell^\infty \to F'$ is Dunford-Pettis.

It follows from Corollary 2.9 by noting that ℓ^∞ does not have the Schur property.

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