Gen. Math. Notes, Vol. 27, No. 1, March 2015, pp. 101-113

ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

A Finite Difference Scheme for the Modified Korteweg-De Vries Equation

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(Received: 31-8-14 / Accepted: 24-1-15)

Abstract

In this paper, the modified Korteweg-de Vries (MKdV) equation is solved numerically using the finite difference method. An energy conservative finite difference scheme was proposed. Accuracy and stability of the difference solution were proved.

Keywords: *MKdV, Finite difference, Solitary waves.*

1 Introduction

In this paper, the finite difference method is employed to obtain the numerical solution to the modified Korteweg-de Vries (mKdV) equation. A scheme is developed for the numerical study of the mKdV equations with initial conditions. The exact and numerical solutions obtained by this scheme are compared. The comparison shows that this scheme provides highly numerical solutions for the MKdV equation. The modified KdV equation has a pulse travelling solution.

Wadati and Ohkuma [4] have used the inverse scattering method to investigate the multiple pole solution of the modified KdV equation. Wazwaz [5] constructed the solution of mkdv equation in the form of Taylor series by using Adomian decomposition method.

2 The Problem and Analytical Solution

The MKdV equation in the form [2]

$$u_t + \varepsilon u^2 u_x + \mu u_{xxx} = 0, \tag{1}$$

where subscripts x and t denote differentiation, is considered with the boundary conditions $u \to 0$ as $x \to \pm\infty$. In this paper, we use periodic boundary conditions for a region $a \le x \le b$. The analytic solution of the MKdV equation can be expressed as

$$u(x,t) = 3c \sec h^2 (p(x - vt - x_0)) \tag{2}$$

where x_0 is an arbitrary constant.

3 Conservation Laws for the MKdV Equation

The MKdV equation possesses four polynomial invariants, corresponding to the conservation of mass, momentum and energy which for the periodic boundary condition can be expressed in the form

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u\,dx \\ I_2 &= \int_{-\infty}^{\infty} u^2\,dx \\ I_3 &= \int_{-\infty}^{\infty} \left(u^4 - \frac{6}{\varepsilon}\mu u_x^2 \right) dx \\ I_4 &= \int_{-\infty}^{\infty} \left(u^6 - \frac{30}{\varepsilon}\mu u^2 u_x^2 + \frac{18}{\varepsilon^2}\mu^2 u_{xx}^2 \right) dx \end{aligned} \tag{3}$$

4 Finite Difference Method

To apply the finite difference method for solving the MKdV equation, firstly we present the following notations for the derivatives

$$\left.\begin{aligned}
(u_i^j)_t &\cong \frac{u_i^{j+1}-u_i^j}{k}\\
(u_i^j)_x &\cong \frac{\theta\left(u_{i+1}^{j+1}-u_{i-1}^{j+1}\right)+(1-\theta)\left(u_{i+1}^j-u_{i-1}^j\right)}{2h}\\
(u_i^j)_{xxx} &\cong \frac{\theta(u_{i+2}^{j+1}-2u_{i+1}^{j+1}+2u_{i-1}^{j+1}+u_{i-2}^{j+1})-\left(1-\theta\right)(u_{i+2}^j-2u_{i+1}^j+2u_{i-1}^j-u_{i-2}^j)}{2h^3}
\end{aligned}\right\} \tag{4}$$

where $0 \le \theta \le 1$, h and k are the spatial and temporal step sizes respectively and $x_i = ih$, $t_j = jk$, i = 0,1,. . . and j = 0,1,. . ., where superscript j denotes a quantity associated with time level t_j and subscript i denotes a quantity associated with space mesh point x_i. The scheme requires two initial time levels, so we use the exact solution (2) at t = 0 and t = k.

Substitute Eq.(4) in Eq.(1), then the resulting algebraic system of equations takes the form

$$\frac{u_i^{j+1}-u_i^j}{k}+\varepsilon\left(u_i^j\right)^2\frac{\theta\left(u_{i+1}^{j+1}-u_{i-1}^{j+1}\right)+(1-\theta)\left(u_{i+1}^j-u_{i-1}^j\right)}{2h}$$
$$+\mu\frac{\theta(u_{i+2}^{j+1}-2u_{i+1}^{j+1}+2u_{i-1}^{j+1}-u_{i-2}^{j+1})+(1-\theta)(u_{i+2}^j-2u_{i+1}^j+2u_{i-1}^j-u_{i-2}^j)}{2h^3}=0, \tag{5}$$

where j=0,1,2,…, i=1,2,…,N-1

5 Linear Stability Analysis

The Von Neumann stability theory [1] will be applied and the growth of a Fourier mode

$$u_j^n = \xi^n e^{ikjh} \tag{6}$$

where k is the mode number and h is the element size, will be determined for a linearization of the numerical scheme. In this nonlinear term $u^2 u_x$ is locally constant. This is equivalent to assuming that the corresponding values u^2 are also constant [3] and equal to U .

Substituting (6) into Eq.(5) we obtain.

$$\xi^{j+1} = g\xi^j \tag{7}$$

where g is the growth factor is thus

$$g = \frac{A + i(1-\theta)B}{A - i\theta B} \tag{8}$$

where A=1, $B = 2\sin jh(-p_1 U + 2p_2(\cos jh - 1))$, $p_1 = \frac{\varepsilon k}{2h}$ and $p_2 = \frac{\mu k}{2h^3}$

Stability can be concluded in different cases:

1. θ = 0, gives an explicit scheme and the linearized scheme is unstable, since $|g| > 1$.
2. θ=1, gives the fully implicit scheme and the linearized scheme is unconditionally stable, since $|g| \le 1$.
3. θ=0.5, gives the Crank-Nicolson scheme and the linearized scheme is unconditionally stable, since $|g| = 1$.

6 Numerical Applications

It has been shown in Section 2 that the MKdV equation has an analytical solution of the form (2). In this work, we present some numerical experiments to assign the numerical solution of single solitary wave, in addition to determine the solution of two and three soliton interactions at different time levels.

6.1 Single Solitary Waves

In this test we choose the initial condition from the exact solution

$$u(x,0) = \sqrt{\frac{6c}{\varepsilon}} \sec h\left[\sqrt{\frac{c}{\mu}}\,(x - x_0)\right], \tag{9}$$

To illustrate the validity of our scheme in case of a single soliton, we use the L_∞-norm to compare the numerical solution with the exact solution, also quantities I_1, I_2, I_3 and I_4 are shown to measure conservation laws for the scheme. In case 1, we choose $\Delta x = 0.1, \Delta t = 0.01$, $\varepsilon = 3$, $\mu = 1$ and $c = 0.845$ as shown in "Table" 1.

Table 1: Invariants and error norm for single solitary wave h=0.1=k=0.01, $\varepsilon = 3$, $\mu = 1$ and $c = 0.3$, $0 \le x \le 80$

T	L_∞-error	I_1	I_2	I_3	I_4
0.1	9.4179E-5	4.44279	2.1908	0.438274	0.0788365
0.2	1.5926 E-4	4.4427	2.1907	0.438217	0.0788886
0.3	2.0619 E-4	4.4426	2.1906	0.43816	0.0788884
0.4	2.3711 E-4	4.4425	2.1905	0.438104	0.0788761
0.5	2.6244 E-4	4.4424	2.1904	0.438047	0.0788634
0.6	2.8110 E-4	4.4423	2.1903	0.43799	0.0788464

0.7	2.9482E-4	4.4422	2.1902	0.43793	0.0788586
0.8	3.0133 E-4	4.4421	2.1901	0.43788	0.0788661
0.9	3.1064 E-4	4.4420	2.1900	0.43782	0.0789231
1.0	3.1027E-4	4.44192	2.18994	0.437763	0.079070

Table 2: Invariants and error norm for single solitary wave The 2nd Scheme h=0.1=k=0.01, $\varepsilon = 3$, $\mu = 1$ and $c = 0.3$, $0 \le x \le 80$

T	L_∞-error	I_1	I_2	I_3	I_4
0.1	9.6546E-5	4.4428	2.19078	0.438262	0.0788272
0.2	6.2638E-4	4.4427	2.19066	0.438193	0.07887
0.3	2.1449E-4	4.4426	2.19055	0.438124	0.0788808
0.4	2.4938E-4	4.4425	2.19043	0.438055	0.0788675
0.5	2.7377E-4	4.4424	2.19032	0.437986	0.0788502
0.6	2.9141E-4	4.4423	2.1902	0.437918	0.078831
0.7	3.0437E-4	4.4423	2.19009	0.437849	0.0788104
0.8	3.1388E-4	4.4422	2.18997	0.43778	0.0787901
0.9	3.2048E-4	4.4421	2.18986	0.437711	0.0787629
1.0	3.2521E-4	4.44198	2.18974	0.437642	0.0787611

The results shown in "Tables" 1 and 2 show that the change in the invariants I_1, I_2, I_3 and I_4 are very small as follows: 8.2×10^{-4}, 1×10^{-3}, 6×10^{-4} and 6.6×10^{-5}, respectively.

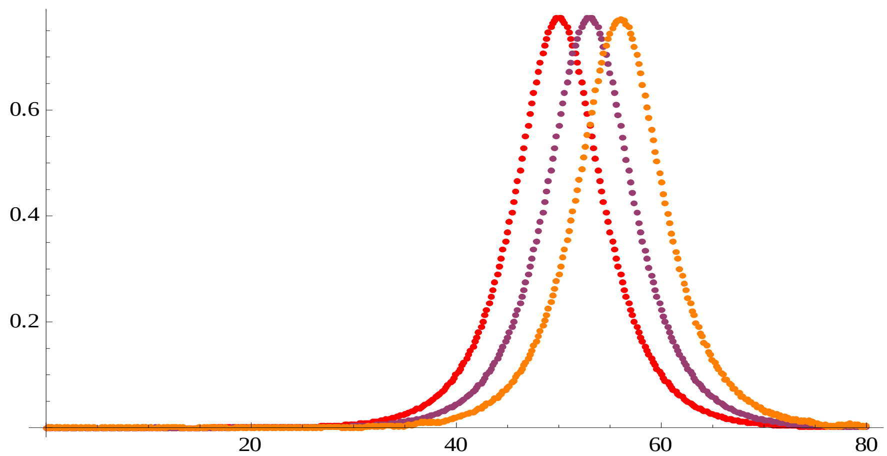


Fig. 1: Single solitary wave of the first scheme with c=0.3, h=0.1, k=0.01 at times: T=0, T=5 and T=10

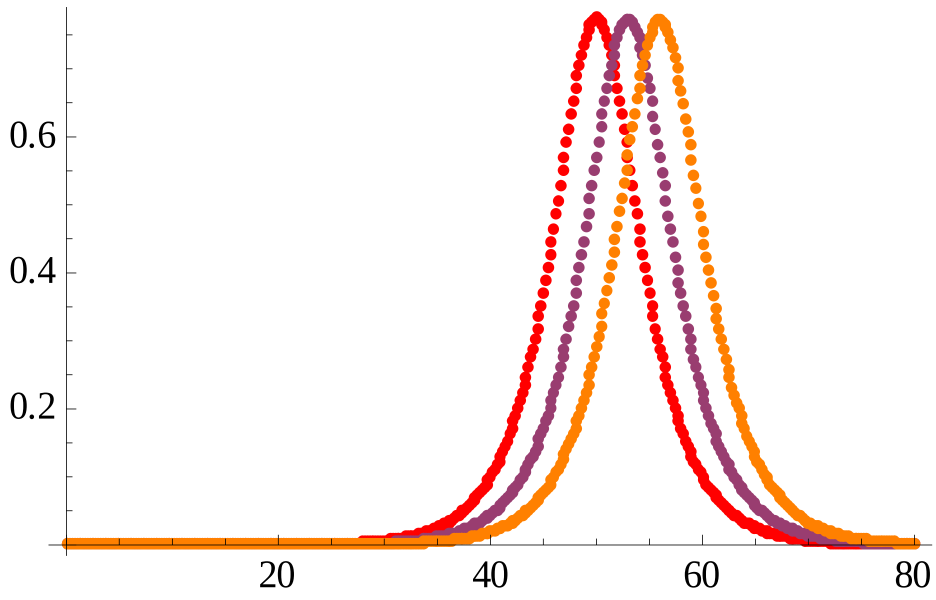


Fig. 2: Single solitary wave of the second scheme with c=0.3, h=0.1, k=0.01 at times: T=0, T=5 and T=10

6.2 Interaction of Two Soliton

In this test we choose the initial condition as the sum of two solitary waves of the form

$$u(x,0) = a_i \operatorname{sech}\left(\sqrt{\frac{c_i}{\mu}}(x - x_i)\right), \qquad a_i = \sqrt{\frac{6c_i}{\varepsilon}}, i = 1,2. \tag{10}$$

Where $c_1 = 0.2$, $c_2 = 0.1$, $x_1 = 15$, $x_2 = 25$. The conserved quantities are given in "Table" 3 and "Table" 4.

Table 3: The computed values of the conservations laws for two soliton of the 1st Scheme with $\Delta x = 0.1, \Delta t = 0.01$, $\varepsilon = 3,\ \mu = 1$, $0 \le x \le 80$

T	I_1	I_2	I_3
1.0	7.52781	2.94444	0.66715
2.0	7.48285	2.9481	0.66283
3.0	7.45274	2.95242	0.665289
4.0	7.39747	2.96155	0.664989
5.0	7.43042	2.97498	0.658588

The conserved quantities for two soliton of the 1st scheme are given in "Table" 3, we found during the interaction simulation. that the computed quantities I_1, I_2 and I_3 change by less than 9×10^{-2}, 3×10^{-2} and 8×10^{-3} .

Table 4: The computed values of the conservations laws for two soliton of the 2nd Scheme with $\Delta x = 0.1, \Delta t = 0.01$, $\varepsilon = 3$, $\mu = 1$, $0 \le x \le 80$

T	I_1	I_2	I_3
1.0	8.88426	3.59982	0.859249
2.0	8.88525	3.59918	0.847121
3.0	8.88445	3.59805	0.83652
4.0	8.88488	3.59732	0.827262
5.0	8.88831	3.59768	0.819233

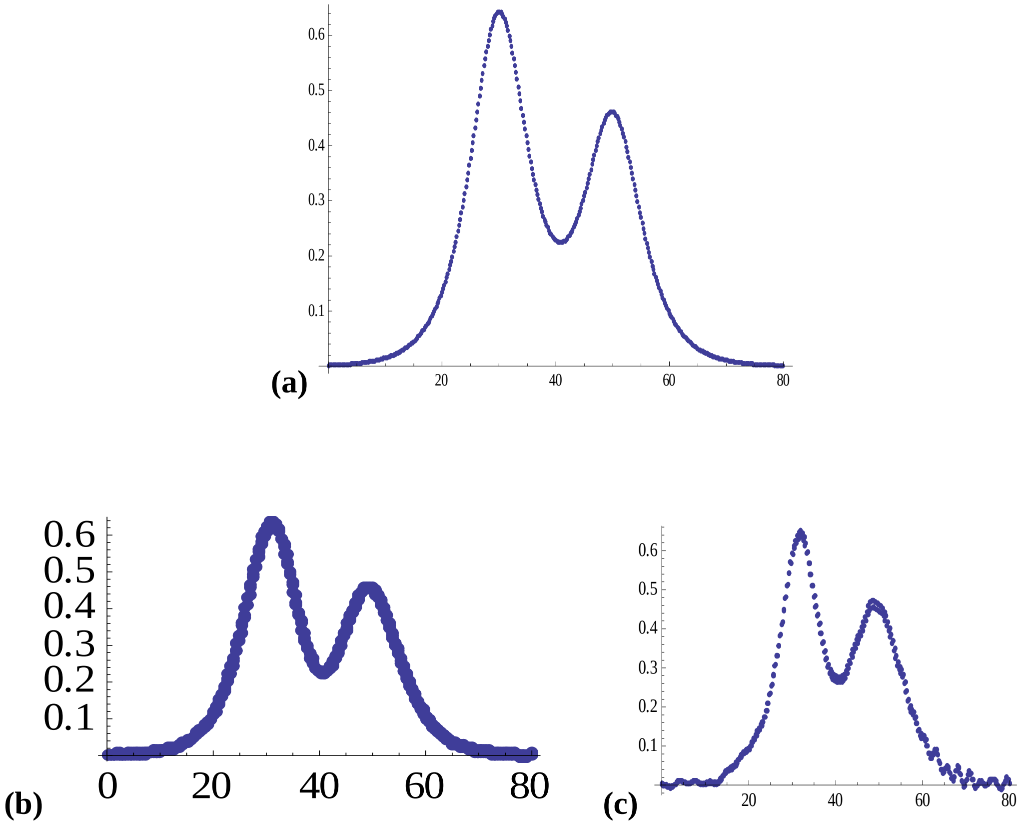


Fig. 3: Interaction of two solitary waves at times: **(a)** =T=0, **(b)** T=2, **(c)** T=5

The conserved quantities for two soliton of the 2nd scheme are given in "Table" 3, we found during the interaction simulation. that the computed quantities I_1, I_2 and I_3 change by less than 4.1×10^{-3}, 3.1×10^{-3} and 4×10^{-2}

6.3 Interaction of Three Soliton

In this test we choose the initial condition for three waves

$$u(x,0)=\sum_{i=1}^{3} a_i \sec h\left[\sqrt{\frac{c_i}{\mu}}\left(x-x_i\right)\right], \qquad a_i=\sqrt{\frac{6c_i}{\varepsilon}}, \quad i=1,2,3,$$

where $c_1=2$, $c_2=1$, $c_3=0.5$, $x_1=15$, $x_2=25$, $x_3=35$. The conserved quantities are given in "Table" 5, 6.

Table 5: The computed values of the conservations laws for three soliton of the 1st Scheme with $\Delta x=0.1, \Delta t=0.01$, $\varepsilon=3,\ \mu=1$, $0\le x\le 80$

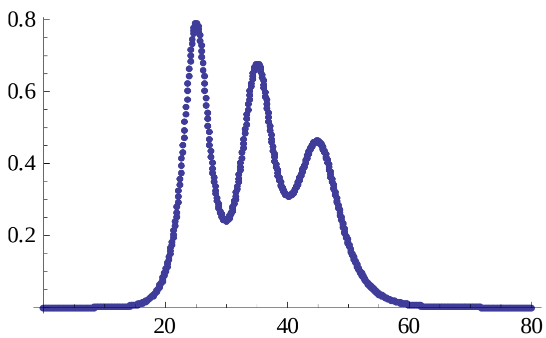
T	I_1	I_2	I_3
0.0	13.3286	6.09900	1.90112
1.0	13.3271	6.09748	1.86964
2.0	13.3260	6.09608	1.84270
3.0	13.3248	6.09481	1.81528
4.0	13.3280	6.09369	1.78710
5.0	13.3203	6.09266	1.75979

The conserved quantities for three soliton of the 1[st] scheme are given in "Table" 5 , we found during the interaction simulation. that the computed quantities I_1, I_2 and I_3 change by less than 8.2×10^{-3}, 6.3×10^{-3} and 0.14 respectively.

Table 6: The computed values of the conservations laws for three soliton of the 2[nd] Scheme with $h=0.1, k=0.01$, $\varepsilon=3,\ \mu=1,\ 0\le x\le 80$

T	I_1	I_2	I_3
0.0	13.3286	6.09900	1.90112
1.0	13.3257	6.09593	1.86863
2.0	13.3227	6.09282	1.84073
3.0	13.3193	6.08958	1.81250
4.0	13.3148	6.08621	1.78366
5.0	13.3110	6.08286	1.75590

The conserved quantities for three soliton of the 2[nd] scheme are given in "Table" 6, we found during the interaction simulation. that the computed quantities I_1, I_2 and I_3 change by less than 1.7×10^{-2}, 1.6×10^{-2} and 0.14



(a)

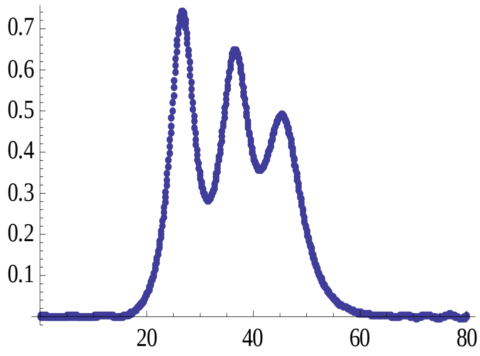
(b) 

Fig. 4: Interaction three solitary waves at times: **(a)** =T=0, **(b)** T=5

7 The Invariant Imbedding Method

The general implicit form is:

$$-p_2\theta u_{i-2}^{j+1} + (-p_1(u_i^j)^2 + 2p_2)\theta u_{i-1}^{j+1} + u_i^{j+1} + (p_1(u_i^j)^2 - 2p_2)\theta u_{i+1}^{j+1} + p_2\theta u_{i+2}^{j+1} =$$
$$p_2(1-\theta)u_{i-2}^j + (p_1(u_i^j)^2 - 2p_2)(1-\theta)u_{i-1}^j + u_i^j + (-p_1(u_i^j)^2 + 2p_2)(1-\theta)u_{i+1}^j - p_2(1-\theta)u_{i+2}^j \quad (11)$$

We solve the scheme (11) by the invariant imbedding method [6]. This system can be written in the form

$$-\mu\theta u_{i-2}^{j+1} + (-\alpha+\beta)\theta u_{i-1}^{j+1} + \gamma u_i^{j+1} + (\alpha-\beta)\theta u_{i+1}^{j+1} + \mu\theta u_{i+2}^{j+1} = F \quad (12)$$

Where

$$F = \mu(1-\theta)u_{i-2}^{j+1} + (\alpha-\beta)(1-\theta)u_{i-1}^{j+1} + \gamma u_i^{j+1} + (-\alpha+\beta)(1-\theta)u_{i+1}^{j+1} - \mu(1-\theta)u_{i+2}^{j+1}$$

Let its solution be

$$u_i^{j+1} = A_i u_{i+2}^{j+1} + B_i u_{i+1}^{j+1} + C_i, \quad I = N-1, N-2, ..., 1 \quad (13)$$

By substituting equation (13) into (12) and comparing the coefficients, one gets the relations

$$A_i = \frac{-\mu\theta}{\gamma - \theta(\mu A_{i-2} + \mu B_{i-2} B_{i-1} + (\alpha - \beta) B_{i-1})}$$
$$B_i = \frac{(\mu B_{i-2} A_{i-1} - (\alpha - \beta) A_{i-1} - (-\alpha + \beta))\theta}{\gamma + ((\alpha - \beta) B_{i-1} - \mu B_{i-2} B_{i-1} - \mu A_{i-2})\theta}$$
$$C_i = \frac{F - \theta((\alpha - \beta) C_{i-1} + \mu C_{i-2} + \mu B_{i-2} C_{i-1})}{\gamma - \theta((\alpha - \beta) B_{i-1} - \mu A_{i-2} - \mu B_{i-2} B_{i-1})} \tag{14}$$

Taking i = 0 in (13) we get A_0=0, B_0=0 and $C_0 = u_0$ where u_0 is known from the boundary condition. Taking i=-1 in (13) we get A_{-1} =1, B_{-1}=0 and C_{-1}=0 ,the scalar A_i and B_i are computed from (14) and are used to find the solution u_i^{j+1} from equation (13).

7.1 Numerical Applications

It has been shown in this Section that the MKdV equation has an analytical solution of the form (2). In this work, we present some numerical experiments to assign the numerical solution of single solitary wave, in addition to determine the solution of two soliton interactions at different time levels.

7.1.1 Single Solitary Waves

To illustrate the validity of our scheme in case of a single soliton, we use the L_2-norm to compare the numerical solution with the exact solution, also quantities I_1, I_2, I_3 and I_4 are shown to measure conservation laws for the scheme. In this case, we choose $\Delta x = 0.2, \Delta t = 0.01$, $\varepsilon = 3$, $\mu = 1$ and $c = 0.3$ with range [0, 80].

Table 7: $\Delta x = 0.2, \Delta t = 0.01$, $\varepsilon = 3$, $\mu = 1$ and $c = 0.3$, $0 \le x \le 80$

T	L_2-error	I_1	I_2	I_3	I_4
1.0	1.9842E-3	4.44199	2.18995	3.8175E-2	7.8484E-2
2.0	2.5119E-3	4.44107	2.18901	3.7979E-2	7.9924E-2
3.0	2.9937E-3	4.44010	2.18807	3.7394E-2	8.5221E-2
4.0	3.5185E-3	4.43974	2.18713	3.6768E-2	8.8609E-2
5.0	4.2136E-3	4.43812	2.18619	3.6296E-2	9.7301E-2

The invariants I_1, I_2, I_3 and I_4 are changed by 3.8×10^{-3}, 3.7×10^{-3}, 1.8×10^{-3}, and 6.8×10^{-2} percent, respectively, throughout "Table" 7.

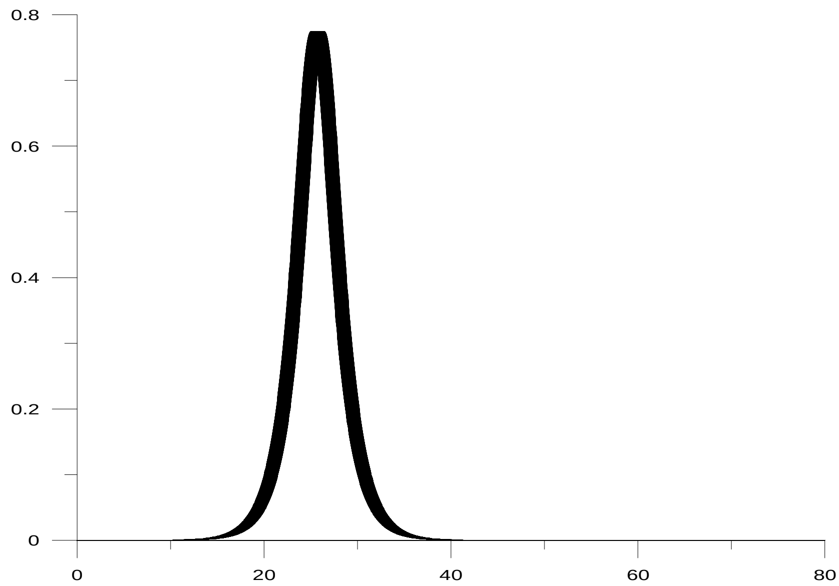


Fig. 5: Single solitary wave, c=0.3, h=0.1, k=0.01

7.1.2 Interaction of Two Solitary Waves

In this test we choose the initial condition as the sum of two solitary waves of the form

$$u(x,0) = a_1 \sec h\left(\sqrt{\frac{c_1}{\mu}}(x - x_1)\right) + a_2 \sec h\left(\sqrt{\frac{c_2}{\mu}}(x - x_2)\right), a_i = \sqrt{\frac{6c_1}{\varepsilon}}, i = 1,2,$$

Where $c_1 = 2$, $c_2 = 1$, $x_1 = 15$, $x_2 = 25$. The conserved quantities are given in "Table" 8.

Table 8: Interaction of two solitary waves

T	I1	I2	I3
1.0	8.63906	9.03663	7.94016
2.0	8.46653	8.65149	6.80785
3.0	8.36797	8.38153	6.02332
4.0	8.24351	8.17557	5.46112
5.0	8.15655	8.00870	5.18743

The conserved quantities are given in "Table" 8. We found that the computed quantities I_1, I_2, and I_3 change by less than 0.48, 1.7and 2.7.

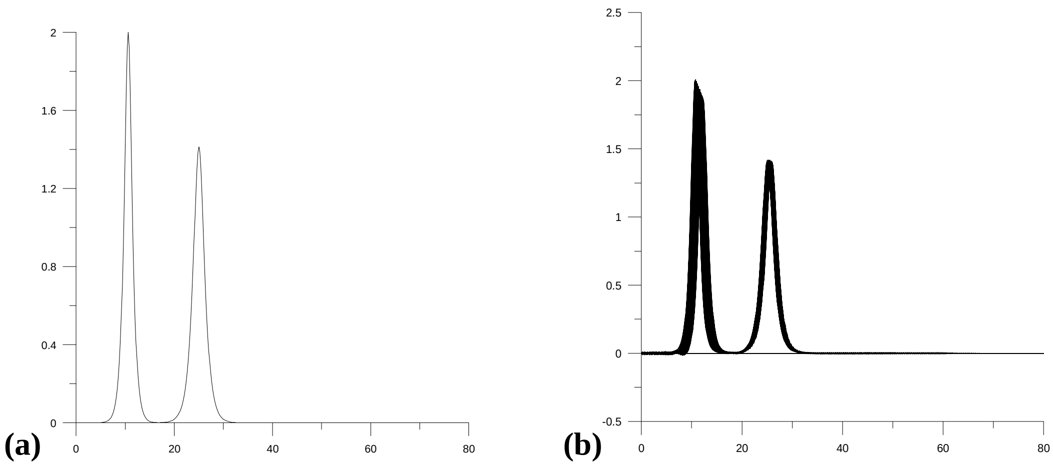


Fig. 6: Interaction two solitary waves at times:
(a) =T=0, **(b)** T=1

7.1.3 Interaction of Three Soliton

In this test we choose the initial condition for three waves

$$u(x,0)=\sum_{i=1}^{3} a_i \sec h\left[\sqrt{\frac{c_i}{\mu}}\left(x-x_i\right)\right], \qquad a_i=\sqrt{\frac{6c_i}{\varepsilon}}, \quad i=1,2,3$$

where c_1=2, c_2=1, c_3=0.5, x_1=15, x_2=25, x_3=35. The conserved quantities are given in "Table" 9.

Table 9: Interaction of three solitary waves

T	I_1	I_2	I_3
1.0	13.07780	11.85946	8.88135
2.0	12.90114	11.46870	7.74215
3.0	12.80053	11.19353	6.95347
4.0	12.67308	10.98244	6.38317
5.0	12.57683	10.80949	6.10943

The conserved quantities are given in "Table" 9. We found that the computed quantities I_1, I_2, and I_3 change by less than 0.5, 1.04 and 2.7.

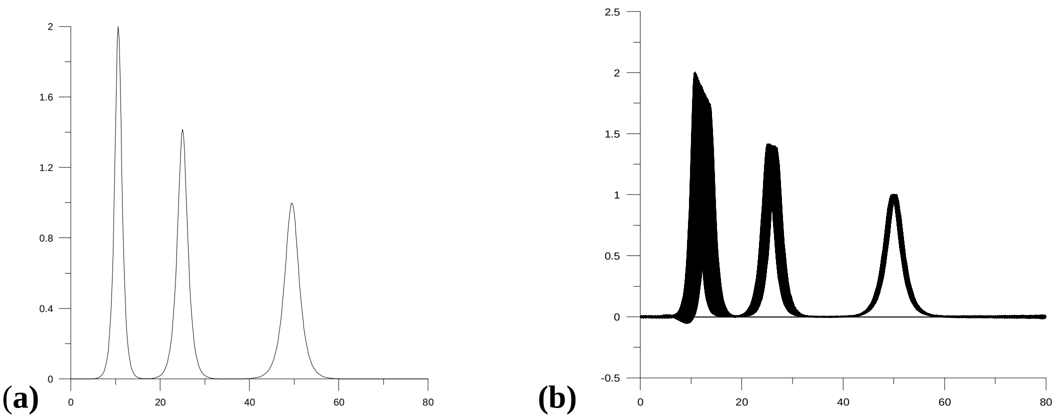


Fig. 7: Interaction three solitary waves at times: **(a)** =T=0, **(b)** T=2

8 Conclusions

In this paper, the finite difference method was applied to study the solitary waves of the MKdV equation. We test our schemes through single solitary wave in which the analytical solution is known, and then extended it to study the interaction of two and three solitons. We have noticed that the schemes keep the conserved quantities are almost constants during the calculations.

References

[1] R.D. Richtmyer and K.W. Morton, *Difference Methods for Initial Value Problems (2nd ed.)*, Interscience, New York, (1967).

[2] M. Wadati, The modified Korteweg–de Vries equation, *J. Phys. Soc. Japan*, 34(1972), 1289-1296.

[3] J.C. Eilbeck and G.R. Mcguire, Numerical study of the regularized long-wave equation, *J. Computational Phys.*, 19(1975), 43-57.

[4] M. Wadati and K. Ohkuma, Multiple pole solutions of the modified Korteweg de Vries equation, *Phys. Soc. Japan*, 51(1982), 2029-2035.

[5] A.M. Wazwaz, Solitary wave solutions for the modified Kdv equation by Adomian decomposition method, *International Journal of Appl. Math.*, 4(2000), 361-368.

[6] P.C. Jain, R. Shankar and T.V. Singh, Numerical solutions of regularized long wave equation, *Comm. Num. Meth. Eng*, 9(1993), 587-594.