Gen. Math. Notes, Vol. 16, No. 2, June, 2013, pp.83-92
ISSN 2219-7184; Copyright ©ICSRS Publication, 2013
www.i-csrs.org
Available free online at http://www.geman.in

On a Subclass of Harmonic Univalent Functions Based on a Generalized Operator

Andreea-Elena Tudor

Department of Mathematics, "Babeş-Bolyai" University
1 Kogălniceanu Street, 400084 Cluj-Napoca, Romania
E-mail: tudor_andreea_elena@yahoo.com

(Received: 11-3-13 / Accepted: 22-4-13)

Abstract

In this paper, using the operator $\mathcal{L}(n, l, m, \alpha)$ studied in [7], we introduce a subclass of harmonic univalent and sense preserving functions for which we obtain coefficient conditions, extreme points, distortion bounds and inclusion results.

Keywords: *Harmonic univalent functions, derivative operator, distorsion bounds, convolution.*

1 Introduction

We denote by $\mathcal{S}_{\mathcal{H}}$ the family of functions $f = h + \overline{g}$ where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \; |b_1| < 1, \tag{1}$$

which are harmonic, univalent and sence preserving in the open unit disk, so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Then

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad |b_1| < 1. \tag{2}$$

We note that the family $\mathcal{S}_{\mathcal{H}}$ reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero

($g \equiv 0$). Silverman [6] introduced the subclass of $\mathcal{S_H}$, denoted by $\mathcal{S}_{\overline{\mathcal{H}}}$, which contains functions of the form $f = h + \overline{g}$ where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \ \text{ and } \ g(z) = \sum_{k=1}^{\infty} |b_k| z^k, \ |b_1| < 1. \tag{3}$$

If $f = h + \overline{g}$,where h and g are of the form (3), and $F = H + \overline{G}$ where

$$H(z) = z - \sum_{k=2}^{\infty} |U_k| z^k \ \text{ and } \ G(z) = \sum_{k=1}^{\infty} |V_k| z^k,$$

then the convolution product of f and F is given by

$$f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_k||U_k| z^k + \sum_{k=1}^{\infty} |b_k||V_k| \overline{z}^k, \ |b_1| < 1.$$

In [7] was introduced the operator $\mathcal{L}(n, l, a, c, \alpha)$ for analytic functions defined by

$$\mathcal{L}(n, l, a, c, \alpha) f(z) = z + \sum_{k=2}^{\infty} \left[\alpha \left(\frac{l+k}{l+1} \right)^n + (1-\alpha) \frac{(a)_{k-1}}{(c)_{k-1}} \right] a_k z^k,$$

where $n, l, a \in \mathbb{N}$, $\alpha \in [0, 1)$, $c \neq 0, -1, -2, \ldots$ and $(x)_k$ the *Pochhammer symbol* given by

$$(x)_k := \begin{cases} 1, & k = 0 \\ x(x+1)(x+2)...(x+k-1), & k \in \mathbb{N}^* \end{cases}$$

For $c = 1$ and $a = m + 1$ we have

$$\mathcal{L}(n, l, m, \alpha) f(z) = z + \sum_{k=2}^{\infty} \left[\alpha \left(\frac{l+k}{l+1} \right)^n + (1-\alpha) C(m, k) \right] a_k z^k, \tag{4}$$

where $C(m, k) = \begin{pmatrix} m+k-1 \\ m \end{pmatrix}$.

Now, for $f = h + \overline{g}$ given by (2), we introduce the modified operator $\mathcal{L}(n, l, m, \alpha)$ of harmonic univalent function f as

$$\mathcal{L}(n, l, m, \alpha) f(z) = \mathcal{L}(n, l, m, \alpha) h(z) + \overline{\mathcal{L}(n, l, m, \alpha) g(z)}, \tag{5}$$

where

$$\mathcal{L}(n, l, m, \alpha) h(z) = z + \sum_{k=2}^{\infty} \left[\alpha \left(\frac{l+k}{l+1} \right)^n + (1-\alpha) C(m, k) \right] a_k z^k$$

and

$$\mathcal{L}(n,l,m,\alpha)g(z) = \sum_{k=1}^{\infty}\left[\alpha\left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(m,k)\right] b_k z^k, |b_1| < 1.$$

We denote by $\mathcal{HL}(n,l,m,\alpha,\gamma)$ the class of harmonic functions f of the form (2), such that

$$\mathrm{Re}\left[\frac{z\left(\mathcal{L}(n,l,m,\alpha)f(z)\right)'}{\mathcal{L}(n,l,m,\alpha)f(z)}\right] \geq \gamma, \quad 0 \leq \gamma < 1.$$

For $m = l$, we obtain the class $\mathcal{HL}(n,l,\alpha,\gamma)$

$$\mathrm{Re}\left[\frac{(l+1)\mathcal{L}(n+1,l,l+1,\alpha)f(z)}{\mathcal{L}(n,l,l,\alpha)f(z)} - l\right] \geq \gamma, \tag{6}$$

where $\mathcal{L}(n,l,m,\alpha)$ is defined by (5).

Also, we denote by $\overline{\mathcal{HL}}(n,l,\alpha,\gamma)$ the class of functions $f = h + \overline{g}$ in $\mathcal{HL}(n,l,\alpha,\gamma)$, where h and g are of the form (3).

We notice that the class $\mathcal{HL}(n,l,\alpha,\gamma)$ includes a variety of well-known subclasses of $\mathcal{S_H}$. For example, $\mathcal{HL}(0,0,1,\gamma)$ represent the class of sense-preserving, harmonic, univalent functions f which are starlike of order γ in U and $\mathcal{HL}(1,0,1,\gamma)$ represent the class of sense-preserving, harmonic, univalent functions f which are convex of order γ in U. These subclasees were introduced and studied by Jahangiri in [2]. Other subclasses studied are $\overline{\mathcal{HL}(n,0,1,\gamma)}$ which is the class of Salagean-type harmonic univalent functions studied by Jahangiri and al. in [3] and $\overline{\mathcal{HL}(n,l,0,\gamma)}$, the class of Ruscheweyh-type harmonic univalent functions studied by Murugusundaramoorthy and Vijaya in [5].

2 Main Results

First we determine a sufficient coefficient bound for functions in $\mathcal{HL}(n,l,\alpha,\gamma)$:

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (2). If*

$$\sum_{k=2}^{\infty}(k-\gamma)\left[\alpha\left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right](|a_k| + |b_k|) + |b_1| \leq 1-\gamma, \tag{7}$$

where $l,n \geq 0,\ a_1 = 1,\ \alpha,\gamma \in [0,1)$, then $f(z)$ is harmonic univalent, sense preserving in U and $f(z) \in \mathcal{HL}(n,l,\alpha,\gamma)$.

Proof.

If we take $|z_1| \le |z_2| < 1$ and consider the inequality (7), we have

$$\begin{aligned}
&|f(z_1) - f(z_2)| \ge |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
&\ge |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} k|a_k||z_2|^{k-1} - \sum_{k=1}^{\infty} k|b_k||z_2|^{k-1}\right) \\
&= |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} k(|a_k| + |b_k|)|z_2|^{k-1} - |b_1|\right) \\
&\ge |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} k(|a_k| + |b_k|) - |b_1|\right) \\
&\ge |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} \frac{(k-\gamma)}{1-\gamma} \left[\alpha \left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right] (|a_k| + |b_k|) - |b_1|\right) \\
&\ge |z_1 - z_2| \left[1 - \left(1 - \frac{|b_1|}{1-\gamma}\right) - |b_1|\right] = \frac{\gamma}{1-\gamma}|b_1||z_1 - z_2| \ge 0
\end{aligned}$$

Hence, $f(z)$ is univalent in U. $f(z)$ is sense preserving in U because

$$\begin{aligned}
&|h'(z)| \ge 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k| \\
&> 1 - \sum_{k=2}^{\infty} \frac{(k-\gamma)}{1-\gamma} \left[\alpha \left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right] |a_k| \\
&\ge \sum_{k=2}^{\infty} \frac{(k-\gamma)}{1-\gamma} \left[\alpha \left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right] |b_k| + \frac{|b_1|}{1-\gamma} \\
&\ge \sum_{k=2}^{\infty} \frac{(k-\gamma)}{1-\gamma} \left[\alpha \left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right] |b_k||z|^{k-1} + \frac{|b_1|}{1-\gamma} \\
&> \sum_{k=2}^{\infty} k|b_k||z|^{k-1} + |b_1| = \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \ge |g'(z)|
\end{aligned}$$

Finally, we show that $f(z) \in \mathcal{HL}(n, l, \alpha, \gamma)$. Using the fact that $\mathrm{Re}\,\omega \ge \gamma$ if and only if $|1 - \gamma + \omega| \ge |1 + \gamma - \omega|$, it suffices to show that

$$\begin{aligned}
&|(1-\gamma-l)\mathcal{L}(n,l,l,\alpha)f(z) + (l+1)\mathcal{L}(n+1,l,l+1,\alpha)f(z)| - \\
&- |(1+\gamma+l)\mathcal{L}(n,l,l,\alpha)f(z) - (l+1)\mathcal{L}(n+1,l,l+1,\alpha)f(z)| \ge 0.
\end{aligned}$$

So, if we set

$$A_k := \left[\alpha \left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right],$$

$$B_k := \left[\alpha\left(\frac{l+k}{l+1}\right)^{n+1} + (1-\alpha)C(l+1,k)\right] = \frac{l+k}{l+1}A_k,$$

we have

$$\begin{aligned}
&\Bigg|(1-\gamma-l)z + \sum_{k=2}^{\infty}(1-\gamma-l)A_k a_k z^k + \sum_{k=1}^{\infty}(1-\gamma-l)A_k\overline{b_k z^k} \\
&+(l+1)z + \sum_{k=2}^{\infty}(l+1)B_k a_k z^k + \sum_{k=1}^{\infty}(l+1)B_k\overline{b_k z^k}\Bigg| \\
&-\Bigg|(1+\gamma+l)z + \sum_{k=2}^{\infty}(1+\gamma+l)A_k a_k z^k + \sum_{k=1}^{\infty}(1+\gamma+l)A_k\overline{b_k z^k} \\
&-(l+1)z - \sum_{k=2}^{\infty}(l+1)B_k a_k z^k - \sum_{k=1}^{\infty}(l+1)B_k\overline{b_k z^k}\Bigg| \\
&\geq (2-\gamma)|z| - \sum_{k=2}^{\infty}(1-\gamma+k)A_k|a_k||z|^k - \sum_{k=1}^{\infty}(1-\gamma+k)A_k|b_k||z|^k \\
&-\gamma|z| - \sum_{k=2}^{\infty}|1+\gamma-k|A_k|a_k||z|^k - \sum_{k=1}^{\infty}|1+\gamma-k|A_k|b_k||z|^k \\
&= 2|z|\left\{(1-\gamma) - \sum_{k=2}^{\infty}(k-\gamma)\left[\alpha\left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right](|a_k|+|b_k|)|z|^{k-1} - |b_1|\right\} \\
&> 2|z|\left\{(1-\gamma) - \sum_{k=2}^{\infty}(k-\gamma)\left[\alpha\left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right](|a_k|+|b_k|) - |b_1|\right\}.
\end{aligned}$$

The last expression is non-negative by (7), and so the proof is complete.

If we take $n, l, \gamma = 0$ and $\alpha = 1$ in the previous theorem, we obtain the following theorem, proved by Jahangiry and Silverman in [4]

Corollary 2.2. *Let $f = h + \overline{g}$ given by (2). If*

$$\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq 1 - |b_1|,$$

then f is sense-preserving, harmonic univalent in U and $f \in \mathcal{S}_{\mathcal{H}}^*$ (the functions in $\mathcal{S}_{\mathcal{H}}$ which are starlike in U).

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty}\frac{1-\gamma}{(k-\gamma)A_k}x_k z^k + \sum_{k=1}^{\infty}\frac{2(1-\gamma)}{[(1-\gamma+k)+|1+\gamma-k|]A_k}\overline{y_k z^k}, \quad (8)$$

where

$$A_k = \left[\alpha\left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right],$$

and

$$\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1,$$

shows that the coefficient bound given by (7) is sharp.

The functions of the form (8) are in $\mathcal{HL}(n,l,\alpha,\gamma)$ because

$$\begin{aligned}&\sum_{k=2}^{\infty}(k-\gamma)A_k(|a_k|+|b_k|)+|b_1|\\ &=\sum_{k=2}^{\infty}(k-\gamma)A_k|a_k| + \sum_{k=1}^{\infty}\frac{(1-\gamma+k)+|1+\gamma-k|}{2}A_k|b_k|\\ &=(1-\gamma)\left(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\right) = 1-\gamma.\end{aligned}$$

In the next theorem we will prove the necessity of condition (7) for functions of the form $f = h + \overline{g}$, where h and g are of the form (3).

Theorem 2.3. *Let $f = h + \overline{g}$ be given by (3). Then $f \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty}(k-\gamma)\left[\alpha\left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right](|a_k|+|b_k|)+|b_1| \le 1-\gamma. \quad (9)$$

Proof.

Since $\overline{\mathcal{HL}}(n,l,\alpha,\gamma) \subset \mathcal{HL}(n,l,\alpha,\gamma)$, we only need to prove the 'only if' part of the theorem. So, for the function f of the form (3), the condition (6) is equivalent to

$$\mathrm{Re}\left\{\frac{(1-\gamma)z - \sum_{k=2}^{\infty}(k-\gamma)A_k|a_k|z^k}{z-\sum_{k=2}^{\infty}A_k|a_k|z^k + \sum_{k=1}^{\infty}A_k|b_k|\overline{z}^k} - \frac{\sum_{k=1}^{\infty}(k-\gamma)A_k|b_k|\overline{z}^k}{z-\sum_{k=2}^{\infty}A_k|a_k|z^k + \sum_{k=1}^{\infty}A_k|b_k|\overline{z}^k}\right\} \ge 0$$

The above condition must hold for all values of z, $|z| = r < 1$. Choosing the values of z on the positive real axis, where $0 \le z = r < 1$, we must have

$$\mathrm{Re}\left\{\frac{(1-\gamma) - \sum_{k=2}^{\infty}(k-\gamma)A_k|a_k|r^{k-1} - \sum_{k=1}^{\infty}(k-\gamma)A_k|b_k|r^{k-1}}{1-\sum_{k=2}^{\infty}A_k|a_k|r^{k-1} + \sum_{k=1}^{\infty}A_k|b_k|r^{k-1}}\right\} \ge 0. \quad (10)$$

If the condition (7) does not hold then the numerator in (10) is negative for r sufficiently close to 1. Hence, there exists a $z_0 = r_0$ in $(0,1)$ for which the quotient in (10) is negative. This contradicts the required condition $f \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma)$.

Theorem 2.4. *Let f be given by 3. Then $f \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_k(z)\right), \tag{11}$$

where

$$h_1(z) = z,\ h_k(z) = z - \frac{1-\gamma}{(k-\gamma)A_k} z^k,\ k \geq 2,$$

$$g_k(z) = z + \frac{2(1-\gamma)}{((1-\gamma+k)+|1+\gamma-k|)A_k} \overline{z}^k,\ k \geq 1,$$

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1,\ X_k \geq 0,\ Y_k \geq 0.$$

Proof.
For functions f of the form (11), we may write

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_k(z)\right) \\ &= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1-\gamma}{(k-\gamma)A_k} X_k z^k \\ &- \sum_{k=1}^{\infty} \frac{2(1-\gamma)}{((1-\gamma+k)+|1+\gamma-k|)A_k} Y_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(k-\gamma)}{1-\gamma} A_k |a_k| + \sum_{k=1}^{\infty} \frac{(1-\gamma+k)+|1+\gamma-k|}{2(1-\gamma)} A_k |b_k| \\ &= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1 \end{aligned}$$

so $f \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma)$.

Conversely, suppose that $f \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma)$. Setting

$$X_k = \frac{(k-\gamma)\left[\alpha\left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right]}{1-\gamma} |a_k| \qquad n \geq 2,$$

$$Y_k = \frac{((1-\gamma+k)+|1+\gamma-k|)\left[\alpha\left(\frac{l+k}{l+1}\right)^n + (1-\alpha)C(l,k)\right]}{2(1-\gamma)} |b_k| \quad n \geq 1,$$

where

$$\sum_{k=1}^{\infty}(X_k+Y_k)=1,$$

we obtain the required condition.

The following theorem gives the distortion bounds for functions in the class $\overline{\mathcal{HL}}(n,l,\alpha,\gamma)$.

Theorem 2.5. *Let $f\in\overline{\mathcal{HL}}(n,l,\alpha,\gamma)$. Then, for $|z|=r<1$, we have*

$$|f(z)|\le(1+|b_1|)r+\frac{1}{\alpha\left(\frac{l+2}{l+1}\right)^n+(1-\alpha)(l+1)}\left(\frac{1-\gamma}{2-\gamma}-\frac{1}{2-\gamma}|b_1|\right)r^2$$

and

$$|f(z)|\ge(1-|b_1|)r-\frac{1}{\alpha\left(\frac{l+2}{l+1}\right)^n+(1-\alpha)(l+1)}\left(\frac{1-\gamma}{2-\gamma}-\frac{1}{2-\gamma}|b_1|\right)r^2$$

Proof.

$$\begin{aligned}|f(z)| &=\left|z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=1}^{\infty}\overline{b}_k\overline{z}^k\right|=\left|z+b_1\overline{z}+\sum_{k=2}^{\infty}\left(a_kz^k+\overline{b}_k\overline{z}^k\right)\right|\\
&\le|1+b_1||z|+\sum_{k=2}^{\infty}|a_k+b_k||z|^k\le(1+|b_1|)r+\sum_{k=2}^{\infty}|a_k+b_k|r^k\\
&\le(1+|b_1|)r+\sum_{k=2}^{\infty}|a_k+b_k|r^2\\
&\le(1+|b_1|)r+\frac{1-\gamma-|b_1|}{(2-\gamma)A_k}\sum_{k=2}^{\infty}\frac{(2-\gamma)A_k}{1-\gamma-|b_1|}(|a_k|+|b_k|)r^2\\
&\le(1+|b_1|)r+\frac{1-\gamma-|b_1|}{(2-\gamma)A_k}\sum_{k=2}^{\infty}\frac{(k-\gamma)A_k}{1-\gamma-|b_1|}(|a_k|+|b_k|)r^2\\
&\le(1+|b_1|)r+\frac{1}{\alpha\left(\frac{l+2}{l+1}\right)^n+(1-\alpha)(l+1)}\left(\frac{1-\gamma}{2-\gamma}-\frac{1}{2-\gamma}|b_1|\right)r^2.\end{aligned}$$

Similarly we obtain the other inequality.

Theorem 2.6. *Let $f(z)\in\overline{\mathcal{HL}}(n,l,\alpha,\gamma)$ and $F(z)\in\overline{\mathcal{HL}}(n,l,\alpha,\delta)$, for $0\le\delta\le\gamma<1$. Then $f(z)*F(z)\in\overline{\mathcal{HL}}(n,l,\alpha,\gamma)\subset\overline{\mathcal{HL}}(n,l,\alpha,\delta)$.*

Proof. Suppose that $f(z) \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma)$ and $F(z) \in \overline{\mathcal{HL}}(n,l,\alpha,\delta)$ then, by Theorem 2.3, we have

$$\sum_{k=2}^{\infty} \frac{(k-\gamma)}{1-\gamma} A_k |a_k||U_k| + \sum_{k=1}^{\infty} \frac{((1-\gamma+k)+|1+\gamma-k|)}{2(1-\gamma)} A_k |b_k||V_k|$$
$$\leq \sum_{k=2}^{\infty} \frac{(k-\gamma)}{1-\gamma} A_k |a_k| + \sum_{k=1}^{\infty} \frac{((1-\gamma+k)+|1+\gamma-k|)}{2(1-\gamma)} B_k |b_k| \leq 1,$$

$$as \ |U_k| < 1, |V_k| < 1.$$

So $f(z) * F(z) \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma)$.
Let now $f(z) \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma)$. We will show that $f(z) \in \overline{\mathcal{HL}}(n,l,\alpha,\delta)$.

$$\sum_{k=2}^{\infty} \frac{(k-\delta)}{1-\delta} A_k |a_k| + \sum_{k=1}^{\infty} \frac{((1-\delta+k)+|1+\delta-k|)}{2(1-\delta)} A_k |b_k|$$
$$\leq \sum_{k=2}^{\infty} \frac{(k-\gamma)}{1-\gamma} A_k |a_k| + \sum_{k=1}^{\infty} \frac{((1-\gamma+k)+|1+\gamma-k|)}{2(1-\gamma)} B_k |b_k| \leq 1,$$

$$for \ 0 \leq \delta \leq \gamma < 1.$$

Therefore $f(z) * F(z) \in \overline{\mathcal{HL}}(n,l,\alpha,\gamma) \subset \overline{\mathcal{HL}}(n,l,\alpha,\delta)$.

3 Acknowledgements

This work was possible with the financial support of the Sectoral Operation Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the project number POSDRU/107/1.5/S/76841 with the title "Modern Doctoral Studies: Internationalization and Interdisciplinarity".

References

[1] N.E. Cho and H.M. Srivastava, Argument estimates for certain analytic functions defined by a class of multiplier transformation, *Math. Comput Modelling*, 37(1-2) (2003), 39-49.

[2] J.M. Jahangiri, Harmonic functions starlike in the unit disc, *J. Math. Anal. Appl.*, 235(1999), 470-477.

[3] J.M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, Sălăgean-type harmonic univalent functions, *South. J. Pure and Appl. Math.*, 2(2) (2002), 77-82.

[4] J.M. Jahangiri and H. Silverman, Harmonic univalent functions with varying arguments, *Int. J. of Appl. Math.*, 8(3) (2002), 267-275.

[5] G. Murugusundaramoorthy and K. Vijaya, On certain classes of harmonic univalent functions involving Ruscheweyh derivatives, *Bulletin of the Calcutta Mathematical Society*, 96(2) (2004), 99-108.

[6] H. Silverman, Harmonic univalent functions with negative coefficients, *J. Math. Anal. Appl.*, 220(1) (1998), 283-289.

[7] A.E. Tudor, A subclass of analytic functions, *Stud. Univ. Babes-Bolyai Math.*, 57(2) (2012), 277-282.