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# **An Algorithm for Solving Bi-Level Integer Linear Fractional Programming Problem based on Fuzzy Approach**

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## **Abstract**

*This paper presents a fuzzy approach for solving the bi-level integer linear fractional programming problem (BILFPP). At the first phase of the solution algorithm and to avoid the complexity of non convexity of this problem, we begin by finding the convex hull of its original set of constraints using the cutting-plane algorithm, and then the Charnes & Cooper transformation is used to convert the BILFPP to an equivalent bi-level linear programming problem (BLPP). At the second phase, a membership function is constructed to develop a fuzzy model for obtaining the optimal solution of the BLPP. Finally, an illustrative numerical example is provided to clarify the proposed approach.*

**Keywords:** *Bi-level programming; integer programming; fractional programming; Fuzzy programming.*

## **1 Introduction**

The multi-level programming problem is a hierarchical optimization problem where a subset of variables is constrained to be a solution of a given optimization problem

parameterized by the remaining variables. For more comprehensive survey, see [1,12 and13].

Bi-level programming (BLP) is a subset of the multi-level programming problem which identified as a mathematical programming problem that solves decentralized planning problems with two decision makers (DMs) in a two- level or hierarchical organization.

The basic concept of the BLP technique is that a first- level decision maker (FLDM) - the leader- sets his goals and/or decisions and then asks each subordinate level of the organization for their optima which are calculated in isolation; the second-level decision maker (SLDM) -the follower- decisions are then submitted and modified by the FLDM with consideration of the over all benefit for the organization; and the process continued until an optimal solution is reached. In other words, although the FLDM independently optimizes its own benefits, the decision may be affected by the reaction of the SLDM. For more details concerning theory and methodology for BLP, see [3, 4, 5,6,7,8,9,16 and 18].

The hierarchical optimization structure appears naturally in many applications when lower level actions depend on upper level decisions. The applications of bi-level and multi-level programming include transportation (taxation, network design, trip demand estimation), management (coordination of multi-divisional firms, network facility location, credit allocation), planning (agricultural policies, electric utility), and optimal design [14].

An algorithm for the integer linear fractional bi-level programming problem has been proposed by Thirwani and Arora in [17]. They examined the case when the objective functions were linear fractional and presented an algorithm for solving the integer case.

Recently, notable studies have been made of bi-level integer nonlinear programming problem (BLINLP). The bi-level integer nonlinear programming problem was introduced by Emam [11]. A fuzzy approach has been suggested and was based mainly on the concept of tolerance membership function together with the branch and bound technique to develop a fuzzy Max-Min decision model for generating Pareto optimal solution for the problem (BLINLP).

The purpose of this paper is to introduce the bi-level linear programming problem with fractional objective functions and suggest an algorithm to solve the problem when the variables are integers. A membership function is constructed to develop a fuzzy model for obtaining the optimal integer solution of the problem under consideration.

It is realized that the proposed fuzzy approach in this paper is more efficient and realistic rather than the method proposed in [17], where the results reflect that the obtained solution is most satisfactory to the decision makers on the bases of their need and desire in the decision environment. The main advantage of the proposed fuzzy approach is that the possibility of rejecting the solution again and again by the FLDM and re-evaluation of the problem repeatedly, by redefining the elicited membership functions, needed to reach to the satisfactory decision does not arise.

This paper is organized as follows:

The bi-level integer linear fractional programming problem (BILFP) is formulated in Section 2 with the solution concept of the problem of concern. A fuzzy approach for solving the integer linear fractional programming problem is described in Section 3 for the first and the second decision-making levels. A negotiation between the two level decision makers is investigated in Section 4. Section 5 contains the steps of the solution algorithm to solve (BILFP). Section 6 provides a numerical example to illustrate the developed algorithm. At the end of the paper in Section 7, a conclusion is given with some open points for future research work in the area of bi-level and multi-level integer linear and nonlinear fractional optimization problems.

## 2 Problem Formulation and the Solution Concept

The bi-level integer linear fractional programming problem (BILFP) may be formulated as follows:

$$(FLDM) \quad \max_{x_1} F_1(x) = \frac{c_1^T x + \alpha_1}{d_1^T x + \beta_1} \quad (1)$$

where  $x_2$  solves

$$(SLDM) \quad \max_{x_2} F_2(x) = \frac{c_2^T x + \alpha_2}{d_2^T x + \beta_2} \quad (2)$$

Subject to

$$M = \{x / Ax \leq b, x \geq 0 \text{ and integer}\} \quad (3)$$

where  $F_1, F_2$  is the objective function the first-level decision maker (FLDM) and second-level decision maker (SLDM), respectively.  $M$  is the set of linear constraints where  $c_1, c_2, d_1$  and  $d_2$  are  $n$ -vectors, and  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  are real numbers,  $A$  is  $(m \times n)$  real matrix and  $b$  is  $m$ -real vector, the vector of decision variables  $x = (x_1, x_2) \in R^n$  is partitioned between the two planners. The first-level decision maker has control over the vector  $x_1 \in R^{n_1}$  and the second-level decision maker has control over the vector  $x_2 \in R^{n_2}$ , where  $n = n_1 + n_2$ .

In what follows, an equivalent bi-level linear fractional programming problem (BLFP) associated with problem (1)-(3) can be stated with the help of cutting-plane technique [15] together with Balinski algorithm [2]. This equivalent BLFP can be written in the following form:

$$(FLDM) \quad \max_{x_1} F_1(x) = \frac{c_1^T x + \alpha_1}{d_1^T x + \beta_1} \quad (4)$$

where  $x_2$  solves

$$(SLDM) \quad \max_{x_2} F_2(x) = \frac{c_2^T x + \alpha_2}{d_2^T x + \beta_2} \quad (5)$$

Subject to

$$x \in [M], \quad (6)$$

where  $[M]$  is the convex hull of the feasible region  $M$  defined by (3) earlier. This convex hull is defined by:

$$[M] = M_R^{(s)} = \{x \in R^n \mid A^{(s)}x \leq b^{(s)}, x \geq 0\} \quad (7)$$

and in addition,

$$A^{(s)} = \begin{bmatrix} A \\ \dots \\ a_1 \\ \cdot \\ \cdot \\ a_s \end{bmatrix} \quad \text{and} \quad b^{(s)} = \begin{bmatrix} b \\ \dots \\ b_1 \\ \cdot \\ \cdot \\ b_s \end{bmatrix} \quad (8)$$

are the original constraint matrix  $A$  and the right-hand side vector  $b$ , respectively, with  $s$ -additional constraints each corresponding to an efficient cut in the form  $a_i x \leq b_i$ . By an efficient cut, we mean that a cut which is not redundant. Furthermore, to find this convex hull  $[M]$  and for more details, the reader is referred to [15].

To tackle problem (4)-(6), a fuzzy approach is presented in the following section.

### 3 Fuzzy Approach for Solving the BLFP (4)-(6)

Generally speaking and from the game theoretic point of view, the bi-level linear programming problem can be seen as a static Stackelberg game [14], where a Stackelberg strategy is used by the leader (or the higher level decision maker, given the rational reaction of the follower (or the lower level decision maker).

Now, problem (BLFP) (4)-(6) can be solved by adopting the leader-follower Stackelberg strategy with the well-known fuzzy decision model of Sakawa and Nishizaki [16]. One first gets the optimal solution that is acceptable to the FLDM and then gives the FLDM decision variable and goal with some leeway to the SLDM for him/her to seek the optimal solution which is closest to the optimal solution of the FLDM. This is due to, the SLDM who should not only optimize his/her objective function but also try to satisfy the FLDMs goal and preference as much as possible.

#### 3.1 FLDM problem

The first -level decision maker programming problem may be formulated as follows:

$$(FLDM) \quad \max_{x_1} F_1(x) = \frac{c_1^T x + \alpha_1}{d_1^T x + \beta_1}, \quad (9)$$

Subject to

$$x \in [M]. \quad (10)$$

Using Charnes & Cooper Transformation Method [10], therefore problem (FLDM) (8)-(9) above can be reformulated as:

$$\max \{c_1^T y_1 + \alpha_1 \rho_1\}, \quad (11)$$

Subject to

$$(y_1, \rho_1) \in Q_1 = \{A^{(s)} y_1 - b^{(s)} \rho_1 \leq 0, d_1^T y_1 + \beta_1 \rho_1 = 1, y_1 \geq 0, \rho_1 > 0\} \quad (12)$$

where

$$\rho_1 = \frac{1}{d_1^T x + \beta_1} \text{ and } y_1 = \rho_1 x_1. \quad (13)$$

**Theorem 1. [10]** *Extreme points of*

$$[M] = M_R^{(s)} = \{x \in R^n \mid A^{(s)} x \leq b^{(s)}, x \geq 0\}$$

*are in one-to-one correspondence with extreme points of*

$$(y_1, \rho_1) \in Q_1 = \{A^{(s)} y_1 - b^{(s)} \rho_1 \leq 0, d_1^T y_1 + \beta_1 \rho_1 = 1, y_1 \geq 0, \rho_1 > 0\}$$

where  $\forall x \in M \Rightarrow d_1^T x + \beta_1 \rho_1 > 0$ .

The following two theorems are important in completing the solution procedure and the proofs can be found in [7].

**Theorem 2. [7]** *The feasible region for the BLFP problem [M] is formed by the continuous union of connected faces of the polyhedron M. Where  $[M] = \{(x_1, x_2^*) / x_2^* \in S(x_1)\}$ ;  $S(x_1)$  denotes the set of optimal solutions to the SLDM problem.*

**Theorem 3. [7]** *An optimal solution to the BLLFP problem occurs at an extreme point of the polyhedron [M].*

*Solving problem (10)-(11), the optimal solution for the FLDM using the WINQSP software package will be obtained as:  $x_1^* = \frac{y_1^*}{\rho_1^*}$*

*To build a membership function, goals and tolerances should be determined first. We should first find the individual best solution  $F_1^B$  and individual worst solution  $F_1^W$  of problem (10)-(11), where*

$$F_1^B = \max_{x \in M} F_1, \quad F_1^W = \min_{x \in M} F_1 \quad (14)$$

*Goals and tolerances can then be reasonably set for individual solution and the difference of the best and worst solution, respectively. This data can then be formulated as the following linear membership function of fuzzy set theory [19]:*

$$\mu[F_1(x)] = \begin{cases} 1 & \text{if } F_1(x) > F_1^B \\ \frac{F_1(x) - F_1^W}{F_1^B - F_1^W} & \text{if } F_1^W \leq F_1(x) \leq F_1^B, \\ 0 & \text{if } F_1^W \geq F_1(x). \end{cases} \quad (15)$$

Now, we can get the solution of the FLDM problem by solving the following mixed-integer Tchebycheff problem with mixed constraints (i.e. linear and non linear):

$$\begin{aligned} & \max \lambda \\ & \text{Subject to} \end{aligned} \tag{16}$$

$$\begin{aligned} & x \in [M], \\ & \mu[F_1(x)] \geq \lambda, \\ & \lambda \in [0,1]. \end{aligned}$$

The FLDM integer solution is assumed to be  $(x_1^F, x_2^F, F_1^F, \lambda)$ .

### 3.2 SLDM problem

Second, the SLDM do the same action like the first level decision maker to solve the following problem by use the algorithm of solution of the integer linear fractional programming problem:

$$\text{(SLDM)} \quad \max_{x_2} F_2(x) = \frac{c_2^T x + \alpha_2}{d_2^T x + \beta_2} \tag{17}$$

Subject to

$$x \in [M].$$

To build membership function, goals and tolerances should be determined first. We should first find the individual best solution  $F_2^B$  and individual worst solution  $F_2^W$  of (17), where

$$F_2^B = \max_{x \in M} F_2, \quad F_2^W = \min_{x \in M} F_2 \tag{18}$$

This data can then be formulated as the following membership function of fuzzy set theory [19]:

$$\mu[F_2(x)] = \begin{cases} 1 & \text{if } F_2(x) > F_2^B \\ \frac{F_2(x) - F_2^W}{F_2^B - F_2^W} & \text{if } F_2^W \leq F_2(x) \leq F_2^B, \\ 0 & \text{if } F_2^W \geq F_2(x). \end{cases} \tag{19}$$

Now, we can get the solution of the SLDM problem by solving the following mixed-integer Tchebycheff problem with mixed constraints (i.e. linear and non linear):

$$\begin{aligned} & \max \beta \\ & \text{Subject to} \end{aligned} \tag{20}$$

$$\begin{aligned} x &\in [M], \\ \mu[F_2(x)] &\geq \beta, \\ \beta &\in [0,1]. \end{aligned}$$

The SLDM integer solution is assumed to be  $(x_1^S, x_2^S, F_2^S, \beta)$ .

## 4 Negotiation between the Two Level Decision Makers

Now, at this stage, the solution of the FLDM and SLDM are disclosed. However, two solutions are usually different because of the nature between two levels objective functions. The FLDM knows that using the optimal decisions  $x_1^F$  as a control factors for the SLDM are not practical. It is more reasonable to have some tolerance that gives the SLDM an extent feasible region to search for his/her optimal solution, and reduce searching time or interactions. In this way, the range of decision variable  $x_1$ , should be around  $x_1^F$  with maximum tolerance  $t$  and the following membership function specify  $x_1$ , as:

$$\mu(x_1) = \begin{cases} \frac{(x_1^F + t) - x_1}{t} & \text{if } x_1^F \leq x_1 \leq x_1^F + t, \\ \frac{x_1 - (x_1^F - t)}{t} & \text{if } x_1^F - t \leq x_1 \leq x_1^F. \end{cases} \quad (21)$$

where  $x_1^F$  is the most preferred solution; the  $(x_1^F - t)$  and  $(x_1^F + t)$  is the worst acceptable decision. To obtain the solution that achieve the benefits for all levels, we should make there negotiation between the FLDM and the SLDM. This negotiation can be represented by the following membership functions for the FLDM and SLDM respectively:

$$\bar{\mu}[F_1(x)] = \begin{cases} 1 & \text{if } F_1(x) > F_1^F, \\ \frac{F_1(x) - \bar{F}_1}{F_1^F - \bar{F}_1} & \text{if } \bar{F}_1 \leq F_1(x) \leq F_1^F, \\ 0 & \text{if } \bar{F}_1 \geq F_1(x), \end{cases} \quad (22)$$

$$\bar{\mu}[F_2(x)] = \begin{cases} 1 & \text{if } F_2(x) > F_2^S, \\ \frac{F_2(x) - \bar{F}_2}{F_2^S - \bar{F}_2} & \text{if } \bar{F}_2 \leq F_2(x) \leq F_2^S, \\ 0 & \text{if } \bar{F}_2 \geq F_2(x). \end{cases} \quad (23)$$

where  $\bar{F}_1 = F_1(x_1^S, x_2^S)$  and  $\bar{F}_2 = F_2(x_1^F, x_2^F)$ .

Finally, in order to generate the optimal solution, which is also an optimal solution with overall satisfaction for all decision-makers, we can solve the following mixed-integer Tchebycheff problem:

$$\begin{aligned}
 & \max \gamma && (24) \\
 & \text{Subject to} \\
 & x \in [M], \\
 & \overline{\mu}[F_1(x)] \geq \gamma, \\
 & \overline{\mu}[F_2(x)] \geq \gamma, \\
 & \mu(x_1) \geq \gamma, \\
 & \gamma \in [0,1], \\
 & t > 0 \text{ and integer.}
 \end{aligned}$$

where  $\gamma$  is the over all satisfaction coming by solving problem (24) above. If the FLDM is satisfied with the solution, then the optimal solution is reached. Otherwise, he/she should provide a new membership function for the control variable and objectives to the SLDM, until an optimal solution is reached.

## 5 Algorithm for Solving the Bi-Level Integer Linear Fractional Programming Problem

The outlines of the steps of the algorithm to solve the bi-level integer linear fractional programming problem based on the fuzzy approach can be summarized in the following:

### Algorithm Development

#### Step 1.

Solve the FLDM problem using the algorithm described in [15] for the solution of ILFP (11)-(12) to find the best solution  $F_1^B$  and the worst solution  $F_1^W$ .

#### Step 2.

Similarly, solve the SLDM problem to find the best solution  $F_2^B$  and the worst solution  $F_2^W$ .

#### Step 3.

Build the membership function  $\mu[F_1(x)]$  for the FLDM and the membership function  $\mu[F_2(x)]$  for the SLDM.

#### Step 4.

Solve the mixed- integer Tchebycheff problems (16) and (20) using the WINQSP package .Denote the optimal solution for FLDM as  $(x_1^F, x_2^F, F_1^F, \lambda)$  and for the SLDM as  $(x_1^S, x_2^S, F_2^S, \beta)$ .



**Step 5.**

Build the membership functions  $\overline{\mu}[F_1(x)]$  and  $\overline{\mu}[F_2(x)]$  whose represent the negotiation between FLDM and SLDM and the membership function  $\mu(x_1)$  for  $x_1^F$  variables.

**Step 6.**

Solve the mixed- integer Tchebycheff (24). This optimal solution is the optimal integer solution of the ILFP problem, then ask the first level decision maker if he is satisfied with the solution go to step 8. Otherwise, go to step 7.

**Step7.**

The SLDM should scarify and provide new membership function for his objectives, then go to step2.

**Step 8.**

The optimal integer solution is reached and then stop.

## 6 Numerical Example

The following bi-level integer linear fractional programming problem is solved to show the feasibility of the proposed fuzzy approach and may be formulated as:

$$(FLDM) \quad \max_{x_1} F_1 = \frac{2x_1 + 3x_2}{x_1 + 4x_2 + 6}$$

$$(SLDM) \quad \max_{x_2} F_2 = \frac{3x_1 + 4x_2}{6x_1 + 4x_2 + 3}$$

Subject to

$$x_1 + x_2 \leq 5,$$

$$3x_1 + x_2 \leq 10,$$

$$x_1, x_2 \geq 0 \text{ and integers.}$$

First, the given bi-level integer linear fractional programming problem can be converted into its equivalent bi-level linear fractional programming problem as follows:

$$(FLDM) \quad \max_{x_1} F_1 = \frac{2x_1 + 3x_2}{x_1 + 4x_2 + 6}$$

$$(SLDM) \quad \max_{x_2} F_2 = \frac{3x_1 + 4x_2}{6x_1 + 4x_2 + 3}$$

Subject to

$$x_1 + x_2 \leq 5,$$

$$3x_1 + x_2 \leq 10,$$

$$2x_1 + x_2 \leq 7,$$

$$x_1 \leq 3,$$

$$x_1, x_2 \geq 0.$$

1. We will solve the first level decision maker FLDM using Charnes and Cooper method[10]. For this reason, let us assume that  $\rho_1 = \frac{1}{x_1 + 4x_2 + 6}$ ,  $\rho_1 x_1 = y_1$ , and  $\rho_1 x_2 = y_2$ , then the FLDM problem will be written as:

$$\begin{aligned} & \max F_1 = 2y_1 + 3y_2 \\ & \text{Subject to} \\ & y_1 + y_2 \leq 5\rho_1, & 11y_1 + 26y_2 \leq 5, \\ & 3y_1 + y_2 \leq 10\rho_1, & 28y_1 + 46y_2 \leq 10, \\ & 2y_1 + y_2 \leq 7\rho_1, & \Rightarrow 19y_1 + 34y_2 \leq 7, \\ & y_1 \leq 3\rho_1, & 3y_1 + 4y_2 \leq 1, \\ & y_1 + 4y_2 + 6\rho_1 = 1, & y_1, y_2 \geq 0. \\ & y_1, y_2 \geq 0 \text{ and } \rho_1 > 0. \end{aligned}$$

The optimal solution is obtained  $(y_1 = 0.2308, y_2 = 0.0769, \rho_1 = 0.07693)$ , which leads to the optimal integer solution of the FLDM as  $(x_1^* = 3, x_2^* = 1)$ , where the individual best solution  $F_1^B = 0.6923$  and the individual worst solution  $F_1^W = 0$ .

2. Similarly, the second level decision maker SLDM is solved again using Charnes and Cooper method [10] by letting  $\rho_2 = \frac{1}{6x_1 + 4x_2 + 3}$ ,  $\rho_2 x_1 = z_1$ , and  $\rho_2 x_2 = z_2$ , then the SLDM problem will be written as :

$$\begin{aligned} & \max F_2 = 3z_1 + 4z_2 \\ & \text{Subject to} \\ & z_1 + z_2 \leq 5\rho_2, & 33z_1 + 23z_2 \leq 5, \\ & 3z_1 + z_2 \leq 10\rho_2, & 69z_1 + 43z_2 \leq 10, \\ & 2z_1 + z_2 \leq 7\rho_2, & \Rightarrow 48z_1 + 31z_2 \leq 7, \\ & z_1 \leq 3\rho_2, & 7z_1 + 4z_2 \leq 1, \\ & 6z_1 + 4z_2 + 3\rho_2 = 1, & z_1, z_2 \geq 0. \\ & z_1, z_2 \geq 0, \text{ and } \rho_2 > 0. \end{aligned}$$

The optimal solution is found  $(z_1 = 0, z_2 = 0.2174, \rho_2 = 0.043466)$ , which leads to the integer optimal solution of SLDM as  $(x_1^* = 0, x_2^* = 5)$ , where the individual best solution  $F_2^B = 0.87$  and the individual worst solution  $F_2^W = 0$ .

Now, a membership function  $\mu[F_1(x)]$  is built to solve the mixed- integer Tchebycheff problem:

$$\begin{aligned} & \max \lambda \\ & \text{Subject to} \\ & x_1 + x_2 \leq 5, \\ & 3x_1 + x_2 \leq 10, \\ & 2x_1 + x_2 \leq 7, \end{aligned}$$

$$\begin{aligned} x_1 &\leq 3, \\ 2.9x_1 + 4.4x_2 - x_1\lambda - 4x_2\lambda - 6\lambda &\geq 0, \\ \lambda &\in [0, 1] \text{ and } x_1, x_2 \geq 0. \end{aligned}$$

The integer solution is  $(x_1^F = 3, x_2^F = 0, F_1^F = 0.67, \lambda = 0.97)$ .

On the other hand, another membership function  $\mu[F_2(x)]$  is built to solve the mixed -integer Tchebycheff problem:

$$\begin{aligned} \max \quad &\beta, \\ \text{Subject to} \quad & \\ &x_1 + x_2 \leq 5, \\ &3x_1 + x_2 \leq 10, \\ &2x_1 + x_2 \leq 7, \\ &x_1 \leq 3, \\ &3.45x_1 + 4.6x_2 - 6x_1\beta - 4x_2\beta - 3\beta \geq 0, \\ &\beta \in [0, 1] \text{ and } x_1, x_2 \geq 0. \end{aligned}$$

The integer solution is  $(x_1^S = 0, x_2^S = 5, F_1^S = 0.87, \beta = 0.999)$ .

We assume the FLDMs control decision with tolerance  $t = 1$  and we solve the mixed- integer Tchebycheff problem:

$$\begin{aligned} \max \quad &\gamma, \\ \text{Subject to} \quad & \\ &x_1 + x_2 \leq 5, \\ &3x_1 + x_2 \leq 10, \\ &x_1 + \gamma \leq 4, \\ &x_1 - \gamma \geq 2, \\ &2x_1 + x_2 \leq 7, \\ &x_1 \leq 3, \\ &0.94x_1 + 5.2x_2 - 6x_1\gamma - 4x_2\gamma - 3\gamma \geq 2.94, \\ &15.8x_1 + 7.7x_2 - x_1\gamma - 4x_2\gamma - 6\gamma \geq 38.4, \\ &\gamma \in [0, 1] \text{ and } x_1, x_2 \geq 0. \end{aligned}$$

Hence, the integer optimal solution is  $(x_1 = 3, x_2 = 1, \gamma = 0.2)$ .

This solution is not acceptable to the FLDM, so the SLDM will scarify and resolve his problem to obtain a new integer solution. Let  $(F_2^B = 0.7, F_2^W = 0)$  and solve the mixed-integer Tchebycheff problem:

$$\begin{aligned} \max \quad &\beta \\ \text{Subject to} \quad & \\ &x_1 + x_2 \leq 5, \end{aligned}$$

$$\begin{aligned}
3x_1 + x_2 &\leq 10, \\
2x_1 + x_2 &\leq 7, \\
x_1 &\leq 3, \\
-1.2x_1 + 1.2x_2 &\leq 2.1, \\
5.3x_1 + 7x_2 - 6x_1\beta - 4x_2\beta - 3\beta &\geq 0, \\
\beta &\in [0, 1] \text{ and } x_1, x_2 \geq 0.
\end{aligned}$$

The optimal integer solution is:  $(x_1^S = 0, x_2^S = 1, F_1^S = 0.57, \beta = 0.999)$

Assuming that the FLDMs control decision with tolerance  $t = 1$  and we solve:

$$\begin{aligned}
&\max \gamma \\
&\text{Subject to} \\
&x_1 + x_2 \leq 5, \\
&3x_1 + x_2 \leq 10, \\
&2x_1 + x_2 \leq 7, \\
&x_1 \leq 3, \\
&-1.2x_1 + 1.2x_2 \leq 2.1, \\
&x_1 + \gamma \leq 4, \\
&x_1 - \gamma \geq 2, \\
&4.6x_1 + 4.9x_2 - x_1\gamma - 4x_2\gamma - 6\gamma \geq 4.9, \\
&3x_1 + 16.2x_2 - 6x_1\gamma - 4x_2\gamma - 3\gamma \geq 9.3, \\
&\gamma \in [0, 1], x_1, x_2 \geq 0.
\end{aligned}$$

The integer optimal solution to the problem of concern is then:  $(x_1 = 3, x_2 = 1, \gamma = 0.64)$ .

## 7 Conclusions

This paper proposed an algorithm for solving the bi-level integer linear fractional programming problem by a fuzzy approach. The solution algorithm described by two main phases: first the solution algorithm should avoid the complexity of non convexity nature of the constraint set by constructing the convex hull equivalent to the original set of constraints using the cutting-plane algorithm, and then the solution process introduces the Charnes & Cooper transformation method to obtain the integer solution. At the second phase, from this obtained integer solution, the fuzzy model is stated to develop the solution algorithm for generating the optimal integer solution for the bi-level integer linear fractional programming problem.

Certainly, there are many other points for future research in this area of bi-level and multi-level integer linear fractional programming and should be studied. One may have to tackle the following open points for future research:

- (i) An algorithm for solving multi-level multiobjective integer fractional programming problems with fuzzy parameters in constraints.
- (ii) An algorithm for solving multi-level multiobjective integer fractional programming problems with fuzzy parameters in objective functions.

- (iii) An algorithm for solving multi-level multiobjective integer fractional programming problems with fuzzy parameters in both objective functions and constraints.
- (iv) A fuzzy goal programming procedure for solving quadratic bi-level integer fractional programming problem is needed.
- (v) Taylor series approach for bi-level integer fractional programming problem must be discussed.
- (vi) However, it is hoped that the fuzzy approach presented here can contribute to future study in the filed of practical hierarchical decision-making problems.

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## References

- [1] G. Anuradha and S.R. Arora, Multi-level multi-objective integer linear programming problem, *Journal of Operations Research*, 10(2008), 297-322.
- [2] M. Balinski, An algorithm for finding all vertices of convex polyhedral sets, *Journal of the Society of Industrial and Applied Mathematics*, 9(1961), 72-88.
- [3] O. Ben-Ayed, Bi-level linear programming, *Computers & Operations Research*, 20(1993), 485- 501.
- [4] W.F. Bialas and M.M. Karwan, Two-level linear programming, *Management Science*, 30(1984), 1004-1020.
- [5] B.P. Bijay and N.M. Bohla, A fuzzy goal programming procedure for solving quadratic bi-level programming problems, *International Journal of Intelligent System*, 18(2003), 529-540.
- [6] H.I. Calvet and C. Gale, The bi-level linear/linear fractional programming problem, *European Journal of Operation Research*, 114(1999), 188-197.
- [7] H.I. Calvet and C. Gale, A note on bi-level linear fractional programming problem, *European Journal of Operational Research*, 152(2004), 296-299.
- [8] H.I. Calvet and C. Gale, Solving linear fractional bi-level programs, *European Journal of Operational Research*, 32(2004), 143-151.
- [9] W. Candler and R. Townsley, A linear two-level programming problem, *Computers & Operations Research*, 9(1982), 59-76.
- [10] A. Charnes and W.W. Cooper, Programming with linear fractional functional, *Naval Research Logistic Quarterly*, 27(1962), 181-186.
- [11] O.E. Emam, A fuzzy approach for bi-level integer non-linear programming problem, *Applied Mathematics and Computation*, 172(2006), 62-71.
- [12] S.S. Hsu, J.L. Yong and L. Stanley, Fuzzy approach for multi-level programming problems, *Computers & Operations Research*, 23(1996), 73-91.
- [13] N.V. Luis and H.C. Paul, Bi-level and multi-level programming: A bibliography review, *Journal of Global Optimization*, 5(1994), 291-306.
- [14] K. Mathur and M.C. Puri, On bi-level fractional programming, *Optimization*, 35(1995), 215-226.

- [15] O.M. Saad, An algorithm for solving integer linear fractional programs, *Journal of Information and Optimization Sciences*, 14(1993), 87-93.
- [16] M. Sakawa and I. Nishizaki, Interactive fuzzy programming for two-level linear and linear fractional production and assignment problem: A case study, *European Journal of Operation Research*, 135(2001), 142-157.
- [17] D. Thirwani and S.R. Arora, An algorithm for the integer linear fractional bi-level programming problem, *Optimization*, 39(1997), 53-67.
- [18] U.P. Wen and S.T. Hsu, A note on a linear bi-level programming algorithm based on bicriteria programming, *Computers & Operations Research*, 16(1989), 79-83.
- [19] L.A. Zadeh, Fuzzy sets, *Information and Control*, 8(1965), 338-353.