LOGISTIC EQUATION WITH THE p-LAPLACIAN AND CONSTANT YIELD HARVESTING

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We consider the positive solutions of a quasilinear elliptic equation with p-Laplacian, logistic-type growth rate function, and a constant yield harvesting. We use sub-super-solution methods to prove the existence of a maximal positive solution when the harvesting rate is under a certain positive constant.

1. Introduction

We consider weak solutions to the boundary value problem

$$\begin{aligned} -\Delta_p u = f(x,u) \equiv au^{p-1} - u^{\gamma-1} - ch(x) \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z := \operatorname{div}(|\nabla z|^{p-2}\nabla z)$; $p > 1$, $\gamma(> p)$, a and c are positive constants, Ω is a bounded domain in R^N; $N \geq 1$, with $\partial\Omega$ of class $C^{1,\beta}$ for some $\beta \in (0,1)$ and connected (if $N = 1$, we assume Ω is a bounded open interval), and $h : \bar{\Omega} \to R$ is a continuous function in $\bar{\Omega}$ satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \not\equiv 0$, $\max_{x\in\bar{\Omega}} h(x) = 1$, and $h(x) = 0$ for $x \in \partial\Omega$. By a weak solution of (1.1), we mean a function $u \in W_0^{1,p}(\Omega)$ that satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \, dx = \int_\Omega \left[au^{p-1} - u^{\gamma-1} - ch(x)\right] w \, dx, \quad \forall w \in C_0^\infty(\Omega). \tag{1.2}$$

From the standard regularity results of (1.1), the weak solutions belong to the function class $C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ (see [4, pages 115–116] and the references therein).

We first note that if $a \leq \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions, then (1.1) has no positive solutions. This follows since if u is a positive solution of (1.1), then u satisfies

$$\int_\Omega |\nabla u|^p dx = \int_\Omega \left[au^{p-1} - u^{\gamma-1} - ch(x)\right] u \, dx. \tag{1.3}$$

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But $\int_\Omega |\nabla u|^p dx \geq \lambda_1 \int_\Omega u^p dx$. Combining, we obtain $\int_\Omega [au^{p-1} - u^{\gamma-1} - ch(x)]u\,dx \geq \lambda_1 \int_\Omega u^p dx$ and hence $\int_\Omega (a-\lambda_1)u^p dx \geq \int_\Omega [u^{\gamma-1} + ch(x)]u\,dx \geq 0$. This clearly requires $a > \lambda_1$.

Next if $a > \lambda_1$ and c is very large, then again it can be proven that there are no positive solutions. This follows easily from the fact that if the solution u is positive, then $\int_\Omega [au^{p-1} - u^{\gamma-1} - ch(x)]dx$ is nonnegative. In fact, from the divergence theorem (see [4, page 151]),

$$\int_\Omega [au^{p-1} - u^{\gamma-1} - ch(x)]dx = -\int_{\partial\Omega} |\nabla u|^{p-2}\nabla u \cdot \nu\,dx \geq 0. \tag{1.4}$$

Thus,

$$c\int_\Omega h(x)dx \leq \int_\Omega [au^{p-1} - u^{\gamma-1}]dx \leq a^{(\gamma-1)/(\gamma-p)}|\Omega|. \tag{1.5}$$

Here in the last inequality, we used the fact that $u(x) \leq a^{1/(\gamma-p)}$ which can be proven by the maximum principle (see [4, page 173]).

This leaves us with the analysis of the case $a > \lambda_1$ and c small which is the focus of the paper.

Theorem 1.1. *Suppose that $a > \lambda_1$. Then there exists $c_0(a) > 0$ such that if $0 < c < c_0$, then (1.1) has a positive $C^{1,\alpha}(\bar{\Omega})$ solution u. Further, this solution u is such that $u(x) \geq (ch(x)/\lambda_1)^{1/(p-1)}$ for $x \in \bar{\Omega}$.*

Theorem 1.2. *Suppose that $a > \lambda_1$. Then there exists $c_1(a) \geq c_0$ such that for $0 < c < c_1$, (1.1) has a maximal positive solution, and for $c > c_1$, (1.1) has no positive solutions.*

Remark 1.3. Theorem 1.2 holds even when $h(x) > 0$ in $\bar{\Omega}$.

We establish Theorem 1.1 by the method of sub-supersolutions. By a supersolution (subsolution) ϕ of (1.1), we mean a function $\phi \in W_0^{1,p}(\Omega)$ such that $\phi = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi \cdot \nabla w\,dx \geq (\leq) \int_\Omega [a\phi^{p-1} - \phi^{\gamma-1} - ch(x)]w\,dx, \quad \forall w \in W, \tag{1.6}$$

where $W = \{v \in C_0^\infty(\Omega) \mid v \geq 0 \text{ in } \Omega\}$. Now if there exist subsolutions and supersolutions ψ and ϕ, respectively, such that $0 \leq \psi \leq \phi$ in Ω, then (1.1) has a positive solution $u \in W_0^{1,p}(\Omega)$ such that $\psi \leq u \leq \phi$. This follows from a result in [3].

Equation (1.1) arises in the studies of population biology of one species with u representing the concentration of the species and $ch(x)$ representing the rate of harvesting. The case when $p = 2$ (the Laplacian operator) and $\gamma = 3$ has been studied in [6]. The purpose of this paper is to extend some of this study to the p-Laplacian case. In [3], the authors studied (1.1) in the case when $c = 0$ (nonharvesting case). However, the $c > 0$ case is a semipositone problem ($f(x,0) < 0$) and studying positive solutions in this case is significantly harder. Very few results exist on semipositone problems involving the p-Laplacian operator (see [1, 2]), and these deal with only radial positive solutions with the domain Ω a ball or an annulus. In Section 2, when $a > \lambda_1$ and c is sufficiently small, we will construct nonnegative subsolutions and supersolutions ψ and ϕ, respectively, such that $\psi \leq \phi$, and establish Theorem 1.1. We also establish Theorem 1.2 in Section 2 and discuss the case when $h(x) > 0$ in $\bar{\Omega}$.

2. Proofs of theorems

Proof of Theorem 1.1. We first construct the subsolution ψ. We recall the antimaximum principle (see [4, pages 155–156]) in the following form. Let λ_1 be the principal eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions. Then there exists a $\delta(\Omega) > 0$ such that the solution z_λ of

$$\begin{gathered} -\Delta_p z - \lambda z^{p-1} = -1 \quad \text{in } \Omega, \\ z = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ is positive for $x \in \Omega$ and is such that $(\partial z_\lambda / \partial \nu)(x) < 0$, $x \in \partial\Omega$.

We construct the subsolution ψ of (1.1) using z_λ such that $\lambda_1 \psi(x)^{p-1} \geq ch(x)$. Fix $\lambda_* \in (\lambda_1, \min\{a, \lambda_1 + \delta\})$. Let $\alpha = \|z_{\lambda_*}\|_\infty$, $K_0 = \inf\{K \mid \lambda_1 K^{p-1} z_{\lambda_*}^{p-1} \geq h(x)\}$, and $K_1 = \max\{1, K_0\}$. Define $\psi = Kc^{1/(p-1)} z_{\lambda_*}$, where $K \geq K_1$ is to be chosen. Let $w \in W$. Then

$$\begin{aligned} &-\int_\Omega |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla w \, dx + \int_\Omega \left[a(\psi)^{p-1} - (\psi)^{\gamma-1} - ch(x)\right] w \, dx \\ &\quad = \int_\Omega \left[-cK^{p-1}\left(\lambda_* z_{\lambda_*}^{p-1} - 1\right) + ac\left(K z_{\lambda_*}\right)^{p-1} - \left(Kc^{1/(p-1)} z_{\lambda_*}\right)^{\gamma-1} - ch(x)\right] w \, dx \\ &\quad \geq \int_\Omega \left[-cK^{p-1}\left(\lambda_* z_{\lambda_*}^{p-1} - 1\right) + ac\left(K z_{\lambda_*}\right)^{p-1} - \left(Kc^{1/(p-1)} z_{\lambda_*}\right)^{\gamma-1} - c\right] w \, dx \\ &\quad = \int_\Omega \left[\left(a - \lambda_*\right)\left(K z_{\lambda_*}\right)^{p-1} - \left(K z_{\lambda_*}\right)^{\gamma-1} c^{(\gamma-p)/(p-1)} + \left(K^{p-1} - 1\right)\right] cw \, dx. \end{aligned} \tag{2.2}$$

Define $H(y) = (a - \lambda_*) y^{p-1} - y^{\gamma-1} c^{(\gamma-p)/(p-1)} + (K^{p-1} - 1)$. Then $\psi(x)$ is a subsolution if $H(y) \geq 0$ for all $y \in [0, K\alpha]$. But $H(0) = K^{p-1} - 1 \geq 0$ since $K \geq 1$ and $H'(y) = y^{p-2}[(a - \lambda_*)(p-1) - c^{(\gamma-p)/(p-1)}(\gamma - 1) y^{\gamma-p}]$. Hence $H(y) \geq 0$ for all $y \in [0, K\alpha]$ if $H(K\alpha) = (a - \lambda_*)(K\alpha)^{p-1} - (K\alpha)^{\gamma-1} c^{(\gamma-p)/(p-1)} + (K^{p-1} - 1) \geq 0$, that is, if

$$c \leq \left(\frac{(a - \lambda_*)(K\alpha)^{p-1} + (K^{p-1} - 1)}{(K\alpha)^{\gamma-1}} \right)^{(p-1)/(\gamma-p)}. \tag{2.3}$$

We define

$$c_1 = \sup_{K \geq K_1} \left(\frac{(a - \lambda_*)(K\alpha)^{p-1} + (K^{p-1} - 1)}{(K\alpha)^{\gamma-1}} \right)^{(p-1)/(\gamma-p)}. \tag{2.4}$$

Then for $0 < c < c_1$, there exists $\bar{K} \geq K_1$ such that

$$c < \left(\frac{(a - \lambda_*)(\bar{K}\alpha)^{p-1} + (\bar{K}^{p-1} - 1)}{(\bar{K}\alpha)^{\gamma-1}} \right)^{(p-1)/(\gamma-p)} \tag{2.5}$$

and hence $\psi(x) = \bar{K} c^{1/(p-1)} z_{\lambda_*}$ is a subsolution.

We next construct the supersolution $\phi(x)$ such that $\phi(x) \geq \psi(x)$. Let $G(y) = ay^{p-1} - y^{\gamma-1}$. Since $G'(y) = y^{p-2}[a(p-1) - (\gamma-1)y^{\gamma-p}]$, $G(y) \leq L = G(y_0)$, where $y_0 = [a(p-1)/(\gamma-1)]^{1/(\gamma-p)}$. Let ϕ be the positive solution of

$$-\Delta_p \phi = L \quad \text{in } \Omega, \tag{2.6}$$

$$\phi = 0 \quad \text{on } \partial\Omega. \tag{2.7}$$

Then for $w \in W$,

$$\begin{aligned}\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla w\,dx &= \int_\Omega Lw\,dx\\ &\geq \int_\Omega [a\phi^{p-1} - \phi^{\gamma-1}]w\,dx\\ &\geq \int_\Omega [a\phi^{p-1} - \phi^{\gamma-1} - ch(x)]w\,dx.\end{aligned} \tag{2.8}$$

Thus ϕ is a supersolution of (1.1). Also since $-\Delta_p\psi \leq a\psi^{p-1} - \psi^{\gamma-1} - ch(x) \leq L = -\Delta_p\phi$, by the weak comparison principle (see [4, 5]), we obtain $\phi \geq \psi \geq 0$. Hence there exists a solution $u \in W_0^{1,p}(\Omega)$ such that $\phi \geq u \geq \psi$. From the regularity results (see [4, pages 115–116] and the references therein), $u \in C^{1,\alpha}(\bar{\Omega})$. □

Remark 2.1. If $\tilde{u}$ is any $C^{1,\alpha}(\bar{\Omega})$ solution of (1.1), then by the weak comparison principle, $\|\tilde{u}\|_\infty \leq \|\phi\|_\infty$, where ϕ is as in (2.6).

Proof of Theorem 1.2. From Theorem 1.1, we know that for c small, there exists a positive solution. Whenever (1.1) has a positive solution u, (1.1) also has a maximal positive solution. This easily follows since ϕ in (2.6) is always a supersolution such that $\phi \geq u$. Next if for $c = \tilde{c}$, we have a positive solution $u_{\tilde{c}}$, then for all $c < \tilde{c}$, $u_{\tilde{c}}$ is a positive subsolution. Hence again using ϕ in (2.6) as the supersolution, we obtain a maximal positive solution for c. From (1.3), it is easy to see that for large c, there does not exist any positive solution. Hence there exists a $c_1(a) > 0$ such that there exists a maximal positive solution for $c \in (0, c_1)$ and no positive solution for $c > c_1$. □

Remark 2.2. The use of the antimaximum principle in the creation of the subsolution helps us to easily modify the proof of Theorem 1.1 to obtain a positive maximal solution for all $c < c_2(a) = \sup_{K\geq 1}(((a-\lambda_*)(K\alpha)^{p-1} + (K^{p-1}-1))/(K\alpha)^{\gamma-1})^{(p-1)/(\gamma-p)}$ even in the case when $h(x) > 0$ in $\bar{\Omega}$. Here $c_2(a) \geq c_0(a)$. (Of course when $h(x) > 0$ in $\bar{\Omega}$, our solution does not satisfy $u(x) \geq (ch(x)/\lambda_1)^{1/(p-1)}$ for $x \in \bar{\Omega}$.) Hence Theorem 1.2 also holds in the case when $h(x) > 0$ in $\bar{\Omega}$.

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