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Some Properties of *N*-Quasinormal Operators

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Abstract

In this article we will give some properties of N-quasinormal operators in Hilbert spaces. The object of this paper is to study conditions on T which imply N-quasinormality. If T_1 and T_2 are N-quasinormal operators, we shall obtain conditions under which their sum is N-quasinormal and if T_1 is N-quasinormal operator and T_2 quasinormal operators, we shall obtain conditions under which their product is N-quasinormal.

Keywords: *Quasinormal operators, N-quasinormal operators.*

1 Introduction

Let us denote by H the complex Hilbert space and with $B(H)$ the space of all bounded linear operators defined in Hilbert space H. Let T be an operator in $B(H)$. The operator T is called quasinormal if: $T(T^*T) = (T^*T)T$. The operator T is called N- quasinormal operator, if $T(T^*T) = N((T^*T)T)$.

Let $T \in B(H)$, $T = U + iV$ where $U = \operatorname{Re} T = \frac{T + T^*}{2}$ and $V = \operatorname{Im} T = \frac{T - T^*}{2i}$ are the real and imaginary parts of T. We shall write $B^2 = TT^*$ and $C^2 = T^*T$ where B and C are non-negative definite.

In this paper we will study some properties of N- quasinormal operators. Exactly we will give conditions under which an operator T is N –quasinormal. Also, we shall that if T_1 and T_2 are N-quasinormal operators, we shall obtain conditions under which their sum is N-quasinormal and if T_1 is N-quasinormal operator and T_2 quasinormal operators, we shall obtain conditions under which their product is N-quasinormal.

2 *N*- Quasinormal Operators

In this section we will show some properties of N-quasinormal operators in Hilbert space.

Theorem 2.1: *If T is an operator such that*

(i) *B commutes with U and V*
(ii) $TB^2 = N\left(C^2 T\right)$.

Then T is N –quasinormal operator.

Proof: Since $BU = UB$, $BV = VB$ we have $B^2U = UB^2$, $B^2V = VB^2$ then

$$B^2T + B^2T^* = TB^2 + T^*B^2$$
$$B^2T - B^2T^* = TB^2 - T^*B^2$$

This gives $B^2T = TB^2 = N\left(C^2T\right) \Rightarrow TT^*T = N\left(T^*TT\right)$.

Hence T is N –quasinormal operator.

Theorem 2.2: *Let T be N –quasinormal operator and* $TB^2 = N\left(C^2T\right)$ *. Then B commutes with U and V.*

Proof: Since $TB^2 = N\left(C^2T\right)$ we have $T\left(TT^*\right) = N\left(\left(T^*T\right)T\right)$.

Hence $\left(TT^*\right)T^* = N\left(T^*\left(T^*T\right)\right)$.

Since T is N –quasinormal operator we have

$$B^2U = TT^*\frac{T+T^*}{2} = \frac{TT^*T+TT^*T^*}{2} =$$
$$\frac{N\left((T^*T)T\right)+N\left(T^*(T^*T)\right)}{2} =$$
$$\frac{N\left((T^*T)T+T^*(T^*T)\right)}{2} =$$
$$\frac{N\left(\frac{1}{N}\left(T\left(TT^*\right)\right)+\frac{1}{N}\left((T^*T)T^*\right)\right)}{2} =$$
$$\frac{T^2T^*+T^*TT^*}{2} = \frac{T+T^*}{2}TT^* = UB^2,$$

Since B is non-negative definite, it follows that $BU = UB$. Similarly $BV = VB$.

Theorem 2.3: *If T is an operator such that* $C^2U = \frac{1}{N}UC^2$, $C^2V = \frac{1}{N}VC^2$. *Then T is N –quasinormal operator.*

Proof: Since $C^2U = \frac{1}{N}UC^2$, $C^2V = \frac{1}{N}VC^2$ then we have

$C^2\left(U+iV\right) = \frac{1}{N}\left(U+iV\right)C^2$ and we have $C^2T = \frac{1}{N}TC^2$ therefore

$\left(T^*T\right)T = \frac{1}{N}T\left(T^*T\right) \Rightarrow T\left(T^*T\right) = N\left(T^*T\right)T$.

Theorem 2.4: *Let T be N –quasinormal operato and* $B^2T = \frac{1}{N}\left(C^2T\right)$ *. Then:*

(i) $C^2U = \frac{1}{N}UC^2$,

(ii) $C^2V = \frac{1}{N}VC^2$.

Proof: (i) Since

$$B^2T = \frac{1}{N}\left(C^2T\right) \Rightarrow \left(TT^*\right)T = \frac{1}{N}\left(\left(T^*T\right)T\right) \Rightarrow T^*\left(TT^*\right) = \frac{1}{N}\left(T^*\left(T^*T\right)\right).$$

Since T is N –quasinormal operator we have

$$C^2U = T^*T\cdot\left(\frac{T+T^*}{2}\right) = \frac{T^*T^2 + T^*TT^*}{2} =$$

$$= \frac{\frac{1}{N}TT^*T + \frac{1}{N}T^{*2}T}{2} = \frac{1}{N}\left(\frac{T+T^*}{2}\right)\cdot T^*T = \frac{1}{N}UC^2.$$

(ii) Similary

$$C^2V = \frac{1}{N}VC^2$$

Theorem 2.5: *Let T_1 and T_2 be two N–quasinormal operators such that* $T_1T_2 = T_2T_1 = T_1^*T_2 = T_2^*T_1 = 0$ *. Then their sum* $T_1 + T_2$ *is N–quasinormal operator.*

Proof:

$$(T_1+T_2)\left[(T_1+T_2)^*(T_1+T_2)\right] =$$
$$(T_1+T_2)\left[(T_1^*+T_2^*)(T_1+T_2)\right] =$$
$$(T_1+T_2)(T_1^*T_1 + T_1^*T_2 + T_2^*T_1 + T_2^*T_2) =$$
$$(T_1+T_2)(T_1^*T_1 + T_2^*T_2)$$
$$T_1T_1^*T_1 + T_1T_2^*T_2 + T_2T_1^*T_1 + T_2T_2^*T_2 =$$
$$T_1T_1^*T_1 + T_2T_2^*T_2 =$$
$$N\left((T_1^*T_1)T_1\right) + N\left((T_2^*T_2)T_2\right) =$$
$$N\left((T_1^*T_1)T_1 + (T_2^*T_2)T_2\right) =$$
$$N\left((T_1+T_2)^*(T_1+T_2)^2\right)$$

Hence $T_1 + T_2$ is N –quasinormal operator.

Theorem 2.6: *Let T_1 be N –quasinormal operator and T_2 quasinormal operator. Then their product T_1T_2 is N–quasinormal operator if the following conditions are satisfied*

(i) $T_1T_2 = T_2T_1$

(ii) $T_1T_2^* = T_2^*T_1$

Proof:

$$\left(T_1T_2\right)\left(T_1T_2\right)*\left(T_1T_2\right)=$$
$$\left(T_1T_2\right)\left(T_2*T_1*\right)\left(T_1T_2\right)=$$
$$\left(T_1T_2\right)\left(T_1*T_2*\right)\left(T_1T_2\right)=$$
$$T_1\left(T_2T_1*\right)\left(T_2*T_1\right)T_2=$$
$$T_1\left(T_1*T_2\right)\left(T_1T_2*\right)T_2=$$
$$T_1T_1*\left(T_2T_1\right)\left(T_2*T_2\right)=$$
$$T_1T_1*\left(T_1T_2\right)\left(T_2*T_2\right)=$$
$$N\left(T_1*T_1^2\right)\left(T_2*T_2^2\right)=$$
$$NT_1*\left(T_1^2T_2*\right)T_2^2=$$
$$N\left(T_1*T_2*\right)\left(T_1^2T_2^2\right)=$$
$$N\left(T_1*T_2*\right)\left(T_1T_2\right)^2=$$
$$N\left(T_2*T_1*\right)\left(T_1T_2\right)^2=$$
$$N\left(T_1T_2\right)*\left(T_1T_2\right)^2$$

Hence T_1T_2 is N –quasinormal operator.

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