ON THE MODULUS OF U-CONVEXITY

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We prove that the moduli of U-convexity, introduced by Gao (1995), of the ultrapower $\widetilde{X}$ of a Banach space X and of X itself coincide whenever X is super-reflexive. As a consequence, some known results have been proved and improved. More precisely, we prove that $u_X(1) > 0$ implies that both X and the dual space X^* of X have uniform normal structure and hence the "worth" property in Corollary 7 of Mazcuñán-Navarro (2003) can be discarded.

1. Introduction

Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be *nonexpansive* provided the inequality

$$\|Tx - Ty\| \leqslant \|x - y\| \tag{1.1}$$

for every $x, y \in C$. Now, a Banach space X is said to have the *fixed point property* if every nonexpansive mapping $T : C \to C$, where C is a nonempty bounded closed convex subset of a Banach space X, has a fixed point.

Many mathematicians have established that, under various geometric properties of the Banach space X often measured by different moduli of convexity, the fixed point property of X is guaranteed.

How the classical modulus of convexity $\delta_X(\cdot)$ of a Banach space X, introduced by J. A. Clarkson in 1936, relates to the fixed point property has been widely studied. It is well known [8, Theorem 5.12, page 122] that if $\delta_X(1) > 0$, then X and X^* have the fixed point property. Recently, García-Falset proved that every weakly nearly uniformly smooth space has the fixed point property. To prove this, he introduced the following coefficient:

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\}, \tag{1.2}$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in $B_X(:= \{x \in X : \|x\| \leqslant 1\})$ and all $x \in S_X(:= \{x \in X : \|x\| = 1\})$. Indeed, he proved that a reflexive Banach space X with $R(X) < 2$ enjoys the fixed point property [7].

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On the other hand, in 1995, Gao defined the following modulus, for $\varepsilon \in [0,2]$:

$$
\begin{aligned}
u_X(\varepsilon) &:= \inf\left\{1 - \frac{1}{2}\|x+y\| : x, y \in S_X,\ f(x-y) \geqslant \varepsilon \text{ for some } f \in \nabla_x\right\} \\
&= \inf\left\{1 - \frac{1}{2}\|x+y\| : x \in S_X, y \in B_X \setminus \{0\},\ f(x-y) \geqslant \varepsilon \text{ for some } f \in \nabla_x\right\}.
\end{aligned}
\tag{1.3}
$$

Here ∇_x denotes the set of all norm 1 supporting functionals f of $x \in S_X$, that is, $f(x) = \|x\| = 1$. It is easy to see that $u_X(\varepsilon) \geqslant \delta_X(\varepsilon)$ for all $\varepsilon \in [0,2]$. The inequality may be strict even when X is a Hilbert space. In fact, $u_H(\varepsilon) = 1 - \sqrt{1-\varepsilon/2}$ for $\varepsilon \in [0,2]$, where H is a Hilbert space. Gao proved that if there exists $\delta > 0$ such that $u_X(1/2 - \delta) > 0$, then X has uniform normal structure [4].

Mazcuñán-Navarro [10] proved a relationship between two of the above notions. Namely, if there exists $\delta > 0$ such that $u_X(1-\delta) > 0$, then $R(X) < 2$ [10, Theorem 5].

This paper is organized as follows: in Section 2 we prove some inequalities concerning the modulus of U-convexity, introduced by Gao, and other constants. By these inequalities, we immediately obtain some results proved by Gao [4] and Mazcuñán-Navarro [10]. Finally, in Section 3, we prove that if a Banach space X is super-reflexive, then the moduli of U-convexity of the ultrapower $\widetilde{X}$ of X and of X itself coincide. Using ultrapower methods we show, a Banach space X and its dual X^* have uniform normal structure whenever $u_X(1) > 0$. The paper concludes with an example showing that such a condition is sharp.

2. The modulus of U-convexity

It was proved in [3] that $u_X(\cdot)$ is continuous on $[0,2)$. Hence we restate [10, Theorem 5] the following.

Theorem 2.1. *Let X be a Banach space with $u_X(1) > 0$. Then $R(X) < 2$.*

Furthermore, the above result follows directly from the following inequality and the continuity of $u_X(\cdot)$.

Proposition 2.2. *Let X be a Banach space. Then*

$$
R(X) \leqslant \inf\{\max\{\varepsilon+1, 2(1-u_X(\varepsilon))\} : \varepsilon \in [0,1]\}. \tag{2.1}
$$

Proof. Suppose the inequality does not hold. Then there exist $\varepsilon \in [0,1)$, a weakly null sequence $\{x_n\}$ in B_X and $x \in S_X$ such that

$$
\liminf_{n\to\infty} \|x+x_n\| > \max\{\varepsilon+1, 2(1-u_X(\varepsilon))\}. \tag{2.2}
$$

Take $f \in \nabla_x$. So $f(x_n) \to 0$ as $n \to \infty$. Hence $f(x-x_n) \geqslant \varepsilon$ for all sufficiently large n and we then have $\|x+x_n\| \leqslant 2(1-u_X(\varepsilon))$, which is a contradiction. □

The following example shows us that $\varepsilon = 1$ is the largest number such that

$$
u_X(\varepsilon) > 0 \Longrightarrow R(X) < 2. \tag{2.3}
$$

Example 2.3. For $p \in (1,\infty)$, we consider the l_p space equipped with the norm

$$\|x\|' = \|x^+\|_p + \|x^-\|_p, \tag{2.4}$$

where x^+ and x^- are positive and negative parts of $x \in l_p$, that is, $(x^+)_n = \max\{x_n, 0\}$ and $(x^-)_n = \max\{-x_n, 0\}$. We write $l_{p,1}$ to denote the space $(l_p, \|\cdot\|')$. This space was introduced and studied by Bynum (see [2]). It is not difficult to see that $R(l_{p,1}) = 2$ and hence $u_{l_{p,1}}(1) = 0$. Moreover, it is well known that $u_{l_{p,1}}(\varepsilon) \geqslant \delta_{l_{p,1}}(\varepsilon) > 0$ for all $\varepsilon > 2^{1/p}$ (see [2]).

Now we let $X = (\oplus l_{p_n,1})_{l_2}$, where $\{p_n\} \subset (1,\infty)$ is a sequence tending to infinity. It is easy to see that $R(X) = 2$ and $u_X(\varepsilon) > 0$ for all $\varepsilon > 1$.

In an attempt to simplify Schäffer's notion of girth and perimeter [12], the James constant

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in B_X\} \tag{2.5}$$

was introduced. It is easy to see that a Banach space X is uniformly nonsquare if and only if $J(X) < 2$.

As we prove in Proposition 2.2, a relationship between the modulus of U-convexity and the James constant is obtained.

Proposition 2.4. *Let X be a Banach space. Then*

$$J(X) \leqslant \inf\{\max\{\varepsilon + 1, 2(1 - u(\varepsilon))\} : \varepsilon \in [0,1]\}. \tag{2.6}$$

In particular, if $u_X(1) > 0$, then X is uniformly nonsquare [4, Theorem 2].

In order to extend this result (see Theorem 2.8), we need the following two lemmas.

Lemma 2.5 (Bishop-Phelps-Bollobás [1]). *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B_X$ and $h \in S_{X^*}$ with $1 - h(z) < \varepsilon^2/4$, then there exist $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Lemma 2.6. *Let*

$$\begin{aligned} u_X'(\varepsilon) = \inf\Big\{ 1 - \frac{1}{2}\|x+y\| : x, y \in S_X,\ f(x) > 1 - \eta, \\ \text{and } f(x-y) \geqslant \varepsilon \text{ for some } f \in S_{X^*},\ \eta > 0 \Big\}. \end{aligned} \tag{2.7}$$

Then for each $\varepsilon \in [0,2)$ and for each $\xi > 0$, there exists $\eta > 0$ such that

$$u'(\varepsilon) + \xi > u(\varepsilon - \eta) - \frac{\eta}{2}. \tag{2.8}$$

Proof. Let $\xi > 0$. Then there exist $\eta > 0$, $x, y \in S_X$, and $f \in S_{X^*}$ such that

$$1 - \frac{1}{2}\|x+y\| < u'(\varepsilon) + \xi, \qquad f(x-y) \geqslant \varepsilon, \qquad f(x) > 1 - \frac{\eta^2}{4}. \tag{2.9}$$

By the Bishop-Phelps-Bollobás theorem, there exist $z \in S_X$ and $g \in \nabla_z$ such that

$$\|g-f\| < \eta, \qquad \|z-x\| < \eta. \tag{2.10}$$

Hence, $1 - 1/2\|x+y\| \geqslant 1 - 1/2\|z+y\| - \eta/2$. Furthermore,

$$g(z-y) = 1 - g(y) = 1 - (g-f)(y) - f(y) \geqslant 1 - \|g-f\| - 1 + \varepsilon > \varepsilon - \eta. \tag{2.11}$$

Therefore, by the definition of $u(\cdot)$, $u'(\varepsilon) + \xi > u(\varepsilon - \eta) - \eta/2$. □

Now by the continuity of $u(\cdot)$ and the fact that $u(\cdot) \geqslant u'(\cdot)$ on $[0,2)$, we have the following.

Corollary 2.7. *$u(\cdot) = u'(\cdot)$ on $[0,2)$.*

Theorem 2.8. *A Banach space X is uniformly nonsquare if and only if there exists $\delta > 0$ such that $u_X(2-\delta) > 0$.*

Proof. Necessity is trivially true, since $u_X(\varepsilon) \geqslant \delta_X(\varepsilon)$ for all $\varepsilon \in [0,2]$. We now prove sufficiency. Since there exists $\delta > 0$ such that $u_X(2-\delta) > 0$, we choose $\eta > 0$ so that $u_X(2-\eta) > \eta$. Suppose that X is not uniformly nonsquare. Then there exists sequence $\{x_n\}, \{y_n\} \subset S_X$ such that

$$\big|\|x_n + y_n\| - 1\big| < \frac{1}{n}, \qquad \big|\|x_n - y_n\| - 1\big| < \frac{1}{n} \tag{2.12}$$

for all $n \in \mathbb{N}$. Let $f_n \in \nabla_{x_n}$. Then $f_n(y_n) \to 0$. Indeed, $|f_n(y_n)| < 1/n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we put

$$x_n' = \frac{x_n + y_n}{\|x_n + y_n\|}, \qquad y_n' = \frac{-x_n + y_n}{\|-x_n + y_n\|}. \tag{2.13}$$

Hence $f_n(x_n') > (n-1)/(n+1)$ and $f_n(x_n' - y_n') > 2(n-1)/(n+1)$ for all $n \in \mathbb{N}$, and $\|x_n' + y_n'\| \to 2$ as $n \to \infty$. Taking n sufficiently large shows $u_X(2-\eta) \leq \eta$, which is a contradiction. In fact, we choose n so large that $1 - 1/2\|x_n' + y_n'\| \leqslant \eta$ and $4/(n+1) \leqslant \eta$. □

Recall that a bounded convex subset K of a Banach space X is said to have *normal structure* if for every convex subset H of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x-y\| : x, y \in H\}. \tag{2.14}$$

A Banach space X is said to have *weak normal structure* if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both notions coincide. A Banach space X is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \sup\{\|x-y\| : x, y \in K\}. \tag{2.15}$$

It was proved by Kirk that every reflexive Banach space with normal structure has the fixed point property (see [9]).

Combining Theorem 2.8 with the "worth" property, introduced by Sims [14], we have the following.

Corollary 2.9. *If there exists $\delta > 0$ such that $u_X(2-\delta) > 0$ and X has the worth property, then X has normal structure.*

In particular, if $u_X(1) > 0$ and X has the worth property, then X has normal structure [10, Corollary 8]. In the next section, we will see that this conclusion still holds regardless of whether or not X has the worth property. Recall that a Banach space is said to have the *worth property* provided that $\lim_n |\|x_n - x\| - \|x_n + x\|| = 0$ whenever $\{x_n\}$ is a weakly null sequence in X and $x \in X$ (see [14]).

3. Normal structures and the modulus of U-convexity

The Banach space ultrapower of a Banach space has proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. First we recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I,\ i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$,
(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I,X_i)$ denote the subspace of the product space $\Pi_{i\in I} X_i$ equipped with the norm $\|(x_i)\| := \sup_{i\in I}\|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \Big\{(x_i) \in l_\infty(I,X_i) : \lim_{\mathcal{U}} ||x_i|| = 0\Big\}. \tag{3.1}$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I,X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows easily from (ii) and the definition of the quotient norm that

$$||(x_i)_{\mathcal{U}}|| = \lim_{\mathcal{U}} ||x_i||. \tag{3.2}$$

In the following, we will restrict our index set I to be $\mathbb{N}$, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ for the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically. For more details see [13].

The main result in this paper is the following.

Theorem 3.1. *Suppose that X is super-reflexive. Then $u_{\widetilde{X}}(\cdot) = u_X(\cdot)$ for all $\varepsilon \in [0,2)$. In particular, if $u_X(\varepsilon) > 0$ for some $\varepsilon \in (0,2)$, then $u_{\widetilde{X}}(\varepsilon) = u_X(\varepsilon)$.*

Proof. It is easy to see that $u_{\tilde{X}}(\varepsilon) \leqslant u_X(\varepsilon)$ for all $\varepsilon \in [0,2)$. It suffices to prove that $u_{\tilde{X}}(\varepsilon) \geqslant u'_X(\varepsilon)$ for all $\varepsilon \in [0,2)$, where $u'_X(\cdot)$ is defined in Lemma 2.6. Let $\tilde{x}, \tilde{y} \in S_{\tilde{X}}$ and $\tilde{f} \in \nabla_{\tilde{x}}$ be such that $\tilde{f}(\tilde{x}-\tilde{y}) \geqslant \varepsilon$. We write $\tilde{x} = (x_n)_{\mathcal{U}}$ and $\tilde{y} = (y_n)_{\mathcal{U}}$, where $x_n, y_n \in X$ for all $n \in \mathbb{N}$. By the super-reflexivity of X, we also write $\tilde{f} = (f_n)_{\mathcal{U}}$, where $f_n \in X^*$ for all $n \in \mathbb{N}$ (see [13]). Then, we have

$$\lim_{\mathcal{U}} ||x_n|| = \lim_{\mathcal{U}} ||y_n|| = \lim_{\mathcal{U}} f_n(x_n) = 1, \qquad \lim_{\mathcal{U}} f_n(y_n) \leqslant 1-\varepsilon. \tag{3.3}$$

Discarding some terms of the above sequences, we may assume that no x_n, y_n or f_n is 0. Then put $x'_n = x_n/\|x_n\|$, $y'_n = y_n/\|y_n\|$, and $f'_n = f_n/\|f_n\|$. Given $\eta > 0$, we have $\{n \in \mathbb{N} : f'_n(x'_n) > 1-\eta\} \in \mathcal{U}$ and $\{n \in \mathbb{N} : 1 - 1/2\|x'_n + y'_n\| > u'_X(\varepsilon) - \eta\} \in \mathcal{U}$. It follows that

$$1 - \frac{1}{2}\|\tilde{x}+\tilde{y}\| = 1 - \frac{1}{2}\lim_{\mathcal{U}} ||x_n + y_n|| \geqslant u'_X(\varepsilon) - \eta. \tag{3.4}$$

This implies that $u_{\tilde{X}}(\varepsilon) \geqslant u'_X(\varepsilon)$ and the proof is complete. □

Recall that a Banach space X is said to be a *U-space* if $u_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2)$. In order to prove that being a U-space is a super-property, that is, every Banach space finitely representable in a U-space is a U-space, Gao and Lau used some equivalent formulations of U-spaces proved through the properties of Asplund spaces (see [6, Theorem 3.7]). We obtain this through a new approach, as a consequence of Theorem 3.1.

Corollary 3.2 [6, Theorem 4.3]. *A Banach space X is a U-space if and only if $\tilde{X}$ is a U-space.*

Proposition 3.3. *If $u_X(1) > 0$, then X and X^* have uniform normal structure.*

Proof. It suffices to prove that X has weak normal structure whenever $u_{\tilde{X}}(1) > 0$ or $u_{\widetilde{X^*}}(1) > 0$, since $u_X(\varepsilon) > 0$ implies that X is super-reflexive, and then $u_{\tilde{X}}(1) = u_X(1) > 0$. Now suppose that X fails to have weak normal structure. Then, by the classical argument, there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$\lim_n ||x - x_n|| = 1 \quad \forall x \in \operatorname{co}\{x_n\}_{n=1}^{\infty}. \tag{3.5}$$

We choose a subsequence of $\{x_n\}_{n=1}^{\infty}$, denoted again by $\{x_n\}_{n=1}^{\infty}$, such that

$$\lim_n ||x_n - x_{n+1}|| = 1, \qquad |f_{n+1}(x_n)| < \frac{1}{n}, \qquad |f_n(x_{n+1})| < \frac{1}{n} \tag{3.6}$$

for all $n \in \mathbb{N}$, where $f_n \in \nabla_{x_n}$. Put $\tilde{x} = (x_n - x_{n+1})$, $\tilde{y} = (x_n)$, and $\tilde{f} = (-f_{n+1})$. Then $\|\tilde{f}\| = \tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x}-\tilde{y}) = \|\tilde{x}\| = \|\tilde{y}\| = 1$. Furthermore,

$$2 \geqslant \|\tilde{x}+\tilde{y}\| = \lim_{\mathcal{U}} ||2x_n - x_{n+1}|| \geqslant \lim_{\mathcal{U}} f_n(2x_n - x_{n+1}) = 2. \tag{3.7}$$

Hence, $u_{\tilde{X}}(1) = 0$.

Next, let $\widetilde{g} = (f_n)$. Hence

$$2 \geqslant \|\widetilde{f}+\widetilde{g}\| \geqslant (\widetilde{f}+\widetilde{g})(\widetilde{x}) = \lim_{\mathcal{U}} \big(- f_{n+1} + f_n \big) \big(x_n - x_{n+1} \big) = 2. \tag{3.8}$$

Moreover, $\widetilde{g}(\widetilde{y}) = 1$ and $\widetilde{f}(\widetilde{y}) = 0$. This implies that

$$u_{X^*}(1) = u_{\widetilde{X^*}}(1) = u_{(\widetilde{X})^*}(1) = 0. \tag{3.9}$$

The proof is finished. □

Theorem 3.4. *If* $u_X(\varepsilon) > \max\{0,(\varepsilon-1)/2\}$ *for some* $\varepsilon \in (0,2)$, *then* X *has uniform normal structure. Furthermore, if* $u_X(\varepsilon) > \max\{0,\varepsilon-1\}$ *for some* $\varepsilon \in (0,2)$, *then both* X *and* X^* *have uniform normal structure.*

Proof. Let $t \in [0,1]$ and follow the proof of Proposition 3.3, but now put $\widetilde{x} = (x_n - x_{n+1})$, $\widetilde{y} = ((1-t)x_n + tx_{n+1})$, and $\widetilde{f} = (-f_{n+1})$. Then $\|\widetilde{f}\| = \widetilde{f}(\widetilde{x}) = \|\widetilde{x}\| = 1$ and $1/2 \leqslant \max\{t, 1-t\} \leqslant \|\widetilde{y}\| \leqslant 1$. Furthermore, we have

$$\begin{aligned}
\widetilde{f}(\widetilde{x}-\widetilde{y}) &= \lim_{\mathcal{U}} \big(- f_{n+1} \big) \big(tx_n - (1+t)x_{n+1} \big) = 1+t, \\
\|\widetilde{x}+\widetilde{y}\| &= \lim_{\mathcal{U}} \big\|(2-t)x_n - (1-t)x_{n+1}\big\| \geqslant \lim_{\mathcal{U}} f_n\big((2-t)x_n - (1-t)x_{n+1}\big) = 2-t.
\end{aligned} \tag{3.10}$$

Hence $u_{\widetilde{X}}(1+t) \leqslant t/2$ and this implies that $u_X(\varepsilon) \leqslant \max\{0,(\varepsilon-1)/2\}$ for all $\varepsilon \in (0,2)$, which is a contradiction.

Next, we put $\widetilde{g} = (tf_{n+1} + (1-t)f_n)$ and $\widetilde{z} = (-x_{n+1})$. It is easy to see that $\widetilde{f}(\widetilde{z}) = 1$, $(\widetilde{f}-\widetilde{g})(\widetilde{z}) = 1+t$, and $1/2 \leqslant \max\{t,1-t\} \leqslant \|\widetilde{g}\| \leqslant 1$. Moreover, we have

$$\begin{aligned}
\|\widetilde{f}+\widetilde{g}\| &= \lim_{\mathcal{U}} \big\| - (1-t) f_{n+1} + (1-t) f_n \big\| \\
&\geqslant \lim_{\mathcal{U}} \big(- (1-t) f_{n+1} + (1-t) f_n \big) \big(- x_{n+1} + x_n \big) \\
&= 2(1-t).
\end{aligned} \tag{3.11}$$

Therefore $u_{X^*}(1+t) \leqslant t$ or $u_{X^*}(\varepsilon) \leqslant \max\{0,\varepsilon-1\}$ for all $\varepsilon \in (0,2)$. Hence if $u_{X^*}(\varepsilon) > \max\{0,\varepsilon-1\}$ for some $\varepsilon \in (0,2)$, then X has normal structure. □

Corollary 3.5 (see [5, Theorem 8] and [11, Corollary 3]). *If* $\delta_X(\varepsilon) > \max\{(\varepsilon-1)/2, 0\}$ *for some* $\varepsilon \in (0,2)$, *then* X *has uniform normal structure.*

Example 3.6. For $p \in (1,\infty)$, we denoted by $l_{p,\infty}$ the l_p space with the norm

$$\|x\| = \max\big\{\|x^+\|_p, \|x^-\|_p\big\}. \tag{3.12}$$

It is known that $l_{p,\infty}$ is a super-reflexive space that fails normal structure [2]. Hence $u_{l_{p,\infty}}(1) = 0$ while $u_{l_{p,\infty}}(\varepsilon) \geqslant \delta_{l_{p,\infty}}(\varepsilon) > 0$ for all $\varepsilon > 1$. This example shows that the condition in Proposition 3.3 is the best possible.

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