

A FREE BOUNDARY PROBLEM DESCRIBING THE SATURATED-UNSATURATED FLOW IN A POROUS MEDIUM

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This paper presents a functional approach to a nonlinear model describing the complete physical process of water infiltration into an unsaturated soil, including the saturation occurrence and the advance of the wetting front. The model introduced in this paper involves a multivalued operator covering the simultaneous saturated and unsaturated flow behaviors and enhances the study of the displacement of the free boundary between these two flow regimes. The model resides in Richards' equation written in pressure form with an initial condition and boundary conditions which in this work express the inflow due to the rain on the soil surface on the one hand, and characterize a certain permeability corresponding to the underground boundary, on the other hand. Existence, uniqueness, and regularity results for the transformed model in diffusive form, that is, for the moisture of the soil, and the existence of the weak solution for the pressure form are proved in the 3D case. The main part of the paper focuses on the existence of the free boundary between the saturated and unsaturated parts of the soil, and this is proved, in the 1D case, for certain stronger assumptions on the initial data and boundary conditions.

1. Introduction

The paper has the purpose of introducing a mathematical model able to describe the complete phenomenon of water infiltration into an unsaturated soil, the evolution of soil moisture up to saturation, and the advance of the interface between the saturated and unsaturated regions. From the hydraulic point of view the problem relies on the Darcian flow of an incompressible fluid in an isotropic, homogeneous porous medium with a constant porosity in the absence of evaporation. Under certain conditions depending on the rate at which rain water is supplied, the initial moisture distribution in the soil, the presence of underground sources, and the boundary permeability, the saturation of the ground surface could occur at the so-called *saturation time*. Consequently a water-front starts to move downwards and this represents the unknown interface between the saturated and unsaturated flow regimes. In these problems the hydraulic functions empirically introduced by soil scientists, but characterizing with good results the hydraulics

properties of various soils, raise a difficult mathematical problem: when the moisture of the soil comes close to the saturation value, the diffusivity expressed as a function of moisture blows up, and so saturation could not be mathematically described, see, for example, [14]. Correspondingly, another hydraulic function standing for a coefficient of the equation written in pressure form vanishes when pressure tends to zero and forces this equation to degenerate. In the mathematical literature devoted to this subject, this particularity was avoided by considering a finite valued diffusivity, see [1, 2, 9, 12, 13]. In [1], for instance, a time-dependent saturated-unsaturated flow was treated in the case of time-dependent water levels and an existence result for the corresponding weak formulation was proved. More recently, in [8], a model of the saturated-unsaturated flow lying on a special definition of the boundary conditions that changes during the phenomenon evolution, has been developed also for a finite value of the diffusivity at saturation.

In [5, 6, 11] the model of unsaturated infiltration with a blowing up diffusivity was treated in the framework of the semigroup theory.

By the model introduced in this paper the difficulty arisen at the interface between the saturated and unsaturated regimes is surpassed, and so the complete mathematical description of the saturated-unsaturated flow is enabled. Starting from the classical model using Richards' equation, the definition of the weak solution in pressure form is given and the connection with the saturated-unsaturated model is made. Then, passing from pressure to moisture by the aid of the hydraulic functions, a specific model is stated by introducing a multivalued operator that characterizes the behaviors at the saturated-unsaturated interface. Using an approximating model, that provides necessary results, the existence and uniqueness of the solution to this specific model and the existence of the weak solution are proved in the 3D case. However, the uniqueness of the weak solution (in pressure form) does not follow under the hypotheses used up to now, and this requires first the assurance of the free boundary existence. Imposing stronger assumptions, some supplementary regularity properties of the solution to the approximating problem are found and these enable the proof of the existence of the free boundary in the 1D case. Finally, the uniqueness of the weak and smooth solution is proved.

2. The mathematical model

The behaviors of an unsaturated porous soil, so partially filled with water, is completely known from the hydraulic point of view if two functions are given: one is the hydraulic conductivity k and the other is the constitutive relationship linking the volumetric water content, or moisture of the soil θ , with the pressure head h . They depend nonlinearly on h .

Since an unsaturated soil is characterized by convenience by negative pressure ($h < 0$) (see [7]), these functions are defined as follows.

(a₁) $k : [h_r, 0) \rightarrow [K_r, K_s]$; $\theta : [h_r, 0) \rightarrow [\theta_r, \theta_s]$, and they are positive and differentiable, with $h_r < 0$.

The value θ_s is the value of moisture at saturation and $\theta_r = \theta(h_r)$ is the residual value of moisture, meaning the moisture resident in a dried soil. From practice it is known that after all drainage forces have ceased, a soil retains a small amount of water which does no longer enter in the water circulation, so it follows that $\theta_r > 0$. The values K_s and K_r are

the values of conductivity at saturation and for a dried soil, respectively . The positive values $\theta_r, \theta_s, K_r,$ and K_s are soil characteristics and they are known.

Generally, this type of processes displays a hysteretic behavior, especially when a cycle of wetting and drying processes happens. But if we assume that only one process takes place, for example, infiltration, then we can disregard the hysteretic aspect (see [7]).

So we may consider that

(a₂) $h \rightarrow \theta(h)$ and $h \rightarrow k(h)$ are single-valued and monotonically increasing functions on $[h_r, 0)$.

The derivative of θ with respect to h is called water capacity C and

(a₃) $h \rightarrow C(h)$ is a positive, bounded, continuous, and monotonically decreasing function on $[h_r, 0)$.

We also assume the following property:

(i_K) there exists $M > 0$ such that $k'(h) \leq MC(h)$.

When a part of the soil begins to saturate itself, the pressure within it becomes positive, $h \geq 0$, and all these functions take constant values, namely,

(a₄) $\theta(h) = \theta_s, k(h) = K_s,$ and $C(h) = 0,$ for $h \geq 0$.

We are now ready to formulate the mathematical model of infiltration of an incompressible fluid (water) into an isotropic and homogeneous porous medium with a constant porosity.

Let Ω be an open bounded subset of \mathbb{R}^N ($N = 1, 2, 3$) with the boundary $\partial\Omega \stackrel{\text{notation}}{=} \Gamma$ sufficiently smooth (e.g., a piecewise $(N - 1)$ -dimensional manifold of class C^2), let $(0, T)$ be a finite time interval, and let $x \in \Omega$ represent the vector $x = (x_1, x_2, x_3)$. To be more specific, we will consider Ω to be the cylinder $\Omega = \{x; (x_1, x_2) \in D, 0 < x_3 < L\}$, where D is an open bounded subset of \mathbb{R}^{N-1} with smooth boundary. We consider that Γ is composed of the disjoint boundaries $\Gamma_u, \Gamma_{\text{lat}},$ and $\Gamma_b,$ all sufficiently smooth, and define $\Gamma_u = \{x \in \Gamma; x_3 = 0\}, \Gamma_b = \{x \in \Gamma; x_3 = L\},$ and $\Gamma = \Gamma_u \cup \Gamma_{\text{lat}} \cup \overline{\Gamma_b}$. We also denote $\Gamma_\alpha = \Gamma_{\text{lat}} \cup \overline{\Gamma_b},$ where $\Gamma_u \cap \Gamma_\alpha = \emptyset$. Correspondingly, we define $\Sigma_\alpha = \Gamma_\alpha \times (0, T), \Sigma_u = \Gamma_u \times (0, T), \Sigma_b = \Gamma_b \times (0, T),$ and $\Sigma_{\text{lat}} = \Gamma_{\text{lat}} \times (0, T)$.

The mathematical model describing the water infiltration into a soil with the properties specified above consists of Richards' equation written for the pressure head $h(x, t)$ (see [7]), with an initial condition and various boundary conditions. In this study we will take into account a more realistic situation, in which infiltration is produced by a rainfall (or irrigation) on the surface of the soil (Γ_u) and the domain considered has a kind of semipermeable boundary (Γ_α). The model reads

$$\begin{aligned}
 C(h) \frac{\partial h}{\partial t} - \nabla \cdot (k(h) \nabla h) + \frac{\partial k(h)}{\partial x_3} &= f \quad \text{in } Q = \Omega \times (0, T), \\
 h(x, 0) &= h_0(x) \quad \text{in } \Omega, \\
 q \cdot \nu &= u(x, t) \quad \text{on } \Sigma_u, \\
 q \cdot \nu &= \alpha(x) K^*(h) + f_0(x, t) \quad \text{on } \Sigma_\alpha.
 \end{aligned}
 \tag{2.1}$$

By q we denoted the flux defined by $q(x, t) = k(h)i_3 - k(h)\nabla h$, where ν is the outward normal to Γ and i_3 is the unit vector along Ox_3 , directed downwards, and α is a bounded, continuous, and positive function on Γ_α . The function K^* is a primitive of k that will be specified later.

For our mathematical purposes, we extend the functions C and k by continuity to the left of h_r such that $0 < C(h) \leq C_r, 0 < k(h) \leq K_r$, and

$$\lim_{h \rightarrow -\infty} \frac{k(h)}{C(h)} = \frac{K_r}{C_r} = \rho > 0, \quad \lim_{h \rightarrow -\infty} k(h) = \lim_{h \rightarrow -\infty} C(h) = 0. \tag{2.2}$$

To accomplish this we may define k and C by

$$k(h) = K_r \exp(h - h_r), \quad C(h) = C_r \exp(h - h_r) \quad \text{for } h < h_r, K_r \leq MC_r. \tag{2.3}$$

Hence we consider the functions C and k defined on \mathbb{R} with properties (a₁), (a₂), (a₃), and (a₄) and (2.2).

We define the primitives of C and K by

$$C^*(h) = \begin{cases} \theta_r + \int_{h_r}^h C(\zeta)d\zeta, & h < 0, \\ \theta_s, & h \geq 0, \end{cases} \tag{2.4}$$

$$K^*(h) = \begin{cases} \int_{-\infty}^h k(\zeta)d\zeta, & h < 0, \\ K_s^* + K_s h, & h \geq 0, K_s^* = K^*(0), \end{cases} \tag{2.5}$$

and denote $\theta = C^*(h)$. We notice that

$$C^* : (-\infty, \infty) \longrightarrow (\theta_{\min}, \theta_s], \quad \text{where } \theta_{\min} = \lim_{h \rightarrow -\infty} C^*(h) = \theta_r - C_r. \tag{2.6}$$

Both functions are continuous and monotonically increasing (C^* on $h < 0$) and with these notations the model becomes

$$\begin{aligned} \frac{\partial C^*(h)}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} &= f \quad \text{in } Q, \\ h(x, 0) &= h_0(x) \quad \text{in } \Omega, \\ q \cdot \nu &= u(x, t) \quad \text{on } \Sigma_u, \\ q \cdot \nu &= \alpha(x)K^*(h) + f_0(x, t) \quad \text{on } \Sigma_\alpha. \end{aligned} \tag{2.7}$$

3. Weak solution

Let V be the space $H^1(\Omega)$ endowed with the usual Hilbertian norm.

Definition 3.1. The function $h \in L^2(0, T; L^2(\Omega))$ is said to be a weak solution to problem (2.7) if $K^*(h) \in L^2(0, T; V)$ and

$$\begin{aligned} & \int_Q \left(-C^*(h)\phi_t(x, t) + \nabla K^*(h) \cdot \nabla \phi(x, t) - k(h) \frac{\partial \phi}{\partial x_3}(x, t) \right) dx dt \\ &= \int_\Omega \phi(x, 0) C^*(h_0(x)) dx - \int_{\Sigma_\alpha} (\alpha(x) K^*(h) + f_0(x, t)) \phi(x, t) d\sigma dt \\ & \quad - \int_{\Sigma_u} u(x, t) \phi(x, t) d\sigma dt + \int_Q f(x, t) \phi(x, t) dx dt \end{aligned} \tag{3.1}$$

for all $\phi \in L^2(0, T; V)$ with $\phi_t \in L^2(0, T; L^2(\Omega))$ and $\phi(x, T) \equiv 0$.

In (3.1) dx is the Lebesgue measure and $d\sigma$ is the surface measure. Further, if any confusion is avoided, we will no longer indicate in the integrands those function arguments which represent the integration variables.

Also we will denote by (\cdot, \cdot) and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega)$, respectively.

Suppose that the domain Q is divided into two very well delimited domains, corresponding to the saturated and unsaturated parts of the soil and the saturated part is above. Denote by $Q_- = \{(x, t); h(x, t) < 0\}$ the unsaturated part, $Q_+ = \{(x, t); h(x, t) > 0\}$ the saturated region, and $Q_0 = \{(x, t); h(x, t) = 0\}$ the free surface (boundary) separating the saturated part Q_+ from the unsaturated one, Q_- . We specify that ν signifies the normal to a boundary, no matter which boundary is in discussion, but we should keep in mind that the respective normal is always directed to the exterior of the domain delimited by that boundary. However, in order to avoid any confusion, we will mark by superscripts the normals to the interface, that is, by ν^+ we mean the normal to Q_0 directed towards Q_- , and by ν^- the normal to Q_0 directed to Q_+ , and we notice that $\nu^+ = -\nu^-$. Moreover, we denote

$$h^-(x_0, t_0) = \lim_{\substack{(x,t) \rightarrow (x_0,t_0) \\ (x,t) \in Q_-}} h(x, t), \quad q^-(x_0, t_0) = - \lim_{\substack{(x,t) \rightarrow (x_0,t_0) \\ (x,t) \in Q_-}} (q \cdot \nu^-)(x, t). \tag{3.2}$$

Similarly, by

$$h^+(x_0, t_0) = \lim_{\substack{(x,t) \rightarrow (x_0,t_0) \\ (x,t) \in Q_+}} h(x, t), \quad q^+(x_0, t_0) = \lim_{\substack{(x,t) \rightarrow (x_0,t_0) \\ (x,t) \in Q_+}} (q \cdot \nu^+)(x, t), \tag{3.3}$$

we denote the corresponding right-hand side limits. We recall that Ox_3 is directed downwards.

PROPOSITION 3.2. *If h is a weak, smooth solution to (2.7), then h is a solution to the following model describing the water infiltration into an unsaturated-saturated soil:*

$$C(h) \frac{\partial h}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} = f \quad \text{in } Q_-, \tag{3.4}$$

$$-K_s \Delta h = f \quad \text{in } Q_+, \tag{3.5}$$

$$h(x, 0) = h_0(x) \quad \text{in } \Omega, \tag{3.6}$$

$$q^+(x, t) = q^-(x, t) \quad \text{on } Q_0, \quad h^+(x, t) = h^-(x, t) = 0 \quad \text{on } Q_0, \tag{3.7}$$

$$q \cdot \nu = u(x, t) \quad \text{on } \Sigma_u, \quad q \cdot \nu = \alpha(x)K^*(h) + f_0(x, t) \quad \text{on } \Sigma_\alpha. \tag{3.8}$$

Proof. If the saturation occurs from above we will be able to represent Q_+ , Q_- , and Q_0 as $Q_+ = \{(x, t); 0 < x_3 < s(t, x_1, x_2)\}$, $Q_- = \{(x, t); s(t, x_1, x_2) < x_3 < L\}$, and $Q_0 = \{(x, t); x_3 = s(t, x_1, x_2)\}$, where $x_3 = s(t, x_1, x_2)$ is a smooth surface.

First, in (3.1), we take ϕ with compact support in Q_- , and then it follows, in the sense of distributions, that

$$-\int_{Q_-} C^*(h) \phi_t dx dt = \int_{Q_-} C_t^*(h) \phi dx dt. \tag{3.9}$$

Then

$$\int_{Q_-} \left(\nabla K^*(h) \cdot \nabla \phi - k(h) \frac{\partial \phi}{\partial x_3} \right) dx dt = \int_{Q_-} \left(-\Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} \right) \phi dx dt \tag{3.10}$$

so that we finally get from (3.1) that

$$\int_{Q_-} \left(C_t^*(h) - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} \right) \phi dx dt = \int_{Q_-} f \phi dx dt, \quad \forall \phi \in C_0^\infty(Q_-), \tag{3.11}$$

which implies that (3.4) is satisfied in the sense of distributions.

Similarly, if we take ϕ with compact support in Q_+ , we get (3.5).

Now we multiply (3.4) by ϕ , integrate it over Q_- , and add it to (3.5) multiplied by ϕ and integrated over Q_+ . After some integrations by parts we obtain

$$\begin{aligned} & \int_{Q_-} \left(-C^*(h) \phi_t + \nabla K^*(h) \cdot \nabla \phi - k(h) \frac{\partial \phi}{\partial x_3} \right) dx dt + \int_{\Sigma_b} q \cdot \nu \phi d\sigma dt \\ & + \int_{\Sigma_{\text{lat}}^-} q \cdot \nu \phi d\sigma dt - \int_{Q_0} q \cdot \nu^- \phi d\sigma dt + \int_{Q_+} K_s \nabla h \cdot \nabla \phi dx dt \\ & + \int_{\Sigma_u} q \cdot \nu \phi d\sigma dt + \int_{\Sigma_{\text{lat}}^+} q \cdot \nu \phi d\sigma dt + \int_{Q_0} q \cdot \nu^+ \phi d\sigma dt \\ & = \int_{Q_-} f \phi dx dt + \int_{Q_+} f \phi dx dt + \int_{\Omega_-} (C^*(h)\phi)(x, 0) dx + \int_{\Omega_+} (C^*(h)\phi)(x, 0) dx. \end{aligned} \tag{3.12}$$

Here, Ω_\pm are the spatial domains corresponding to Q_\pm , and Σ_{lat}^\pm are the lateral boundaries corresponding to Q_\pm , with $\Sigma_{\text{lat}}^+ \cup \overline{\Sigma_{\text{lat}}^-} = \Sigma_{\text{lat}}$, $\Sigma_{\text{lat}}^+ \cap \Sigma_{\text{lat}}^- = \emptyset$.

Taking into account (3.1) we get

$$\begin{aligned}
 & \int_{\Omega} \phi(x,0)C^*(h_0)dx - \int_{\Sigma_{\alpha}} (\alpha K^*(h) + f_0)\phi d\sigma dt - \int_{\Sigma_{u_i}} u\phi d\sigma dt \\
 & + \int_{\Sigma_b} q \cdot \nu \phi d\sigma dt + \int_{\Sigma_{\text{lat}}^-} q \cdot \nu \phi d\sigma dt - \int_{Q_0} q^- \phi d\sigma dt \\
 & + \int_{\Sigma_u} q \cdot \nu \phi d\sigma dt + \int_{\Sigma_{\text{lat}}^+} q \cdot \nu \phi d\sigma dt + \int_{Q_0} q^+ \phi d\sigma dt \\
 & = \int_{\Omega_-} (C^*(h)\phi)(x,0)dx + \int_{\Omega_+} (C^*(h)\phi)(x,0)dx,
 \end{aligned} \tag{3.13}$$

for each ϕ with the properties from Definition 3.1. Since ϕ is arbitrary, we obtain $q^+ = q^-$ on Q_0 , $q \cdot \nu = \alpha K^*(h) + f_0$ on Σ_{α} , $q \cdot \nu = u$ on Σ_u , and $C^*(h_0(x)) = (C^*(h))(x,0)$. The condition related to zero-pressure continuity on Q_0 is implied by the assumption that h is smooth and by the definition of Q_0 . It must be emphasized that the three conditions on the boundary Q_0 are necessary since the free boundary is unknown. \square

4. Transformed problem

It will be convenient to work with the variable θ , hence, to this end we introduce the inverse of C^*

$$h = (C^*)^{-1}(\theta), \quad \theta_{\min} < \theta < \theta_s, \quad (C^*)^{-1}(\theta_s) = [0, +\infty) \tag{4.1}$$

and replace all over in (2.7) h by (4.1). So we get the multivalued function

$$\beta^*(\theta) = K^*((C^*)^{-1}(\theta)), \quad \theta \in (\theta_{\min}, \theta_s), \quad \beta^*(\theta_s) = [K_s^*, +\infty), \tag{4.2}$$

and the conductivity expressed as a function of θ

$$K(\theta) = k((C^*)^{-1}(\theta)), \quad \theta \in (\theta_{\min}, \theta_s]. \tag{4.3}$$

For $\theta \in (\theta_{\min}, \theta_s)$, we can calculate the derivative of $\beta^*(\theta)$, denoted by $\beta(\theta)$, which turns out to be given by

$$\beta(\theta) = \frac{k((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))} \geq \frac{K_r}{C_r} = \rho, \tag{4.4}$$

and assume that it is convex in order to respect the physical model.

Taking into account (2.2) we can extend these functions to the left of θ_{\min} by setting

$$\beta(\theta) = \rho, \quad K(\theta) = 0 \quad \text{for } \theta \leq \theta_{\min}, \tag{4.5}$$

so that we finally get

$$\beta^*(\theta) = \begin{cases} \rho\theta, & \theta \leq \theta_{\min}, \\ K^*((C^*)^{-1}(\theta)), & \theta_{\min} < \theta < \theta_s, \\ [K_s^*, +\infty), & \theta = \theta_s, \end{cases} \tag{4.6}$$

$$K(\theta) = \begin{cases} 0, & \theta \leq \theta_{\min}, \\ k((C^*)^{-1}(\theta)), & \theta_{\min} < \theta \leq \theta_s. \end{cases} \tag{4.7}$$

For $\theta < \theta_s$, we also have $\lim_{\theta \nearrow \theta_s} \beta^*(\theta) = \lim_{\theta \nearrow \theta_s} K^*((C^*)^{-1}(\theta)) = \lim_{h \nearrow 0} K^*(h) = K_s^*$. The function β^* defined by (4.6) satisfies

- (i) $(\beta^*(\theta) - \beta^*(\bar{\theta}))(\theta - \bar{\theta}) \geq \rho(\theta - \bar{\theta})^2$, for all $\theta, \bar{\theta} \in (-\infty, \theta_s]$,
- (ii) $\lim_{\theta \rightarrow -\infty} \beta^*(\theta) = -\infty$.

Due to (i_K) it follows that $\theta \rightarrow K(\theta)$ is Lipschitz, that is, (ii_K) $|K(\theta) - K(\bar{\theta})| \leq M|\theta - \bar{\theta}|$, for all $\theta, \bar{\theta} \leq \theta_s$.

Condition (i) can be very easily checked for $\theta, \bar{\theta} < \theta_s$, or $\theta = \bar{\theta} = \theta_s$. If $\theta = \theta_s$ and $\bar{\theta} < \theta_s$, we have

$$\begin{aligned} (\beta^*(\theta_s) - \beta^*(\bar{\theta}))(\theta_s - \bar{\theta}) &\geq (K_s^* - \beta^*(\bar{\theta}))(\theta_s - \bar{\theta}) \\ &= \left(\lim_{\theta \nearrow \theta_s} K^*((C^*)^{-1}(\theta)) - \beta^*(\bar{\theta}) \right) (\theta_s - \bar{\theta}) \geq \rho(\theta_s - \bar{\theta})^2. \end{aligned} \tag{4.8}$$

With these notations the mathematical model describing the saturated-unsaturated case is reduced to the nonlinear diffusion equation

$$\frac{\partial \theta}{\partial t} - \Delta \beta^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \quad \text{in } Q, \tag{4.9}$$

$$\theta(x, 0) = \theta_0(x) \quad \text{in } \Omega, \tag{4.10}$$

$$(K(\theta)i_3 - \nabla \beta^*(\theta)) \cdot \nu = u \quad \text{on } \Sigma_u, \tag{4.11}$$

$$(K(\theta)i_3 - \nabla \beta^*(\theta)) \cdot \nu = \alpha \beta^*(\theta) + f_0 \quad \text{on } \Sigma_\alpha. \tag{4.12}$$

From this point we may further have two approaches. If we are interested in the study of the occurrence of saturation in a porous medium and in the advance of the free boundary between the saturated and unsaturated domains, we will analyze model (4.9) for $\theta \leq \theta_s$, with $\beta^*(\theta)$ being the multivalued operator given by (4.6), with properties (i), (ii), and K being the continuous function given by (4.7) satisfying (ii_K).

But in soil sciences, the interest is often in the unsaturated flow only, so that model (4.9) is studied for $\theta < \theta_s$, with β^* being the function defined strictly in this domain, satisfying (i) but with blowing up at θ_s . This situation has already been considered in [6] and for a stratified soil in [5].

The theoretical results presented in the work can be applied to the parametric hydraulic model of Broadbridge and White introduced in [14], which in our notations read as follows:

$$K(\theta) = \frac{(c-1)\theta^2}{c-\theta}, \quad \beta(\theta) = \frac{c(c-1)}{(c-\theta)^2}. \tag{4.13}$$

Here, $c \in (1, \infty)$ is a parameter that indicates the degree of nonlinearity of the medium, that is, when c is close to 1, $\theta \rightarrow 1$ and the soil is strongly nonlinear. Moreover, $K(0) = 0$ and $\beta(0) = (c - 1)/c > 0$, corresponding to the conditions imposed above for the case $\theta_{\min} = 0$.

Functional framework. We consider the space $V = H^1(\Omega)$, with the norm defined by

$$\|\psi\|_V = \left(\int_{\Omega} |\nabla \psi|^2 dx + \int_{\Gamma_\alpha} \alpha(x) |\psi|^2 d\sigma \right)^{1/2} \tag{4.14}$$

which is equivalent to the usual norm on $H^1(\Omega)$. Let $V' = (H^1(\Omega))'$ be the dual of V . It is convenient to endow the dual V' with the scalar product

$$\langle \theta, \bar{\theta} \rangle_{V'} = (\theta, \psi), \quad \forall \theta, \bar{\theta} \in V', \tag{4.15}$$

where $\psi \in V$ satisfies the boundary value problem

$$-\Delta \psi = \bar{\theta}, \quad \frac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \quad \text{on } \Gamma_\alpha, \quad \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \Gamma_u, \tag{4.16}$$

meaning that

$$\int_{\Omega} \nabla \psi \cdot \nabla \phi dx + \int_{\Gamma_\alpha} \alpha \psi \phi d\sigma = \bar{\theta}(\phi), \quad \forall \phi \in V, \tag{4.17}$$

($\partial/\partial \nu$ is the normal derivative). Obviously $\|\psi\|_V = \|\bar{\theta}\|_{V'}$. We set

$$D(A) = \{ \theta \in L^2(\Omega); \exists \eta \in V \text{ and } \eta(x) \in \beta^*(\theta(x)) \text{ a.e. } x \in \Omega \} \tag{4.18}$$

and we define the multivalued operator $A : D(A) \subset V' \rightarrow V'$ by

$$(A\theta, \psi) = \int_{\Omega} \left(\nabla \eta \cdot \nabla \psi - K(\theta) \frac{\partial \psi}{\partial x_3} \right) dx + \int_{\Gamma_\alpha} \alpha \eta \psi d\sigma, \quad \forall \psi \in V. \tag{4.19}$$

We still define $B \in L(L^2(\Gamma_u); V')$ and $f_\Gamma \in L^2(0, T; V')$ by

$$\begin{aligned} Bu(\psi) &= - \int_{\Gamma_u} u \psi d\sigma, \quad \forall \psi \in V, \\ f_\Gamma(t)(\psi) &= - \int_{\Gamma_\alpha} f_0 \psi d\sigma, \quad \forall \psi \in V, \end{aligned} \tag{4.20}$$

and so we are led to the Cauchy problem

$$\frac{d\theta}{dt} + A\theta \ni f + Bu + f_\Gamma \quad \text{a.e. } t \in (0, T), \tag{4.21}$$

$$\theta(0) = \theta_0(x) \quad \text{in } \Omega. \tag{4.22}$$

Equation (4.21) can still be written equivalently in the form

$$\int_{\Omega} \left(\frac{\partial \theta}{\partial t} \psi + \nabla \eta \cdot \nabla \psi - K(\theta) \frac{\partial \psi}{\partial x_3} \right) dx \tag{4.23}$$

$$= \int_{\Omega} f \psi \, dx - \int_{\Gamma_{\alpha}} (\alpha \eta + f_0) \psi \, d\sigma - \int_{\Gamma_u} u \psi \, d\sigma, \quad \forall \psi \in V, \, t \in (0, T),$$

for some $\eta \in \beta^*(\theta)$. One can show that if θ is a strong solution to the Cauchy problem (4.21)-(4.22), then it satisfies (4.9)-(4.12) (see [6]).

4.1. The approximating problem. We now assume that β^* is defined to be of class C^3 on $\theta < \theta_s$ and such that it preserves the condition that $\lim_{\theta \rightarrow -\infty} \beta^*(\theta) = -\infty$. That means we have imposed that the functions C and k should be of class C^2 on $h < 0$. Although this assumption is concordant with the physical model and could be imposed without any loss of generality, it is not absolutely necessary since we can approximate a non sufficiently smooth function by sequences of functions having the necessary smoothness, and pass then to the limit in the approximating equations.

So, in order to prove the existence and uniqueness results, we approximate β^* by the continuous function

$$\beta_{\varepsilon}^*(\theta) = \begin{cases} \beta^*(\theta), & \theta < \theta_s, \\ K_s^* + \frac{\theta - \theta_s}{\varepsilon}, & \theta \geq \theta_s, \end{cases} \tag{4.24}$$

for each $\varepsilon > 0$, so $\beta_{\varepsilon}^*(\theta)$ satisfies properties (i), (ii), and

$$(iii) \lim_{\theta \rightarrow -\infty} \beta_{\varepsilon}^*(\theta) = +\infty.$$

Another way to approximate β^* is the following, for which β_{ε}^* is continuous and differentiable except for $\theta_s - \varepsilon$:

$$\beta_{\varepsilon}^*(\theta) = \begin{cases} \beta^*(\theta), & \theta < \theta_s - \varepsilon, \\ \beta^*(\theta_s - \varepsilon) + \frac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon} [\theta - (\theta_s - \varepsilon)], & \theta \geq \theta_s - \varepsilon. \end{cases} \tag{4.25}$$

Hence we have to study the approximating problem

$$\frac{d\theta_{\varepsilon}}{dt} + A_{\varepsilon} \theta_{\varepsilon} = f + Bu + f_{\Gamma} \quad \text{a.e. } t \in (0, T), \tag{4.26}$$

$$\theta_{\varepsilon}(0) = \theta_0(x) \quad \text{in } \Omega, \tag{4.27}$$

where $A_{\varepsilon} : D(A_{\varepsilon}) \subset V' \rightarrow V'$ is a single-valued operator defined by

$$(A_{\varepsilon} \theta, \psi) = \int_{\Omega} \left(\nabla \beta_{\varepsilon}^*(\theta) \cdot \nabla \psi - K(\theta) \frac{\partial \psi}{\partial x_3} \right) dx + \int_{\Gamma_{\alpha}} \alpha \beta_{\varepsilon}^*(\theta) \psi \, d\sigma, \quad \forall \psi \in V, \tag{4.28}$$

with the domain

$$D(A_\varepsilon) = \{\theta \in L^2(\Omega); \beta_\varepsilon^*(\theta) \in V\}. \tag{4.29}$$

4.2. Main results. The results presented below refer to the properties of the operator A_ε (Proposition 4.1), existence, uniqueness, and regularity of the solution to the approximating problem (Proposition 4.2 and Theorem 4.3), existence and uniqueness of the solution to the exact Cauchy problem (Theorem 4.4), some properties of the solution (Corollaries 4.5 and 4.6), and the existence of the weak solution (Corollary 4.8). In all these proofs we will use the approximation (4.24).

PROPOSITION 4.1. *Under the conditions (i)–(iii), (ii_K) the operator A_ε is quasi m -accretive in V' , meaning that*

$$\langle (\lambda I + A_\varepsilon)\theta - (\lambda I + A_\varepsilon)\bar{\theta}, \theta - \bar{\theta} \rangle_{V'} \geq 0 \tag{4.30}$$

for $\lambda > 0$ large enough, and

$$R(\lambda I + A_\varepsilon) = V' \tag{4.31}$$

for some λ sufficiently large.

Since the proofs of Propositions 4.1 and 4.2 and Theorem 4.3 are essentially the same as those of [6, Proposition 1, Theorem 1, and Corollary 1, (a)], they will be omitted, and for their details, we refer the reader to this work.

Let $j_\varepsilon : \mathbb{R} \rightarrow (-\infty, \infty)$ be defined by

$$j_\varepsilon(r) = \int_0^r \beta_\varepsilon^*(\xi) d\xi. \tag{4.32}$$

By Proposition 4.1 and by standard existence results for nonlinear accretive differential equations (see [4]), we obtain the following proposition.

PROPOSITION 4.2. *Let*

$$f \in W^{1,1}(0, T; V'), \quad f_0 \in W^{1,1}(0, T; L^2(\Gamma_\alpha)), \quad u \in W^{1,1}(0, T; L^2(\Gamma_u)), \tag{4.33}$$

$$\theta_0 \in D(A_\varepsilon)$$

and assume that conditions (i)–(iii), (ii_K) hold. Then, for each ε , there exists a unique strong solution θ_ε to problem (4.26)–(4.27) such that

$$\begin{aligned} \theta_\varepsilon &\in L^\infty(0, T; D(A_\varepsilon)) \cap W^{1,\infty}(0, T; V'), \\ \beta_\varepsilon^*(\theta_\varepsilon) &\in L^\infty(0, T; V), \quad \theta_\varepsilon \in L^\infty(0, T; V), \\ j_\varepsilon(\theta_\varepsilon) &\in L^\infty(0, T; L^1(\Omega)). \end{aligned} \tag{4.34}$$

Moreover, the solution satisfies the estimates

$$\begin{aligned} & \|\theta_\varepsilon(t)\|_{V'}^2 + \int_0^t \|\theta_\varepsilon(\tau)\|_{V'}^2 d\tau \\ & \leq \gamma_1(\alpha_m) \left(\|\theta_0\|_{V'}^2 + \int_0^T \|f(\tau)\|_{V'}^2 d\tau + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 d\tau \right), \end{aligned} \tag{4.35}$$

$$\begin{aligned} \|\theta_\varepsilon(t)\|^2 & \leq \int_\Omega j_\varepsilon(\theta_\varepsilon(t)) dx + \int_0^t \left\| \frac{d\theta_\varepsilon}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_{V'}^2 d\tau \\ & \leq \gamma_2(\alpha_m) \left(\int_\Omega j_\varepsilon(\theta_0) dx + \int_0^T \|f(\tau)\|_{V'}^2 d\tau \right. \\ & \quad \left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 d\tau \right). \end{aligned} \tag{4.36}$$

In the above estimates, $\alpha_m = \min_{x \in \Gamma_\alpha} \alpha(x)$, $\gamma_1(\alpha_m) = O(1/\alpha_m)$, and $\gamma_2(\alpha_m) = O(1/\alpha_m)$. Also we notice that it follows that $K(\theta_\varepsilon) \in L^\infty(0, T; V)$.

Let now $j : \mathbb{R} \rightarrow (-\infty, \infty]$ be defined by

$$j(r) = \begin{cases} \int_0^r \beta^*(\xi) d\xi, & r \leq \theta_s, \\ +\infty, & r > \theta_s, \end{cases} \tag{4.37}$$

where, by convention, $\beta^*(\theta_s) = \lim_{\substack{\xi \rightarrow \theta_s \\ \xi < \theta_s}} \beta^*(\xi) = K_s^*$.

Then (see [11]) j is a proper, convex, and lower semicontinuous function on \mathbb{R} and

$$\partial j(r) = \beta^*(r). \tag{4.38}$$

In particular, β^* is a maximal monotone operator from \mathbb{R} to \mathbb{R} .

Denote $M_{\theta_s} = \{\theta \in L^2(\Omega); \theta \leq \theta_s \text{ a.e. on } \Omega\}$ and $M_j = \{\theta \in L^2(\Omega); j(\theta) \in L^1(\Omega)\}$. Then $M_{\theta_s} \subset \overline{M_j}$.

THEOREM 4.3. *Let*

$$f \in L^2(0, T; V'), \quad u \in L^2(0, T; L^2(\Gamma_u)), \quad f_0 \in L^2(0, T; L^2(\Gamma_\alpha)), \quad \theta_0 \in M_{\theta_s}. \tag{4.39}$$

Then, problem (4.26)-(4.27) has, for each $\varepsilon > 0$, a unique solution that satisfies estimates (4.35)-(4.36) and

$$\theta_\varepsilon \in W^{1,2}(0, T; V'), \quad \beta_\varepsilon^*(\theta) \in L^2(0, T; V), \quad \theta_\varepsilon \in L^2(0, T; V). \tag{4.40}$$

Proof. If $\theta_0 \leq \theta_s$, we have

$$j_\varepsilon(\theta_0) = \int_0^{\theta_0} \beta_\varepsilon^*(\theta) d\theta \leq \int_0^{\theta_s} \beta_\varepsilon^*(\theta) d\theta \leq \int_0^{\theta_s} \beta^*(\theta) d\theta = j(\theta_s) \leq K_s^* \theta_s. \tag{4.41}$$

This means that $\theta_0 \in D(\varphi_\varepsilon)$, where $\varphi_\varepsilon(\theta) = \int_\Omega j_\varepsilon(\theta) dx$ and $\overline{D(\varphi_\varepsilon)} = \overline{D(A_\varepsilon)}$.

Due to density arguments, let $\{f_n\}$, $\{u_n\}$, and $\{f_n^0\}$ be three sequences such that

$$\begin{aligned} f_n &\in W^{1,1}(0, T; V'), & f_n &\longrightarrow f \text{ in } L^2(0, T; V'), \\ u_n &\in W^{1,1}(0, T; L^2(\Gamma_u)), & u_n &\longrightarrow u \text{ in } L^2(0, T; L^2(\Gamma_u)), \\ f_n^0 &\in W^{1,1}(0, T; L^2(\Gamma_\alpha)), & f_n^0 &\longrightarrow f_0 \text{ in } L^2(0, T; L^2(\Gamma_\alpha)), \end{aligned} \tag{4.42}$$

and let $\theta_0 \in L^2(\Omega)$, $\theta_0 \leq \theta_s$. Then there exists $\{(\theta_0)_n\} \subset D(A_\varepsilon)$ such that $(\theta_0)_n \rightarrow \theta_0$ in V' . Then, for each $\varepsilon > 0$, there is a unique solution $(\theta_\varepsilon)_n$ to the approximating problem

$$\begin{aligned} \frac{d(\theta_\varepsilon)_n}{dt} + A_\varepsilon(\theta_\varepsilon)_n &= f_n + Bu_n + f_n^\Gamma \quad \text{a.e. } t \in (0, T), \\ (\theta_\varepsilon)_n(0) &= (\theta_0)_n(x) \quad \text{in } \Omega, \end{aligned} \tag{4.43}$$

that satisfies (4.34)–(4.36) and

$$\begin{aligned} &\|(\theta_\varepsilon)_n(t) - (\theta_\varepsilon)_m(t)\|_{V'}^2 + \int_0^t \|(\theta_\varepsilon)_n(\tau) - (\theta_\varepsilon)_m(\tau)\|_{V'}^2 d\tau \\ &\leq \gamma_3(\alpha_m) \left(\|(\theta_0)_n - (\theta_0)_m\|_{V'}^2 + \int_0^T \|f_n(\tau) - f_m(\tau)\|_{V'}^2 d\tau \right. \\ &\quad \left. + \int_0^T \|u_n(\tau) - u_m(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \|f_n^0(\tau) - f_m^0(\tau)\|_{L^2(\Gamma_\alpha)}^2 d\tau \right). \end{aligned} \tag{4.44}$$

By passing to the limit as $n \rightarrow 0$, we obtain the results of [Theorem 4.3](#), as claimed. Moreover, we deduce that $K(\theta_\varepsilon) \in L^2(0, T; V)$. □

In [Theorem 4.3](#), letting ε tend to 0, we obtain the following existence result.

THEOREM 4.4. *Let f , u , f_0 , and θ_0 satisfy (4.39). Then there exists a unique solution θ to the exact problem (4.21)–(4.22) with the following properties:*

$$\begin{aligned} \theta &\in L^2(0, T; V) \cap W^{1,2}(0, T; V'), & \beta^*(\theta) &\in L^2(0, T; V), \\ K(\theta) &\in L^2(0, T; V), & j(\theta) &\in L^\infty(0, T; L^1(\Omega)). \end{aligned} \tag{4.45}$$

Proof. Assume that (4.39) holds. Then the approximating problem (4.26)–(4.27) has a strong solution θ_ε satisfying (4.40), (4.35), and (4.36).

Since for $\theta_0 \leq \theta_s$ we have $j_\varepsilon(\theta_0) \leq K_s^* \theta_s$, the right-hand side term in (4.36) is bounded independently of ε . Hence, from (4.35) and (4.36), we deduce that $\{\theta_\varepsilon\}$ lies in a bounded subset of $L^\infty(0, T; L^2(\Omega))$, $\{d\theta_\varepsilon/dt\}$ lies in a bounded subset of $L^2(0, T; V')$, and $\{\beta_\varepsilon^*(\theta_\varepsilon)\}$ is in a bounded subset of $L^2(0, T; V)$. Using (ii), we get that $(\beta_\varepsilon^*)^{-1}$ is Lipschitz, and hence $\{\theta_\varepsilon\}$ is in a bounded subset of $L^2(0, T; V)$ too.

So, from the boundedness of the sequences previously mentioned, we conclude that there exists a subsequence (that will be denoted θ_ε too) such that

$$\begin{aligned} \theta_\varepsilon &\rightharpoonup \theta \quad \text{weakly in } L^2(0, T; V) \text{ and weak star in } L^\infty(0, T; L^2(\Omega)), \\ \frac{d\theta_\varepsilon}{dt} &\rightharpoonup \frac{d\theta}{dt} \quad \text{weakly in } L^2(0, T; V'). \end{aligned} \tag{4.46}$$

Since $V = H_0^1(\Omega)$ is compactly embedded in $H = L^2(\Omega)$ by Lions-Aubin compactness theorem (see [10]), we conclude that $\{\theta_\varepsilon\}$ is compact in $L^2(0, T; L^2(\Omega))$, that is,

$$\begin{aligned} \theta_\varepsilon &\rightarrow \theta \quad \text{strongly in } L^2(0, T; L^2(\Omega)), \text{ as } \varepsilon \rightarrow 0, \\ \theta_\varepsilon(t) &\rightharpoonup \theta(t) \quad \text{weakly in } V', \text{ for each } t \in [0, T]. \end{aligned} \tag{4.47}$$

We obtain also that $\theta(0) = \theta_0$. Since $\theta \rightarrow K(\theta)$ is continuous from $L^2(0, T; V)$ to $L^2(0, T; L^2(\Omega))$, it follows that

$$K(\theta_\varepsilon) \rightarrow K(\theta) \quad \text{strongly in } L^2(0, T; L^2(\Omega)), \text{ as } \varepsilon \rightarrow 0. \tag{4.48}$$

From (4.36) we may assume that

$$\beta_\varepsilon^*(\theta_\varepsilon) \rightharpoonup \eta \quad \text{weakly in } L^2(0, T; V). \tag{4.49}$$

Moreover, by the trace theorem, it follows that $\beta_\varepsilon^*(\theta_\varepsilon) \rightarrow \eta$ strongly in $L^2(\Sigma)$.

We will prove that $\eta \in \beta^*(\theta)$ a.e. on Ω . Indeed, from the obvious inequality

$$\int_\Omega j_\varepsilon(\theta) dx \leq \int_\Omega j_\varepsilon(\theta_\varepsilon) dx + \int_\Omega \beta_\varepsilon^*(\theta)(\theta - \theta_\varepsilon) dx, \quad \forall \theta \in L^2(\Omega), \tag{4.50}$$

we have by passing to the limit as $\varepsilon \rightarrow 0$ that

$$\liminf_{\varepsilon \rightarrow 0} \int_\Omega j_\varepsilon(\theta) dx \leq \liminf_{\varepsilon \rightarrow 0} \int_\Omega j_\varepsilon(\theta_\varepsilon) dx. \tag{4.51}$$

Since $j_\varepsilon(\theta) > 0$, we have by Fatou's lemma that

$$\int_\Omega j(\theta) dx \leq \liminf_{\varepsilon \rightarrow 0} \int_\Omega j_\varepsilon(\theta) dx. \tag{4.52}$$

On the other hand, we have

$$\int_Q \beta_\varepsilon^*(\theta_\varepsilon)(\theta_\varepsilon - z) dx dt \geq \int_Q j_\varepsilon(\theta_\varepsilon) dx dt - \int_Q j_\varepsilon(z) dx dt, \tag{4.53}$$

for all $z \in L^2(Q)$, $z \leq \theta_s$ a.e. on Q . Passing to the limit as $\varepsilon \rightarrow 0$, we obtain that

$$\int_Q \eta(\theta - z) dx \geq \int_Q j(\theta) dx - \int_Q j(z) dx, \quad \forall z \in L^2(Q), z \leq \theta_s \text{ a.e. on } Q. \tag{4.54}$$

This means that $\eta \in \beta^*(\theta)$ a.e. on Ω .

In the above calculations we used the fact that $j_\varepsilon(z) \rightarrow j(z)$, for all $z \in L^2(\Omega)$, which is an obvious assertion for $z \leq \theta_s$. Now, if $z > \theta_s$, we obtain

$$j_\varepsilon(z) = \int_0^z \beta_\varepsilon^*(r) dr = K_s^* z + \frac{(z - \theta_s)^2 - \theta_s^2}{2\varepsilon} \rightarrow +\infty = j(z) \quad \text{as } \varepsilon \rightarrow 0. \tag{4.55}$$

By (4.26), we obtain

$$\begin{aligned} & \int_Q \left(\frac{\partial \theta_\varepsilon}{\partial t} \psi + \nabla \beta_\varepsilon^*(\theta) \cdot \nabla \psi - K(\theta_\varepsilon) \frac{\partial \psi}{\partial x_3} \right) dx dt \\ &= \int_Q f \psi dx dt - \int_{\Sigma_\alpha} (\alpha \beta_\varepsilon^*(\theta) + f_0) \psi d\sigma dt - \int_{\Sigma_u} u \psi d\sigma dt, \quad \forall \psi \in V. \end{aligned} \tag{4.56}$$

Passing to the limit as $\varepsilon \rightarrow 0$, we get

$$\begin{aligned} & \int_Q \left(\frac{\partial \theta}{\partial t} \psi + \nabla \eta \cdot \nabla \psi - K(\theta) \frac{\partial \psi}{\partial x_3} \right) dx dt \\ &= \int_Q f \psi dx dt - \int_{\Sigma_\alpha} (\alpha \eta + f_0) \psi d\sigma dt - \int_{\Sigma_u} u \psi d\sigma dt, \quad \forall \psi \in V, \end{aligned} \tag{4.57}$$

so θ is a solution to (4.21)-(4.22). The uniqueness follows from the estimate of the difference between the two solutions. \square

COROLLARY 4.5. *Let f, u, f_0 , and θ_0 satisfy (4.39). Then, the solution θ to (4.21)-(4.22) has the property that $\theta(x, t) \leq \theta_s$, a.e. in Q .*

Proof. By (4.36), we have that

$$\int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|^2 d\tau \leq \int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_V^2 d\tau \leq c_0. \tag{4.58}$$

If we denote by $\chi_\varepsilon^+(x, t)$ the characteristic function of Q_ε^+ and denote $Q_\varepsilon^- = \{(x, t) \in Q; \theta_\varepsilon(x, t) \leq \theta_s\}$ and $Q_\varepsilon^+ = \{(x, t) \in Q; \theta_\varepsilon(x, t) > \theta_s\}$, we have

$$\int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|^2 d\tau = \int_{Q_\varepsilon^-} (\beta_\varepsilon^*(\theta_\varepsilon))^2 dx dt + \int_{Q_\varepsilon^+} (\beta_\varepsilon^*(\theta_\varepsilon))^2 dx dt \leq c_0 \tag{4.59}$$

wherefrom, using that $\beta_\varepsilon^*(\theta_\varepsilon) \leq K_s^*$ on Q_ε^- , we get

$$\int_Q \chi_\varepsilon^+(x, t) \left(K_s^* + \frac{\theta_\varepsilon - \theta_s}{\varepsilon} \right)^2 dx dt \leq c_1. \tag{4.60}$$

This implies after some calculations that

$$\int_Q \chi_\varepsilon^+(x, t) (\theta_\varepsilon - \theta_s)^2 dx dt \leq c_2 \varepsilon^2 \tag{4.61}$$

with c_0, c_1 , and c_2 some constants. But

$$\liminf_{\varepsilon \rightarrow 0} \chi_\varepsilon^+(x, t) \geq \chi^+(x, t) \quad \text{a.e. on } Q, \tag{4.62}$$

hence

$$\liminf_{\varepsilon \rightarrow 0} \chi_\varepsilon^+(x, t) (\theta_\varepsilon - \theta_s)^2 \geq \chi^+(x, t) (\theta - \theta_s)^2 \quad \text{a.e. on } Q. \tag{4.63}$$

By Fatou’s lemma, we have

$$\int_Q \chi^+(x, t) (\theta - \theta_s)^2 dx dt \leq \liminf_{\varepsilon \rightarrow 0} \int_Q \chi_\varepsilon^+(x, t) (\theta_\varepsilon - \theta_s)^2 dx dt = 0, \tag{4.64}$$

where $\chi^+(x, t)$ is the characteristic function of $Q^+ = \{(x, t) \in Q; \theta(x, t) > \theta_s\}$. This yields that $\chi^+(x, t) = 0$, meaning that $\theta(x, t) \leq \theta_s$, a.e. $(x, t) \in Q$. \square

Concerning θ_r , the relationship $\theta(x, t) \geq \theta_r$ does not generally hold. Actually that would have been expected, because the fact that the moisture in a soil does not go under the residual value θ_r is determined by factors that have not been taken into account in our model. But instead of this, we can prove under some hypotheses that $\theta(x, t) \geq \theta_{\min}$ a.e. If, in particular, $\theta_{\min} = 0$, meaning that the relation $\theta_r = C_r$ takes place, then $\theta(x, t) \geq 0$.

COROLLARY 4.6. *Let f, u, f_0 , and θ_0 satisfy (4.39) and assume that*

$$\theta_0 \geq \theta_{\min} \quad \text{in } \Omega, \quad f \geq 0 \quad \text{in } Q, \quad u \leq 0 \quad \text{on } \Sigma_u, \quad f_0 \leq 0 \quad \text{on } \Sigma_\alpha. \tag{4.65}$$

Then, the solution θ to (4.21)-(4.22) is in $L^\infty(Q)$ and satisfies $\theta(x, t) \geq \theta_{\min}$ a.e. in Q .

Proof. Assume that $\theta_0 \geq \theta_{\min}$, meaning that the negative part $(\theta_0 - \theta_{\min})^- = 0$. We have to show that $(\theta_\varepsilon - \theta_{\min})^- = 0$ too. We multiply (4.26) by $(\theta_\varepsilon - \theta_{\min})^-$ and integrate over $\Omega \times (0, t)$. Using Stampacchia’s lemma (see, e.g., [4]),

$$\nabla \theta^- = \begin{cases} -\nabla \theta, & \text{a.e. on } \theta < 0, \\ 0, & \text{a.e. on } \theta \geq 0, \end{cases} \tag{4.66}$$

we have since θ_{\min} is a constant

$$\begin{aligned} & \int_0^t \int_\Omega \left\{ -\frac{1}{2} \frac{d}{d\tau} [(\theta_\varepsilon - \theta_{\min})^-]^2 + \nabla \beta_\varepsilon^*(\theta_\varepsilon) \cdot \nabla (\theta_\varepsilon - \theta_{\min})^- - K(\theta_\varepsilon) \frac{\partial (\theta_\varepsilon - \theta_{\min})^-}{\partial x_3} \right\} dx d\tau \\ &= \int_0^t \int_\Omega f(\theta_\varepsilon - \theta_{\min})^- dx d\tau - \int_0^t \int_{\Gamma_\alpha} (\alpha \beta_\varepsilon^*(\theta_\varepsilon) + f_0) (\theta_\varepsilon - \theta_{\min})^- d\sigma d\tau \\ &\quad - \int_0^t \int_{\Gamma_u} u(\theta_\varepsilon - \theta_{\min})^- d\sigma d\tau. \end{aligned} \tag{4.67}$$

After integrating the first term on the left-hand side with respect to t , we get

$$\begin{aligned}
 & -\frac{1}{2} \int_{\Omega} [(\theta_{\varepsilon}(t) - \theta_{\min})^{-}]^2 dx - \int_0^t \int_{\Omega} \beta_{\varepsilon}(\theta_{\varepsilon}) \left| \nabla(\theta_{\varepsilon} - \theta_{\min})^{-} \right|^2 dx d\tau \\
 & \quad - \int_0^t \int_{\Omega} K(\theta_{\varepsilon}) \frac{\partial(\theta_{\varepsilon} - \theta_{\min})^{-}}{\partial x_3} dx d\tau + \int_0^t \int_{\Gamma_{\alpha}} \alpha \beta_{\varepsilon}^*(\theta_{\varepsilon}) (\theta_{\varepsilon} - \theta_{\min})^{-} d\sigma d\tau \\
 & = \int_0^t \int_{\Omega} f(\theta_{\varepsilon} - \theta_{\min})^{-} dx d\tau \\
 & \quad - \int_0^t \int_{\Gamma_{\alpha}} f_0(\theta_{\varepsilon} - \theta_{\min})^{-} d\sigma d\tau - \int_0^t \int_{\Gamma_u} u(\theta_{\varepsilon} - \theta_{\min})^{-} d\sigma d\tau.
 \end{aligned} \tag{4.68}$$

We took into account that $\theta\theta^{-} = -(\theta^{-})^2$. But, since $\beta_{\varepsilon}^*(\theta_{\varepsilon}) = \int_{\theta_{\min}}^{\theta_{\varepsilon}} \beta_{\varepsilon}(\xi) d\xi \geq \rho(\theta_{\varepsilon} - \theta_{\min})$, we have

$$-\rho \int_0^t \int_{\Gamma_{\alpha}} \alpha_m [(\theta_{\varepsilon} - \theta_{\min})^{-}]^2 d\sigma d\tau \leq \int_0^t \int_{\Gamma_{\alpha}} \alpha \beta^*(\theta_{\varepsilon}) (\theta_{\varepsilon} - \theta_{\min})^{-} d\sigma d\tau. \tag{4.69}$$

Therefore, setting $\rho_{\alpha} = \min(\rho, \alpha_m \rho)$, we get

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} [(\theta_{\varepsilon}(t) - \theta_{\min})^{-}]^2 dx + \rho_{\alpha} \int_0^t \left\| (\theta_{\varepsilon}(t) - \theta_{\min})^{-} \right\|_V^2 d\tau \\
 & = - \int_0^t \int_{\Omega} K(\theta_{\varepsilon}) \frac{\partial(\theta_{\varepsilon} - \theta_{\min})^{-}}{\partial x_3} dx dt - \int_0^t \int_{\Omega} f(\theta_{\varepsilon} - \theta_{\min})^{-} dx d\tau \\
 & \quad + \int_0^t \int_{\Gamma_{\alpha}} f_0(\theta_{\varepsilon} - \theta_{\min})^{-} d\sigma dt + \int_0^t \int_{\Gamma_u} u(\theta_{\varepsilon} - \theta_{\min})^{-} d\sigma d\tau.
 \end{aligned} \tag{4.70}$$

Further, using the hypotheses, (4.5), and the fact that $|K(\theta_{\varepsilon})| \leq M|\theta_{\varepsilon} - \theta_{\min}|$, we have

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} [(\theta_{\varepsilon}(t) - \theta_{\min})^{-}]^2 dx + \rho_{\alpha} \int_0^t \left\| (\theta_{\varepsilon}(t) - \theta_{\min})^{-} \right\|_V^2 d\tau \\
 & \leq \frac{M^2}{\rho_{\alpha}} \int_0^t \left\| (\theta_{\varepsilon}(\tau) - \theta_{\min})^{-} \right\|^2 d\tau + \frac{\rho_{\alpha}}{2} \int_0^t \left\| (\theta_{\varepsilon}(\tau) - \theta_{\min})^{-} \right\|_V^2 d\tau.
 \end{aligned} \tag{4.71}$$

This implies

$$\left\| (\theta_{\varepsilon}(t) - \theta_{\min})^{-} \right\|^2 \leq \frac{2M^2}{\rho_{\alpha}} \int_0^t \left\| (\theta_{\varepsilon}(\tau) - \theta_{\min})^{-} \right\|^2 d\tau, \tag{4.72}$$

wherefrom we deduce according to Gronwall's lemma that $\|(\theta_{\varepsilon}(t) - \theta_{\min})^{-}\|^2 = 0$, meaning that $\theta_{\varepsilon}(x, t) \geq \theta_{\min}$ a.e. on Ω , for each $t \in [0, T]$.

Passing to the limit as $\varepsilon \rightarrow 0$, we obtain $\theta(x, t) \geq \theta_{\min}$ a.e. on Q . □

Remark 4.7. In a similar way, it follows under certain conditions that $\theta(x, t) \leq \theta_M(t) = Pt + P_0$ a.e. on Ω , for each t , where $\theta_M(t)$ is a solution to (4.26)-(4.27), $P \geq 0$, and $P_0 = \sup_{x \in \Omega} \theta_0(x)$. This result is implied by the proof of $\|(\theta_{\varepsilon}(t) - \theta_M(t))^{+}\|^2 = 0$.

COROLLARY 4.8. Assume $f \in L^2(0, T; V')$, $u \in L^2(0, T; L^2(\Gamma_u))$, $f_0 \in L^2(0, T; L^2(\Gamma_\alpha))$, and $\theta_0 \in M_{\theta_s}$. Then problem (2.7) has a weak solution $h \in L^2(0, T; V)$.

Proof. Under assumption (4.39), we obtain a solution θ satisfying the conclusions of Theorem 4.4, and we define

$$h(x, t) = (C^*)^{-1}(\theta(x, t)), \quad \theta < \theta_s, \quad h(x, t) \in [0, +\infty) = (C^*)^{-1}(\theta_s). \quad (4.73)$$

We will show that it is a weak solution to (2.7). We apply K^* to (4.73) and obtain

$$K^*(h) = \beta^*(\theta), \quad \text{for } h < 0, \quad \theta < \theta_s, \quad K^*(h) \in \beta^*(\theta_s), \quad h \geq 0. \quad (4.74)$$

From (4.73) we get $\theta = C^*(h)$ and we introduce it in (4.23). We have

$$\begin{aligned} & \int_Q \left(\frac{\partial C^*(h)}{\partial t} \phi + \nabla \eta \cdot \nabla \phi - K(C^*(h)) \frac{\partial \phi}{\partial x_3} \right) dx dt \\ &= \int_Q f \phi dx dt - \int_{\Sigma_\alpha} (\alpha \eta + f_0) \phi d\sigma dt - \int_{\Sigma_u} u \phi d\sigma dt, \end{aligned} \quad (4.75)$$

for $\eta \in V$ such that $\eta \in \beta^*(\theta)$. After integrating the first term on the left-hand side with respect to t , we deduce that

$$\begin{aligned} & \int_Q \left(-C^*(h) \phi_t dx dt + \nabla K^*(h) \cdot \nabla \phi - k(h) \frac{\partial \phi}{\partial x_3} \right) dx dt \\ &= \int_\Omega \phi(x, 0) C^*(h_0) dx - \int_{\Sigma_\alpha} (\alpha \eta + f_0) \phi d\sigma dt - \int_{\Sigma_u} u \phi d\sigma dt - \int_Q f \phi dx dt. \end{aligned} \quad (4.76)$$

Since $\beta^*(\theta) = K^*((C^*)^{-1}(\theta)) \in L^2(0, T; V)$ and $K^*((C^*)^{-1}(\theta))$ satisfies

$$(K^*((C^*)^{-1}(\theta)) - K^*((C^*)^{-1}(\bar{\theta}))) (\theta - \bar{\theta}) \geq K_r ((C^*)^{-1}(\theta) - (C^*)^{-1}(\bar{\theta}))^2, \quad (4.77)$$

for all $\theta, \bar{\theta} \leq \theta_s$, it follows that $h = (C^*)^{-1}(\theta) \in L^2(0, T; V)$. \square

5. Existence of free boundary

To prove that there exists a free boundary $s = s(t, x_1, x_2)$ that determines a strict delimitation of the domains Q_+ and Q_- with Q_+ above Q_- , the idea is to prove that the function h is monotonically decreasing with respect to x_3 , that is, $w_h = \partial h / \partial x_3 \leq 0$. Consequently, the equation $h(x, t) = 0$ can be solved with respect to x_3 and has a unique solution $x_3 = s(t, x_1, x_2)$.

To come to this end, we first need to have some supplementary regularities for the solution θ_ε to the approximating problem (4.26)-(4.27), considered for a smoother approximation of class $C^3(\mathbb{R})$.

PROPOSITION 5.1. *Let*

$$f \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(Q), \quad u \in W^{1,2}(0, T; L^2(\Gamma_u)) \cap L^\infty(\Sigma_u),$$

$$f_0 \in W^{1,2}(0, T; L^2(\Gamma_\alpha)) \cap L^\infty(\Sigma_\alpha),$$
(5.1)

$$u \leq 0 \quad \text{on } \Sigma_u,$$
(5.2)

$$\theta_0 \in L^2(\Omega) \quad \text{such that } \beta_\varepsilon^*(\theta_0) \in H^1(\Omega).$$
(5.3)

Then, the solution to the approximating problem (4.26)-(4.27) has the supplementary property

$$\beta_\varepsilon^*(\theta_\varepsilon) \in L^2(0, T; H^2(\Omega)).$$
(5.4)

Proof. Under the hypotheses (5.1) and (5.3) which are stronger than those imposed in Proposition 4.2, it follows that θ_ε is a unique strong solution to the approximating problem (4.26)-(4.27) and satisfies the conclusions of Proposition 4.2. Moreover, the requirement (5.3) implies that $\theta_\varepsilon \in D(A_\varepsilon)$.

Consider again the Cauchy problem (4.26)-(4.27). By a similar argument as done for the exact problem, its strong solution θ_ε is a solution to the boundary value problem

$$\frac{\partial \theta_\varepsilon}{\partial t} - \Delta \beta_\varepsilon^*(\theta_\varepsilon) + \frac{\partial K(\theta_\varepsilon)}{\partial x_3} = f \quad \text{in } Q,$$

$$\theta_\varepsilon(x, 0) = \theta_0(x) \quad \text{in } \Omega,$$

$$(K(\theta_\varepsilon)i_3 - \nabla \beta_\varepsilon^*(\theta_\varepsilon)) \cdot \nu = u \quad \text{on } \Sigma_u,$$

$$(K(\theta_\varepsilon)i_3 - \nabla \beta_\varepsilon^*(\theta_\varepsilon)) \cdot \nu = \alpha \beta_\varepsilon^*(\theta_\varepsilon) + f_0 \quad \text{on } \Sigma_\alpha.$$
(5.5)

Since the function β_ε^* is continuous and monotonically increasing, we may define its inverse. Moreover, we assumed that β_ε^* is in this case of class $C^3(\mathbb{R})$, so that its first derivative β_ε is bounded on \mathbb{R} and satisfies

$$\rho \leq \beta_\varepsilon(\theta_\varepsilon) \leq \rho_\varepsilon < \infty$$
(5.6)

for each $\varepsilon > 0$. Hence, denoting $\eta = \beta_\varepsilon^*(\theta_\varepsilon)$ with $\eta(0) = \beta_\varepsilon^*(\theta_0)$, we have

$$\theta_\varepsilon = (\beta_\varepsilon^*)^{-1}(\eta),$$
(5.7)

$$\frac{\partial \theta_\varepsilon}{\partial t} = \frac{\partial}{\partial t} ((\beta_\varepsilon^*)^{-1}(\eta)) = \omega(t) \frac{\partial \eta}{\partial t},$$
(5.8)

and $\zeta(\eta) = K((\beta_\varepsilon^*)^{-1}(\eta))$, where $\omega(\eta) = 1/\beta_\varepsilon((\beta_\varepsilon^*)^{-1}(\eta))$. We also know that

$$0 < \rho_\varepsilon^1 \leq \omega(\eta) \leq \rho_2 < \infty, \quad \rho_\varepsilon^1 = \frac{1}{\rho_\varepsilon}, \rho_2 = \frac{1}{\rho}.$$
(5.9)

Using the conclusions of Proposition 4.2, we deduce that

$$\eta \in L^\infty(0, T, V), \quad \omega(\eta)\eta_t \in L^\infty(0, T, V'), \quad \zeta(\eta) \in L^\infty(0, T, V).$$
(5.10)

Therefore, because $\omega(\eta)$ is bounded, we still obtain $\eta_t \in L^\infty(0, T; V')$. We introduce (5.7) in (5.5) and obtain

$$\omega(\eta) \frac{\partial \eta}{\partial t} - \Delta \eta + \frac{\partial \zeta(\eta)}{\partial x_3} = f \quad \text{in } Q, \tag{5.11}$$

$$\eta_0(x) = \eta(x, 0) = \beta_\varepsilon^*(\theta_0) \quad \text{in } \Omega, \tag{5.12}$$

$$(\zeta(\eta) i_3 - \nabla \eta) \cdot \nu = u \quad \text{on } \Sigma_u, \quad (\zeta(\eta) i_3 - \nabla \eta) \cdot \nu = \alpha \eta + f_0 \quad \text{on } \Sigma_\alpha. \tag{5.13}$$

We multiply equation (5.11) by η_t ($\eta_t = \partial \eta / \partial t$) and integrate over $\Omega \times (0, t)$. We obtain

$$\begin{aligned} & \int_0^t \int_\Omega \omega(\eta) \eta_\tau^2 d\tau dx + \frac{1}{2} \int_0^t \frac{d}{dt} \left(\int_\Omega |\nabla \eta(\tau)|^2 dx \right) d\tau - \int_0^t \int_\Omega \zeta(\eta) \frac{\partial \eta_\tau}{\partial x_3} d\tau dx \\ & = - \int_0^t \int_{\Gamma_u} u \eta_\tau d\sigma d\tau - \int_0^t (\alpha \eta + f_0) \eta_\tau d\sigma d\tau + \int_0^t \int_\Omega f \eta_\tau dx d\tau. \end{aligned} \tag{5.14}$$

We calculate

$$\begin{aligned} - \int_0^t \int_{\Gamma_u} u \eta_\tau d\sigma d\tau & = - \int_0^t \int_{\Gamma_u} \left(\frac{\partial}{\partial t} (u\eta) - \eta \frac{\partial u}{\partial \tau} \right) d\sigma d\tau \\ & = - \int_{\Gamma_u} u(t) \eta(t) d\sigma + \int_{\Gamma_u} u_0 \eta_0 d\sigma + \int_0^t \int_{\Gamma_u} \eta \frac{\partial u}{\partial \tau} d\sigma d\tau. \end{aligned} \tag{5.15}$$

Proceeding in the same manner for all other terms on the right-hand side, we get after some calculations roughly the estimate

$$\begin{aligned} & \int_0^t \int_\Omega \eta_\tau^2 dx d\tau + \int_\Omega |\nabla \eta(\tau)|^2 dx \\ & \leq c_0 \int_\Omega |\nabla \eta_0|^2 dx + \|\zeta(\eta(t))\| \left\| \frac{\partial \eta(t)}{\partial x_3} \right\| + \|\zeta(\eta_0)\| \left\| \frac{\partial \eta_0}{\partial x_3} \right\| \\ & \quad + \int_0^t \left\| \frac{\partial \zeta(\eta(\tau))}{\partial x_3} \right\| \left\| \frac{\partial \eta(\tau)}{\partial x_3} \right\| d\tau + \int_0^t \|\eta(\tau)\|_{L^2(\Gamma_u)} \left\| \frac{\partial u(\tau)}{\partial \tau} \right\|_{L^2(\Gamma_u)} d\tau \\ & \quad + \int_0^t \|\eta(\tau)\|_{L^2(\Gamma_\alpha)} \left\| \frac{df_0(\tau)}{d\tau} \right\|_{L^2(\Gamma_\alpha)} d\tau + \|u(t)\|_{L^2(\Gamma_u)} \|\eta(t)\|_{L^2(\Gamma_u)} \\ & \quad + \|u_0\|_{L^2(\Gamma_u)} \|\eta_0\|_{L^2(\Gamma_u)} + \|f_0(t)\|_{L^2(\Gamma_\alpha)} \|\eta(t)\|_{L^2(\Gamma_\alpha)} + \|f_0(0)\|_{L^2(\Gamma_\alpha)} \|\eta_0\|_{L^2(\Gamma_\alpha)} \\ & \quad + \|f(t)\| \|\eta(t)\| + \|f(0)\| \|\eta_0\| + \int_0^t \|\eta(\tau)\| \left\| \frac{\partial f(\tau)}{\partial \tau} \right\| d\tau. \end{aligned} \tag{5.16}$$

But, from the hypotheses, it follows that $u_0, f_0(0), f(0)$, and η_0 make sense and finally we conclude that the right-hand side in (5.16) is bounded. This implies that

$$\eta_t \in L^2(0, T; L^2(\Omega)), \quad \eta \in L^\infty(0, T; H^1(\Omega)). \tag{5.17}$$

A little problem in the previous calculation is that we do not know a priori that $\eta_t \in L^2(0, T; L^2(\Omega))$. Hence, rigorously, (5.11) should have been approximated by a finite-difference equation for $(\eta(t + \delta) - \eta(t))/\delta$, and in the same manner, the result obtained would have been

$$\int_0^t \|\eta(\tau + \delta) - \eta(\tau)\|^2 d\tau \leq c\delta^2 \tag{5.18}$$

that implies $\eta \in W^{1,2}(0, T; L^2(\Omega))$ (see [3, page 21]).

Then, from (5.11), we have that

$$\|\Delta\eta\| \leq \left\| \omega(\eta) \frac{\partial\eta}{\partial t} \right\| + \left\| \frac{\partial\zeta(\eta)}{\partial x_3} \right\| + \|f\| \tag{5.19}$$

and we can deduce that

$$\Delta\eta \in L^2(0, T; L^2(\Omega)) \tag{5.20}$$

and consequently (taking into account the boundary conditions)

$$\eta \in L^2(0, T; H^2(\Omega)). \tag{5.21}$$

Finally we have to keep in mind that $\theta_\varepsilon \in L^\infty(0, T; H^1(\Omega))$ and $\beta_\varepsilon^*(\theta_\varepsilon) \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$. □

PROPOSITION 5.2. *Under the hypotheses of Proposition 5.1,*

$$\theta_\varepsilon \in L^{4/3}(0, T; H^2(\Omega)). \tag{5.22}$$

Proof. We start from (5.7) and calculate the partial derivative of θ_ε , denoted further θ_{x_i} (we omit the subscript ε) with respect to x_i . First we remind the reader that we work with the smooth approximate of class C^3 , whose derivatives up to the third order (denoted β_ε , β'_ε , and β''_ε) are bounded. We have

$$\theta_{x_i} = \frac{\eta_{x_i}}{\beta_\varepsilon((\beta_\varepsilon^*)^{-1}(\eta))} \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)). \tag{5.23}$$

Then

$$\theta_{x_i x_j} = \frac{\eta_{x_i x_j} \beta_\varepsilon((\beta_\varepsilon^*)^{-1}(\eta)) - \eta_{x_i} \eta_{x_j} \beta'_\varepsilon((\beta_\varepsilon^*)^{-1}(\eta)) / \beta_\varepsilon((\beta_\varepsilon^*)^{-1}(\eta))}{[\beta_\varepsilon((\beta_\varepsilon^*)^{-1}(\eta))]^2} \tag{5.24}$$

and we need to estimate its norm. Since $\eta_{x_i x_j} \in L^2(0, T; L^2(\Omega))$, we will deal only with the product $\eta_{x_i} \eta_{x_j}$. From the previous proposition we have $\eta \in L^2(0, T; H^2(\Omega))$, which implies the following sequence:

$$\eta_{x_i}(t) \in H^1(\Omega) \subset L^6(\Omega) \subset L^4(\Omega) \subset L^2(\Omega), \quad \forall N \leq 3. \tag{5.25}$$

Now $\eta_{x_i}(t) \in L^4(\Omega)$ implies $\eta_{x_i}^2(t) \in L^2(\Omega)$ so that we can write

$$\|\eta_{x_i}(t)\eta_{x_j}(t)\|^2 = \int_{\Omega} \eta_{x_i}^2(t)\eta_{x_j}^2(t)dx \leq \left(\int_{\Omega} \eta_{x_i}^4(t)dx\right)^{1/2} \left(\int_{\Omega} \eta_{x_j}^4(t)dx\right)^{1/2}. \quad (5.26)$$

But $\eta_{x_i}(t) \in L^6(\Omega)$ implies $\eta_{x_i}^3(t) \in L^2(\Omega)$ so that we have

$$\int_{\Omega} \eta_{x_i}^4(t)dx = \int_{\Omega} \eta_{x_i}(t)\eta_{x_i}^3(t)dx \leq \|\eta_{x_i}(t)\| \|\eta_{x_i}^3(t)\| \quad (5.27)$$

or

$$\int_{\Omega} \eta_{x_i}^4(t)dx \leq \|\eta(t)\|_{H^1(\Omega)} \|\eta(t)\|_{H^2(\Omega)}^3. \quad (5.28)$$

Then we obtain

$$\begin{aligned} \|\theta_{x_i x_j}(t)\|^2 &\leq d_0 \|\eta_{x_i x_j}(t)\|^2 + d_1 \|\eta_{x_i}(t)\eta_{x_j}(t)\|^2 \\ &\leq d_0 \|\eta(t)\|_{H^2(\Omega)}^2 + d_1 \|\eta(t)\|_{H^1(\Omega)} \|\eta(t)\|_{H^2(\Omega)}^3, \end{aligned} \quad (5.29)$$

and therefore we get

$$\|\theta_{\varepsilon}(t)\|_{H^2(\Omega)}^2 \leq d_2 \|\eta(t)\|_{H^2(\Omega)} + d_3 \|\eta(t)\|_{H^2(\Omega)}^3. \quad (5.30)$$

Finally, this yields

$$\int_0^t \|\theta_{\varepsilon}(\tau)\|_{H^2(\Omega)}^{4/3} d\tau \leq d_4 \int_0^t \|\eta(\tau)\|_{H^2(\Omega)}^2 d\tau < \infty \quad (5.31)$$

with d_0, d_1, d_2, d_3 , and d_4 some constants. □

We now pass to the proof of the monotonicity of the partial derivative of θ with respect to x_3 .

A better insight can be gained in one dimension, so that we will prove the result for the case $N = 1$. In this case we denote $z = x_3$, the domain Ω becomes $\Omega = (0, L)$, $\Gamma_u = \{z; z = 0\}$, $\Gamma_{\alpha} = \{z; z = L\}$, $\alpha(L) = \alpha = \alpha_M$, $f_0(x, t) = f_0(t)$, and system (3.4) reads

$$\begin{aligned} C(h)h_t - (k(h)h_z)_z + (k(h))_z &= f \quad \text{in } Q_- = \{z; s(t) < z < L\}, \\ -K_s \Delta h &= f \quad \text{in } Q_+ = \{z; 0 < z < s(t)\}, \\ h(z, 0) &= h_0(z) \quad \text{in } (0, L), \\ q^+(s(t), t) &= q^-(s(t), t), \quad h^+(s(t), t) = h^-(s(t), t), \\ K_s - K_s h_z(0, t) &= -u(t) \stackrel{\text{notation}}{=} u_-(t), \\ k(h(L, t)) - k(h(L, t))h_z(L, t) &= \alpha K^*(h(L, t)) + f_0(t). \end{aligned} \quad (5.32)$$

Solving this problem, one determines the free boundary $z = s(t)$ from the equation $h(z, t) = 0$. The subscript z means the partial derivative with respect to z .

Obviously, in order to ensure the existence of a free boundary that determines a clear separation of the saturated region from the unsaturated one, some conditions have to be fulfilled and this is presented in the following result.

PROPOSITION 5.3. Let $N = 1$, $K_s \leq \alpha_M K_s^*$,

$$\begin{aligned}
 f &\in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(Q), & u &\in W^{1,2}(0, T; L^2(\Gamma_u)) \cap L^\infty(\Sigma_u), \\
 f_0 &\in W^{1,2}(0, T; L^2(\Gamma_\alpha)) \cap L^\infty(\Sigma_\infty), & \theta_0 &\in L^2(\Omega), \text{ such that } \beta_\varepsilon^*(\theta_0) \in H^1(\Omega), \\
 \frac{\partial \theta_0}{\partial z}(z, 0) &\leq 0 \text{ in } \Omega, & f_z(z, t) &\leq 0 \text{ in } Q, & u_-(t) &\geq K_s, \\
 f_0(t) &\geq \sup_{r \in [0, \theta_\varepsilon]} \{K(r) - \alpha_M \beta^*(r)\}.
 \end{aligned}
 \tag{5.33}$$

Then $w = \partial\theta/\partial z$ is negative a.e. on $(0, L) \times (0, T)$ and $z \rightarrow \theta(z, t)$ is monotonically decreasing on $[0, T]$, for each $t \in [0, T]$.

Proof. We mean by β^* the minimal section of β^* . By Proposition 5.1, without any loss of generality, we may assume that θ_ε is smooth enough and for that we work with the approximate which has the property that $\beta_\varepsilon'' \in C^0(\mathbb{R})$.

By the hypotheses, it follows that θ_ε is a solution satisfying the conclusions of Propositions 5.1 and 5.2.

We denote $w_\varepsilon = \partial\theta_\varepsilon/\partial z$ and note that by assumption $w_\varepsilon(z, 0) \leq 0$. From the boundary conditions we have

$$\begin{aligned}
 K(\theta_\varepsilon) - \beta_\varepsilon(\theta_\varepsilon)w_\varepsilon &= u_- \quad \text{for } z = 0, \\
 K(\theta_\varepsilon) - \beta_\varepsilon(\theta_\varepsilon)w_\varepsilon &= \alpha\beta_\varepsilon^*(\theta_\varepsilon) + f_0 \quad \text{for } z = L.
 \end{aligned}
 \tag{5.34}$$

If we assume that saturation begins from above, then we will necessarily have that $u_-(t) \geq K_s \geq K(\theta_\varepsilon(0, t))$. Then we have

$$w_\varepsilon(0, t) = \frac{K(\theta_\varepsilon(0, t)) - u_-(t)}{\beta_\varepsilon(\theta_\varepsilon(0, t))} \leq 0
 \tag{5.35}$$

implying that $w_\varepsilon^+(0, t) = 0$.

Then, if we assume that for $z = L$ we have

$$f_0(t) \geq \sup_{r \in [0, \theta_\varepsilon]} (K(r) - \alpha_M \beta^*(r)) \geq K(r) - \alpha_M \beta^*(r), \quad \forall r \in [0, \theta_\varepsilon],
 \tag{5.36}$$

we will have from the boundary condition on $z = L$ and taking into account that $K_s \leq \alpha_M K_s^*$ that

$$w_\varepsilon(L, t) = \frac{K(\theta_\varepsilon(L, t)) - (\alpha\beta_\varepsilon^*(\theta_\varepsilon(L, t)) + f_0(t))}{\beta_\varepsilon(\theta_\varepsilon(L, t))} \leq 0,
 \tag{5.37}$$

that is, $w_\varepsilon^+(L, t) = 0$.

We differentiate (5.5) with respect to z :

$$\frac{\partial w_\varepsilon}{\partial t} - \Delta(\beta_\varepsilon(\theta_\varepsilon)w_\varepsilon) + \frac{\partial}{\partial z}(K'(\theta_\varepsilon)w_\varepsilon) = f_z.
 \tag{5.38}$$

Then we multiply it by w_ε^+ and integrate over $\Omega \times (0, t)$. We have

$$\begin{aligned} \frac{1}{2} \|w_\varepsilon^+(t)\|^2 + \int_0^t \int_0^L \left[\frac{\partial}{\partial z} (\beta_\varepsilon(\theta_\varepsilon) w_\varepsilon) \frac{\partial w_\varepsilon^+}{\partial z} - K'(\theta_\varepsilon) w_\varepsilon \frac{\partial w_\varepsilon^+}{\partial z} \right] dz d\tau \\ + \int_0^t \left[K'(\theta_\varepsilon) w_\varepsilon - \frac{\partial}{\partial z} (\beta_\varepsilon(\theta_\varepsilon) w_\varepsilon) \right] w_\varepsilon^+ \Big|_{z=0}^{z=L} d\tau = \int_0^t \int_0^L f_z w_\varepsilon^+ dz d\tau. \end{aligned} \tag{5.39}$$

But, using Stampacchia's lemma and $w^2 = (w^+)^2$, we have

$$\begin{aligned} \int_0^t \int_0^L \frac{\partial}{\partial z} (\beta_\varepsilon(\theta_\varepsilon) w_\varepsilon) \frac{\partial w_\varepsilon^+}{\partial z} dz d\tau &= \int_0^t \int_0^L \left(\beta_\varepsilon(\theta_\varepsilon) w_\varepsilon^2 + \beta_\varepsilon(\theta_\varepsilon) \frac{\partial w_\varepsilon}{\partial z} \right) \frac{\partial w_\varepsilon^+}{\partial z} dz d\tau \\ &= \int_0^t \int_0^L \frac{1}{3} \left(\beta'_\varepsilon(\theta_\varepsilon) \frac{\partial}{\partial z} (w_\varepsilon^+)^3 + \beta_\varepsilon(\theta_\varepsilon) \left(\frac{\partial w_\varepsilon^+}{\partial z} \right)^2 \right) dz d\tau. \end{aligned} \tag{5.40}$$

Moreover

$$\int_0^t \int_0^L \beta'_\varepsilon(\theta_\varepsilon) \frac{\partial}{\partial z} (w_\varepsilon^+)^3 dz d\tau = \int_0^t \beta'_\varepsilon(\theta_\varepsilon) (w_\varepsilon^+)^3 \Big|_0^L d\tau - \int_0^t \int_0^L \beta''_\varepsilon(\theta_\varepsilon) (w_\varepsilon^+)^4 dz d\tau \tag{5.41}$$

and $K'(\theta_\varepsilon) \leq M$, so we get

$$\begin{aligned} \frac{1}{2} \|w_\varepsilon^+(t)\|^2 + \rho \int_0^t \int_0^L \left(\frac{\partial w_\varepsilon^+}{\partial z} \right)^2 dz d\tau &\leq \frac{1}{2} \int_0^t \left(\rho \left\| \frac{\partial w_\varepsilon^+}{\partial z} \right\|^2 + \frac{M^2}{\rho} \|w_\varepsilon^+(t)\|^2 \right) d\tau \\ &\quad + \frac{1}{3} \int_0^t \int_0^L \beta''_\varepsilon(\theta_\varepsilon) (w_\varepsilon^+)^4 dz d\tau. \end{aligned} \tag{5.42}$$

Finally, we obtain

$$\begin{aligned} \frac{1}{2} \|w_\varepsilon^+(t)\|^2 + \frac{\rho}{2} \int_0^t \int_0^L \left(\frac{\partial w_\varepsilon^+}{\partial z} \right)^2 dz d\tau \\ \leq \frac{M^2}{2\rho} \int_0^t \|w_\varepsilon^+(\tau)\|^2 d\tau + \frac{1}{3} \int_0^t \int_0^L \beta''_\varepsilon(\theta_\varepsilon) (w_\varepsilon^+(\tau))^4 d\tau, \end{aligned} \tag{5.43}$$

where $\beta''_\varepsilon(\theta_\varepsilon)$ is bounded by a constant denoted β''_M . We now write the following relation:

$$\frac{1}{2} (w_\varepsilon^+(t))^2 = \int_0^z w_\varepsilon^+(t) \frac{\partial w_\varepsilon^+(t)}{\partial s} ds \leq \left(\int_0^L (w_\varepsilon^+(t))^2 dz \right)^{1/2} \left(\int_0^L \left(\frac{\partial w_\varepsilon^+(t)}{\partial z} \right)^2 dz \right)^{1/2}. \tag{5.44}$$

By [Proposition 5.1](#), we have that $\nabla\theta_\varepsilon \in L^\infty(0, T; H^1(\Omega))$, implying that the same result is inherited by θ_ε^+ . Then $w_\varepsilon^+ \in L^\infty(0, T; H^1(\Omega))$. We denote

$$\gamma(t) = \left(\int_0^L \left(\frac{\partial w_\varepsilon^+(t)}{\partial z} \right)^2 dz \right)^{1/2} = \left\| \frac{\partial w_\varepsilon^+(t)}{\partial z} \right\| = \left\| \frac{\partial^2 \theta_\varepsilon^+(t)}{\partial z^2} \right\| \leq \|\theta_\varepsilon(t)\|_{H^2(\Omega)}, \tag{5.45}$$

so

$$\frac{1}{2} (w_\varepsilon^+(t))^2 \leq \|w_\varepsilon^+(t)\| \gamma(t) \leq c_3 \gamma(t). \tag{5.46}$$

In [\(5.43\)](#) we still can write the last term on the right-hand side as

$$\begin{aligned} \int_0^t \int_0^L \beta_\varepsilon''(\theta_\varepsilon) (w_\varepsilon^+(\tau))^4 dz d\tau &\leq \beta_M'' \int_0^t \int_0^L (w_\varepsilon^+(\tau))^2 (w_\varepsilon^+(\tau))^2 dz d\tau \\ &\leq c_4 \int_0^t \int_\Omega \gamma(\tau) (w_\varepsilon^+(\tau))^2 dz d\tau, \end{aligned} \tag{5.47}$$

where, by [Proposition 5.2](#), it follows that $\gamma \in L^{4/3}(0, T) \subset L^1(0, T)$.

Recalling again [\(5.43\)](#), we finally get

$$\frac{1}{2} \|w_\varepsilon^+(t)\|^2 + \frac{\rho}{2} \int_0^t \int_0^L \left(\frac{\partial w_\varepsilon^+}{\partial z} \right)^2 dz d\tau \leq \int_0^t \left(\frac{M^2}{2\rho} + c_4 \gamma(\tau) \right) \|w_\varepsilon^+(\tau)\|^2 d\tau. \tag{5.48}$$

Further, we apply Gronwall's lemma and we obtain

$$w_\varepsilon(x, t) \leq 0, \quad \forall t \in (0, T). \tag{5.49}$$

Since the solution θ_ε having all these properties tends strongly to θ in $L^2(Q)$, by passing to the limit as $\varepsilon \rightarrow 0$, we obtain that $w(x, t) \leq 0$. □

The conclusion is that if we define for each $t \in [0, T]$,

$$s(t) = \sup \{z; \theta(z, t) < \theta_s\}, \tag{5.50}$$

the curve $z = s(t)$ separates the regions Q_- and Q_+ .

6. Uniqueness of the weak and smooth solution

For weak solutions only, the uniqueness is not rather obvious. But if the weak solution is sufficiently smooth such that it may imply a smooth separation surface, the uniqueness can be proved.

Under the conditions of [Proposition 5.3](#), in the 1D case the sets $\{(z, t); h(z, t) < 0\}$, $\{(z, t); h(z, t) > 0\}$, and $\{(z, t); h(z, t) = 0\}$ are open and it follows that any solution h satisfies

$$\begin{aligned} h_{zz} &= -\frac{f}{K_s}, & 0 < z < s(t), \\ -h_z(0, t) &= \frac{u_-(t) - K_s}{K_s}, & 0 \leq t \leq T, \\ h(s(t), t) &= 0. \end{aligned} \tag{6.1}$$

Hence

$$h_z(z, t) = \frac{K_s - u_-(t)}{K_s} - \frac{1}{K_s} \int_0^z f(\xi, t) d\xi, \tag{6.2}$$

and finally

$$h(z, t) = \frac{u_-(t) - K_s}{K_s} (s(t) - z) + \frac{1}{K_s} \int_z^{s(t)} d\zeta \int_0^\zeta f(\xi, t) d\xi, \quad 0 < z < s(t). \tag{6.3}$$

Since $s(t)$ is defined by $\theta(s(t), t) = \theta_s$ and θ is unique, it follows that h is uniquely defined on $0 < z < s(t)$, that is, in $\{(z, t); h(z, t) > 0\}$.

In Q_- we have

$$(C^*(h))_t - \Delta K^*(h) + (K(h))_z = f \quad \text{in } Q_-, \tag{6.4}$$

with flux boundary conditions on $\{z; z = L\} \cup \{z; z = s(t)\}$. Equivalently

$$\theta_t - \Delta \beta^*(h) + (K(h))_z = f \quad \text{in } \{\theta < \theta_s\}. \tag{6.5}$$

Since $C^*(h)$ is uniquely defined, so is h .

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