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Numerical Solution of Eighth Order Boundary Value Problems with Variational Iteration Method

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Abstract

In this article, variational iteration method (VIM) is considered to solve eighth order boundary value problems. To illustrate the ability and reliability of the method, some examples are given, revealing its effectiveness and simplicity.

Keywords: *Variational iteration method; Boundary value problems*

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1 Introduction

The variational iteration method is a powerful tool for solving nonlinear differential equations by an iterative formula. This method was first proposed by He [1–3], and has been extensively worked out over a number of years by numerous authors. This method solves the problems without any need to discrete

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the variables. Therefore, there is no need to compute the round off errors and one is not faced with necessity of large computer memory and time.

Consider the following boundary value problem

$$\begin{aligned}
&y^{8}(x)+f(x)y(x)=g(x), \qquad x\in[a,b],\\
&y(a)=\alpha_0, \qquad y(b)=\alpha_1,\\
&y^{(1)}(a)=\gamma_0, \qquad y^{(1)}(b)=\gamma_1,\\
&y^{(2)}(a)=\delta_0, \qquad y^{(2)}(b)=\delta_1,\\
&y^{(3)}(a)=\eta_0, \qquad y^{(3)}(b)=\eta_1,
\end{aligned} \tag{1}$$

Where $\alpha_i, \gamma_i, \delta_i$ and η_i; $i=0,1$ are finite real constants while the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$.
To illustrate the method, consider the following general functional equation

$$Lu(t)+N(t)=g(t), \tag{2}$$

Where L is a linear operator, N is a non-linear operator and $g(t)$ is a known analytical function. According to the variational iteration method, we can construct the following correction functional

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(\xi)\{Lu_n(\xi)+N\tilde{u}_n(\xi)-g(\xi)\}d\xi, \tag{3}$$

Where λ is a general Lagrange multiplier which can be identified optimally via variational theory, u_0 is an initial approximation with possible unknowns, and $\tilde{u}_n$ is considered as restricted variation, i.e., $\delta\tilde{u}_n=0$ [4].

2 numerical examples

In this section, we present examples of eighth order boundary value problems and results will be compared with the exact solutions.

Example1. Consider the following eighth order boundary value problems [5]:

$$y^{(8)}(x)+x\,y(x)=-\left(48+15x+x^3\right)e^x, \quad 0<x<1, \tag{4}$$

With the boundary conditions:

$$\begin{aligned} &y(0)=0, && y(1)=0,\\ &y^{(1)}(0)=1, && y^{(1)}(1)=-e,\\ &y^{(2)}(0)=0, && y^{(2)}(1)=-4e,\\ &y^{(3)}(0)=-3, && y^{(3)}(1)=-9e, \end{aligned} \qquad (5)$$

The analytical solution of the above problem is given by,

$$y(x)=x(1-x)e^{x}\,. \qquad (6)$$

In the view of the variational iteration method, we construct a correction functional in the following form:

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(\xi)\left\{y_n^{(8)}(\xi)+\xi\tilde{y}_n(\xi)+\left(48+15\xi+\xi^3\right)e^{\xi}\right\}d\xi, \qquad (7)$$

To find the optimal $\lambda(s)$, calculation variation with respect to y_n, we have the following stationary conditions

$$\begin{aligned} &\delta y_n:\lambda^{(8)}(\xi)=0,\\ &\delta y_n^{(7)}:\lambda(\xi)\big|_{\xi=x}=0,\\ &\delta y_n^{(6)}:\lambda'(\xi)\big|_{\xi=x}=0,\\ &\vdots\\ &\delta y_n:1-\lambda^{(7)}(\xi)\big|_{\xi=x}=0. \end{aligned} \qquad (8)$$

The Lagrange multiplier, therefore can identified as $\lambda=\dfrac{-(x-\xi)^{(7)}}{7!}$.

Substituting the identified multiplier into Eq. (7), we have the following iteration formula:

$$y_{n+1}(x)=y_n(x)-\int_0^x \frac{(x-\xi)^{(7)}}{7!}\left\{y_n^{(8)}(\xi)+\xi y_n(\xi)+\left(48+15\xi+\xi^3\right)e^{\xi}\right\}d\xi, \qquad (9)$$

Starting with the initial approximation $y_0 = x - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{8}x^5$ in Eq. (9) successive approximations $y_i(x)$'s will be achieved. The plot of exact solution Eq. (4), the 6th order of approximate solution obtained using the VIM and absolute error between the exact and numerical solutions of this example are shown in Fig. 1.

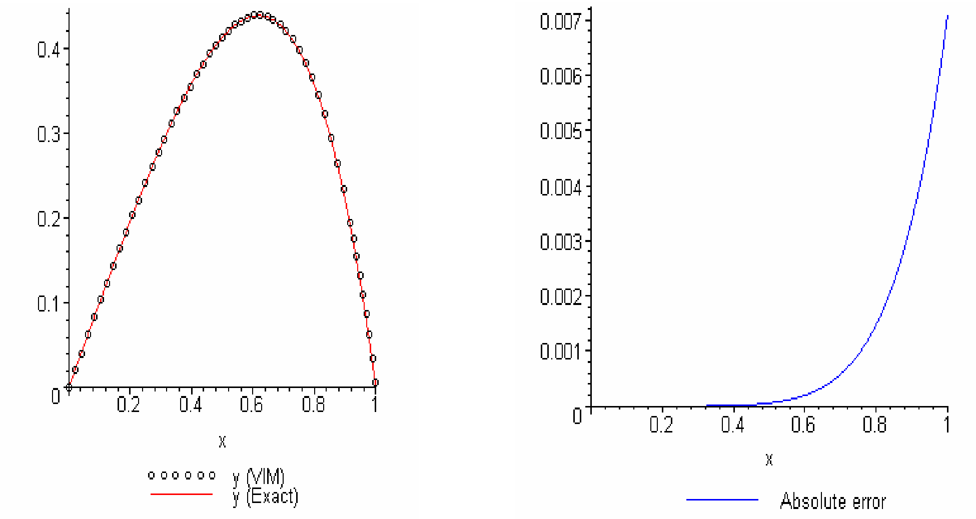

Fig. 1. The plots of approximate solution, exact solution and absolute error for Example 1.

Example 2. . Now we consider the following eighth order boundary value problems:

$$y^{(8)}(x) - y(x) = -8(2x\cos x + 7\sin x), \quad 0 < x < 1, \tag{10}$$

With the boundary conditions:

$$\begin{aligned} &y(0)=0, && y(1)=0,\\ &y^{(1)}(0)=-1, && y^{(1)}(1)=2\sin(1),\\ &y^{(2)}(0)=0, && y^{(2)}(1)=-4\cos(1)-2\sin(1),\\ &y^{(3)}(0)=7, && y^{(3)}(1)=6\cos(1)-6\sin(1), \end{aligned} \tag{11}$$

The analytical solution of the above problem is given by,

$$y(x)=(x^2-1)\sin x\,. \tag{12}$$

For solving by VIM we obtain the recurrence relation

$$y_{n+1}(x)=y_n(x)-\int_0^x \frac{(x-\xi)^{(7)}}{7!}\left\{y^{(8)}(\xi)-y(\xi)+8(2\xi\cos\xi+7\sin\xi)\right\}d\xi, \tag{13}$$

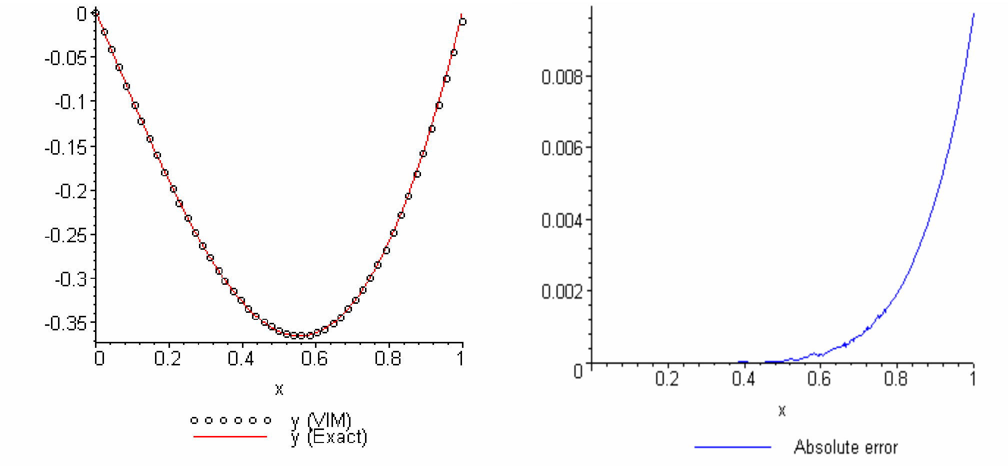

Fig. 2. The plots of approximate solution, exact solution and absolute error for Example 2.

Using the initial approximation $y_0=-x+\frac{7}{6}x^3-\frac{7}{40}x^5$ in Eq.(13), approximations $y_i(x)$'s will be calculated, successively. The plot of exact solution Eq.(10), the 6th order of approximate solution obtained using the VIM and absolute error between the exact and numerical solutions of this example are shown in Fig. 2.

3 Conclusion

We introduced a simple method with high accuracy for solving eighth order boundary value problems. This approach is simple in applicability as it does not require linearization, discretization or perturbation like other numerical and approximate methods. This method will be developed by authors for solving eighth order boundary value problems. The results showed that the convergence and accuracy of the variational iteration method for numerically analyzed eighth order boundary value problems was in a good agreement with the analytical solutions. The computations associated with the examples in this paper were performed using maple 13.

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