A QUASI-LINEAR PARABOLIC SYSTEM OF CHEMOTAXIS

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We consider a quasi-linear parabolic system with respect to unknown functions u and v on a bounded domain of n-dimensional Euclidean space. We assume that the diffusion coefficient of u is a positive smooth function $A(u)$, and that the diffusion coefficient of v is a positive constant. If $A(u)$ is a positive constant, the system is referred to as so-called Keller-Segel system. In the case where the domain is a bounded domain of two-dimensional Euclidean space, it is shown that some solutions to Keller-Segel system blow up in finite time. In three and more dimensional cases, it is shown that solutions to so-called Nagai system blow up in finite time. Nagai system is introduced by Nagai. The diffusion coefficients of Nagai system are positive constants. In this paper, we describe that solutions to the quasi-linear parabolic system exist globally in time, if the positive function $A(u)$ rapidly increases with respect to u.

1. Introduction

Our purpose is to show the well-posedness of a system of parabolic equations proposed in mathematical biology. Its origin is in Keller and Segel [9], describing the chemotactic aggregation of cellular slime molds which move preferentially toward relatively high concentrations of a chemical secreted by the amoebae themselves. Here, we study the simplified system,

$$\begin{aligned} u_t &= \nabla \cdot \big(A(u)\nabla u - \chi u \nabla v\big) \quad \text{in } \Omega\times(0,\infty),\\ \tau v_t &= d\Delta v - av + bu \quad \text{in } \Omega\times(0,\infty),\\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty),\\ u(\cdot,0) &= u_0, \quad v(\cdot,0) = v_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

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where $\Omega \subset \mathbf{R}^N$ $(N = 1,2,3,\ldots)$ is a bounded domain with smooth boundary $\partial\Omega$. u_0 and v_0 are smooth non-negative functions on $\overline{\Omega}$ satisfying

$$u_0 \not\equiv 0 \quad \text{in } \Omega, \qquad \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = \frac{\partial \Delta v_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

τ, χ, d, a and b are positive constants, and $A(\cdot)$ is a smooth function defined on $[0,\infty)$.

If $A(\cdot)$ is a positive constant, this system describes the mean field of many self-gravitating particles, and the cases $\tau > 0$ and $\tau = 0$ are referred to as Keller-Segel system and Nagai system, respectively. In this case, if $N = 2$, there is a threshold of the initial value in L^1 norm for existence of the solution globally in time, denoted by c_*, so that if $\|u_0\|_1 < c_*$, then the solution exists globally in time and is bounded ([2, 5, 11, 14]), while blowup of the solution occurs in finite time in the case of $\|u_0\|_1 > c_*$ ([8, 11, 12, 17]). This phenomenon was conjectured by [4] and gave a motivation of our previous works on the mass quantization of collapses in the blowup solution. See [19, 20] for more details. On the other hand, in [13, 16], Nagai and the first author studied

$$\begin{gathered} u_t = \nabla \cdot (\nabla u - \chi u \nabla \phi(v)) \quad \text{in } \Omega \times (0,\infty), \\ 0 = d\Delta v - av + bu \quad \text{in } \Omega \times (0,\infty), \end{gathered} \tag{1.3}$$

and gave necessary and sufficient conditions for the existence of the solution globally in time, assuming that the sensitivity function $\phi = \phi(v)$ takes the form $\phi(v) = v^p$ for $p > 0$ or $\phi(v) = \log v$. Also, Biler [3] studied the same problem for more general ϕ. However, system (1.1) has not been studied so much and in this paper, we show that the solution exists globally in time if

$$\inf_{0 \le u} A(u) > 0, \qquad \liminf_{u \to \infty} \frac{A(u)}{u^{\tilde{\gamma}}} > 0 \tag{1.4}$$

hold for some $\tilde{\gamma} > \max((N-2)/N, 0)$. Those conditions imply

$$A(u) \ge \alpha u^{\gamma} + \beta \quad (u \ge 0) \tag{1.5}$$

for some $\alpha > 0$, $\beta > 0$, $\gamma \in (\max(0,(N-2)/N), \min(\tilde{\gamma},1))$. Henceforth, we put $\tau = d = a = b = 1$ for simplicity.

To state the main theorem, we need a few notations. First, given $T \in (0,\infty]$, we set $Q_T = \Omega \times (0,T)$ and take the function spaces $H^{\ell}(\overline{\Omega})$ and $H^{\ell,\ell/2}(\overline{Q_T})$ as in [10], where ℓ is a non-integral positive number. That is, $H^{\ell}(\overline{\Omega})$ denotes the Banach space of continuous functions $\{w(x)\}$ defined on $\overline{\Omega}$, provided with continuous derivatives up to order $[\ell]$ and the finite

$$|w|_{\Omega}^{(\ell)} = \langle w \rangle_{\Omega}^{(\ell)} + \sum_{j=0}^{[\ell]} \langle w \rangle_{\Omega}^{(j)} = \sum_{|\delta|=[\ell]} \sup_{x',x \in \Omega} \frac{|D_x^{\delta} w(x) - D_x^{\delta} w(x')|}{|x - x'|^{\ell-[\ell]}} + \sum_{j=0}^{[\ell]} \sum_{|\delta|=j} \sup_{x \in \Omega} |D_x^{\delta} w(x)|, \tag{1.6}$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_N)$, $|\delta| = \delta_1 + \delta_2 + \cdots + \delta_N$, and $D_x^{\delta} = D_{x_1}^{\delta_1} D_{x_2}^{\delta_2} \cdots D_{x_N}^{\delta_N}$. Next,

$H^{\ell,\ell/2}(\overline{Q_T})$ denotes the Banach space of continuous functions $\{w(x,t)\}$ defined on $\overline{Q_T}$, provided with continuous derivatives $\{D_t^r D_x^\delta w(x,t)\}$ for $2r+|\delta|<\ell$ and the finite

$$\begin{aligned} |w|_{Q_T}^{(\ell)} &= \langle w\rangle_{x,Q_T}^{(\ell)} + \langle w\rangle_{t,Q_T}^{(\ell/2)} + \sum_{j=0}^{[\ell]} \langle w\rangle_{Q_T}^{(j)} \\ &= \sum_{2r+|\delta|=[\ell]} \sup_{(x',t),(x,t)\in Q_T} \frac{|D_t^r D_x^\delta w(x,t) - D_t^r D_x^\delta w(x',t)|}{|x-x'|^{\ell-[\ell]}} \\ &\quad + \sum_{2r+|\delta|=[\ell]} \sup_{(x,t'),(x,t)\in Q_T} \frac{|D_t^r D_x^\delta w(x,t) - D_t^r D_x^\delta w(x,t')|}{|t-t'|^{(\ell-[\ell])/2}} \\ &\quad + \sum_{j=0}^{[\ell]} \sum_{2r+|\delta|=j} \sup_{(x,t)\in Q_T} |D_t^r D_x^\delta w(x,t)|. \end{aligned} \tag{1.7}$$

Then, the first theorem guarantees the local well-posedness of (1.1).

Theorem 1.1 (time-local solution). *If (1.2) holds and*

$$\inf_{u\geq 0} A(u) = \beta > 0 \tag{1.8}$$

in (1.1), then there is $T>0$ such that (1.1) has a unique classical solution (u,v) in Q_T satisfying

$$|u|_{Q_T}^{2+\theta} < +\infty, \qquad |v|_{Q_T}^{2+\theta} < +\infty \tag{1.9}$$

for some $\theta \in (0,1)$.

The main theorem is now stated as follows.

Theorem 1.2 (time-global solution). *If (1.2) and (1.4) hold in (1.1), then it has a unique classical solution (u,v) in Q_∞ satisfying*

$$|u|_{Q_\infty}^{2+\theta} < +\infty, \qquad |v|_{Q_\infty}^{2+\theta} < +\infty \tag{1.10}$$

for some $\theta \in (0,1)$.

In Section 2, we consider some linear system related to (1.1). By using some estimates of the solutions and a standard contraction mapping principle, we show Theorem 1.1.

In Section 3, we introduce Lyapunov function of (1.1). By using the Lyapunov function, a maximal regularity in [6] and Moser's technique in [1], we show Theorem 1.2.

2. Time local solution

We take $T>0$, $p>N+2$, and $\ell \in (0, 1-(N+2)/p)$. Given

$$u \in H^{1+\ell,\ell/2}(\overline{Q_T}) \equiv \left\{ w \in H^{\ell,\ell/2}(\overline{Q_T}) \mid \nabla_x u \in H^{\ell,\ell/2}(\overline{Q_T})^N \right\}, \tag{2.1}$$

we apply [10, Theorem IV.5.3] and get the unique solution

$$(U,v) \in H^{3+\ell,(3+\ell)/2}(\overline{Q_T})^2 \tag{2.2}$$

to the system

$$\begin{aligned} U_t &= \nabla \cdot (A(u)\nabla U - U\nabla v) \quad \text{in } Q_T, \\ v_t &= \Delta v - v + u \quad \text{in } Q_T, \\ \frac{\partial U}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ U(\cdot,0) &= u_0, \quad v(\cdot,0) = v_0 \quad \text{in } \Omega. \end{aligned} \tag{2.3}$$

Thus, we can introduce the operator $\mathcal{F}$ on $\{w \in H^{1+\ell,\ell/2}(\overline{Q_T}) \mid w \geq 0 \text{ in } \overline{Q_T}\}$ by

$$\mathcal{F}u = U. \tag{2.4}$$

Let

$$\begin{aligned} B(M,T) \equiv \Big\{ & u \in H^{1+\ell,\ell/2}(\overline{Q_T}) \cap C([0,T];\ W^{2,p}(\Omega)) \mid u \geq 0 \text{ in } \overline{Q_T}, \\ & \partial u/\partial \nu = 0 \ \text{ on } \partial\Omega \times [0,T],\ \max_{0\leq t\leq T} \left(\|\Delta u\|_p^p + \|u\|_p^p\right)^{1/p} \leq M \Big\} \end{aligned} \tag{2.5}$$

for

$$M = 2^{1/p}\left(||\Delta u_0||_p^p + ||u_0||_p^p\right)^{1/p}, \tag{2.6}$$

where $\|\cdot\|_q$ and $\|\cdot\|_{m,q}$ denote the standard norms in $L^p(\Omega)$ and $W^{m,p}(\Omega)$, respectively.

Henceforth, C_i ($i = 1,2,3,4,5$) denote positive constants determined by M and T, which are monotone increasing in T. First, we show the following.

Lemma 2.1. *If $u \in B(M,T)$, then the solution v to*

$$v_t = \Delta v - v + u \quad \text{in } Q_T, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{2.7}$$

satisfies that

$$\max_{0\leq t\leq T}\left(||\nabla\Delta v(\cdot,t)||_p + ||v(\cdot,t)||_p\right) \leq C_1. \tag{2.8}$$

Proof. Let $\Delta_{N,q}$ be the Laplacian with homogeneous Neumann boundary condition, realized as an m-accretive operator in $L^q(\Omega)$, where $q \in (1,\infty)$. Its domain is given by

$$D(\Delta_{N,q}) \equiv \left\{ w \in W^{2,q}(\Omega) \mid \frac{\partial w}{\partial \nu} = 0 \ \text{ on } \partial\Omega \right\}, \tag{2.9}$$

and henceforth we will write it as Δ_N for simplicity. Then, $-\Delta_N + 1$ generates the analytic semi-group $\{e^{t(\Delta_N-1)}\}_{t\geq 0}$ ([7]), and it holds that

$$v(\cdot,t)=e^{(\Delta_N-1)t}v_0+\int_0^t e^{(\Delta_N-1)(t-s)}u(\cdot,s)ds. \tag{2.10}$$

On the other hand, we have

$$\|(-\Delta_N+1)^{3/2}v_0\|_p\leq C_2 \tag{2.11}$$

from the assumption, and if $u\in B(M,T)$, then it holds that $u(\cdot,t)\in D(\Delta_N)$ and

$$\|(-\Delta_N+1)u(\cdot,t)\|_p\leq\left(\|\Delta u(\cdot,t)\|_p^p+\|u(\cdot,t)\|_p^p\right)^{1/p}\leq M \tag{2.12}$$

for $t\in[0,T]$. Thus, we have from (2.10), (2.11), and (2.12) that

$$\begin{aligned}\|(-\Delta_N+1)^{3/2}v(\cdot,t)\|_p&\leq C_2+\int_0^t\|(-\Delta_N+1)^{1/2}e^{(\Delta_N-1)(t-s)}(-\Delta_N+1)u(\cdot,s)\|_p ds\\&\leq C_2+\int_0^t\frac{C_3}{\sqrt{t-s}}M\,ds\leq C_2+\frac{1}{2}\sqrt{T}C_3M,\end{aligned} \tag{2.13}$$

and in particular,

$$\|\nabla\Delta v(\cdot,t)\|_p\leq C_4 \tag{2.14}$$

follows for $t\in[0,T]$. It also holds that

$$\|v(\cdot,t)\|_p\leq C_5 \tag{2.15}$$

and the proof is complete. □

Now, we show the following.

Lemma 2.2. *There exists $T_1>0$ such that*

$$\mathcal{F}B(M,T)\subset B(M,T) \tag{2.16}$$

holds for any $T\in(0,T_1]$.

Proof. Henceforth, C_i $(i=6,7,8,\ldots,20)$ denote positive constants depending only on M. We have $W^{2,p}(\Omega)\subset W^{1,\infty}(\Omega)$ and it holds that

$$\sup_{0\leq t\leq S}\left(\|u(\cdot,t)\|_{1,\infty}+\|v(\cdot,t)\|_{1,\infty}\right)\leq C_6 \tag{2.17}$$

for $u\in B(M,S)$ if $S\in(0,1]$.

From the first equation of (2.3), we obtain that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega |U|^p dx &= \int_\Omega \nabla\cdot(A(u)\nabla U - U\nabla v)|U|^{p-2}U\,dx\\ &= -(p-1)\int_\Omega A(u)|U|^{p-2}|\nabla U|^2 dx + (p-1)\int_\Omega |U|^{p-2}U\nabla v\cdot\nabla U\,dx\\ &\le -\beta(p-1)\int_\Omega |U|^{p-2}|\nabla U|^2 dx + (p-1)\int_\Omega |U|^{p-1}|\nabla v||\nabla U|dx\\ &\le -\frac{1}{2}\beta(p-1)\int_\Omega |U|^{p-2}|\nabla U|^2 dx + C_7\int_\Omega |\nabla v|^2|U|^p dx.\end{aligned}\tag{2.18}$$

We have $W^{2,p}(\Omega)\subset H^{1+\ell}(\overline{\Omega})$, and hence Lemma 2.1 implies the boundedness of ∇v. Therefore, it follows that

$$\frac{d}{dt}\int_\Omega |U|^p dx \le C_8\int_\Omega |U|^p dx.\tag{2.19}$$

The first equation of (2.3) implies also that

$$\begin{aligned}\frac{1}{p(p-1)}\frac{d}{dt}\int_\Omega |\Delta U|^p dx &= \frac{1}{p-1}\int_\Omega |\Delta U|^{p-2}\Delta U\cdot\Delta U_t\,dx\\ &= -\frac{1}{p-1}\int_\Omega \nabla U_t\cdot\nabla\left(|\Delta U|^{p-2}\Delta U\right)dx\\ &= -\int_\Omega \nabla U_t\cdot\left(|\Delta U|^{p-2}\Delta\nabla U\right)dx\\ &= -\int_\Omega \left[\nabla\left(\nabla\cdot(A(u)\nabla U - U\nabla v)\right)\cdot\nabla\Delta U\right]|\Delta U|^{p-2}dx.\end{aligned}\tag{2.20}$$

Here, we have

$$\begin{aligned}&\nabla\{\nabla\cdot(A(u)\nabla U - U\nabla v)\}\\ &\quad= A(u)\nabla\Delta U + (\Delta U)\nabla A(u) + \nabla\left(\nabla A(u)\cdot\nabla U\right)\\ &\qquad - \nabla(\nabla U\cdot\nabla v) - (\Delta v)\nabla U - U\nabla\Delta v\\ &\quad= A(u)\nabla\Delta U + (\Delta U)\nabla A(u) + \sum_{i=1}^N \frac{\partial\nabla A(u)}{\partial x_i}\frac{\partial U}{\partial x_i}\\ &\qquad + \sum_{i=1}^N \frac{\partial A(u)}{\partial x_i}\frac{\partial\nabla U}{\partial x_i} - \sum_{i=1}^N \frac{\partial\nabla U}{\partial x_i}\frac{\partial v}{\partial x_i} - \sum_{i=1}^N \frac{\partial U}{\partial x_i}\frac{\partial\nabla v}{\partial x_i}\\ &\qquad - (\Delta v)\nabla U - U\nabla\Delta v\\ &\quad= I + II + III + IV - V - VI - VII - VIII\end{aligned}\tag{2.21}$$

and those terms are estimated as follows:

$$
\begin{aligned}
&-\int_\Omega (I\cdot\nabla\Delta U)|\Delta U|^{p-2}dx \le -\beta\int_\Omega |\nabla\Delta U|^2|\Delta U|^{p-2}dx,\\
&-\int_\Omega (II\cdot\nabla\Delta U)|\Delta U|^{p-2}dx\\
&\quad = -\int_\Omega (\nabla A(u)\cdot\nabla\Delta U)|\Delta U|^{p-2}\Delta U\,dx\\
&\quad \le C_6\|A'\|_{L^\infty([0,C_6])}\cdot\int_\Omega |\nabla\Delta U||\Delta U|^{p-1}dx\\
&\quad \le \frac{\beta}{10}\int_\Omega |\nabla\Delta U|^2|\Delta U|^{p-2}dx + C_9\int_\Omega |\Delta U|^p dx,\\
&-\int_\Omega (III\cdot\nabla\Delta U)|\Delta U|^{p-2}dx\\
&\quad \le \|A'\|_{L^\infty([0,C_6])}\int_\Omega\sum_{i,j=1}^N\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right||\nabla U||\nabla\Delta U||\Delta U|^{p-2}dx\\
&\qquad + \|A''\|_{L^\infty([0,C_6])}\int_\Omega |\nabla u|^2|\nabla U||\nabla\Delta U||\Delta U|^{p-2}dx\\
&\quad \le C_{10}\left(\|\Delta u\|_p^p+\|u\|_p^p\right)^{1/p}\|\nabla U\|_\infty\\
&\qquad \cdot\left\{\int_\Omega |\nabla\Delta U|^2|\Delta U|^{p-2}dx\right\}^{1/2}\|\Delta U\|_p^{(p-2)/2}\\
&\qquad + C_{10}C_6^2\|\nabla U\|_p\left\{\int_\Omega |\nabla\Delta U|^2|\Delta U|^{p-2}dx\right\}^{1/2}\|\Delta U\|_p^{(p-2)/2}\\
&\quad \le \frac{\beta}{10}\int_\Omega |\nabla\Delta U|^2|\Delta U|^{p-2}dx + C_{11}\left(\|\Delta U\|_p^p+\|U\|_p^p\right),\\
&-\int_\Omega (IV\cdot\nabla\Delta U)|\Delta U|^{p-2}dx\\
&\quad \le \|A'\|_{L^\infty([0,C_6])}\int_\Omega |\nabla u|\left(\sum_{i,j=1}^N\left|\frac{\partial^2 U}{\partial x_i\partial x_j}\right|\right)|\nabla\Delta U||\Delta U|^{p-2}dx\\
&\quad \le C_6\|A'\|_{L^\infty([0,C_6])}\left\{\int_\Omega\left(\sum_{i,j=1}^N\left|\frac{\partial^2 U}{\partial x_i\partial x_j}\right|\right)^p dx\right\}^{1/p}\\
&\qquad \cdot\left\{\int_\Omega |\nabla\Delta U|^2|\Delta U|^{p-2}dx\right\}^{1/2}\|\Delta U\|_p^{(p-2)/2}\\
&\quad \le \frac{\beta}{10}\int_\Omega |\nabla\Delta U|^2|\Delta U|^{p-2}dx + C_{12}\left(\|\Delta U\|_p^p+\|U\|_p^p\right),\\
&-\int_\Omega (-V\cdot\nabla\Delta U)|\Delta U|^{p-2}dx\\
&\quad \le \int_\Omega |\nabla v|\left(\sum_{i,j=1}^N\left|\frac{\partial^2 U}{\partial x_i\partial x_j}\right|\right)|\nabla\Delta U||\Delta U|^{p-2}dx
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
&\le C_6\left\{\int_\Omega\left(\sum_{i,j=1}^N\left|\frac{\partial^2 U}{\partial x_i\partial x_j}\right|\right)^p dx\right\}^{1/p}\\
&\quad\cdot\left\{\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx\right\}^{1/2}\|\Delta U\|_p^{(p-2)/2}\\
&\le\frac{\beta}{10}\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx+C_{13}\left(\|\Delta U\|_p^p+\|U\|_p^p\right),\\
-&\int_\Omega(-VI\cdot\nabla\Delta U)|\nabla\Delta U|^{p-2}dx\\
&\le\int_\Omega|\nabla U|\left(\sum_{i,j=1}^N\left|\frac{\partial^2 v}{\partial x_i\partial x_j}\right|\right)|\nabla\Delta U||\Delta U|^{p-2}dx\\
&\le C_{14}\|v\|_{2,p}\|\nabla U\|_\infty\left\{\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx\right\}^{1/2}\|\Delta U\|_p^{(p-2)/p}\\
&\le\frac{\beta}{10}\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx+C_{15}\left(\|\Delta U\|_p^p+\|U\|_p^p\right),\\
-&\int_\Omega(-VII\cdot\nabla\Delta U)|\Delta U|^{p-2}dx\\
&\le\int_\Omega|\Delta v||\nabla U||\nabla\Delta U||\Delta U|^{p-2}dx\\
&\le C_{16}\|v\|_{2,p}\|\nabla U\|_\infty\left\{\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx\right\}^{1/2}\|\Delta U\|_p^{(p-2)/2}\\
&\le\frac{\beta}{10}\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx+C_{17}\left(\|\Delta U\|_p^p+\|U\|_p^p\right),\\
-&\int_\Omega(-VIII\cdot\nabla\Delta U)|\Delta U|^{p-2}\\
&\le\int_\Omega U|\nabla\Delta v||\nabla\Delta U||\Delta U|^{p-2}dx\\
&\le\left\{\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx\right\}^{1/2}\|\nabla\Delta v\|_p\|\Delta U\|_p^{(p-2)/2}\|U\|_\infty\\
&\le\frac{\beta}{10}\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx+C_{18}\left(\|\Delta U\|_p^p+\|U\|_p^p\right).
\end{aligned}
\tag{2.23}
$$

Those inequalities are summarised as

$$
\frac{1}{p(p-1)}\frac{d}{dt}\int_\Omega|\Delta U|^p dx+\frac{3\beta}{10}\int_\Omega|\nabla\Delta U|^2|\Delta U|^{p-2}dx\le C_{19}\left(\|\Delta U\|_p^p+\|U\|_p^p\right),\tag{2.24}
$$

and therefore, we have

$$
\frac{d}{dt}\left(\|\Delta U\|_p^p+\|U\|_p^p\right)\le C_{20}\left(\|\Delta U\|_p^p+\|U\|_p^p\right)\tag{2.25}
$$

by (2.19). Taking

$$T_1 = \min\left(\frac{\log 2}{C_{20}}, S\right), \tag{2.26}$$

we have for $T \in (0, T_1]$ that

$$\sup_{0\le t\le T}\left(||\Delta U(\cdot,t)||_p^p + ||U(\cdot,t)||_p^p\right) \le e^{C_{20}T}\left(\|\Delta u_0\|_p^p + \|u_0\|_p^p\right) \le \frac{1}{2}M^p e^{C_{20}T} \le M^p, \tag{2.27}$$

and the proof is complete. □

Next, we show the following.

Lemma 2.3. *There is $T_2 \in (0, T_1]$ satisfying*

$$\max_{0\le t\le T_2}||\mathcal{F}u_1(\cdot,t) - \mathcal{F}u_2(\cdot,t)||_p^p \le \frac{1}{2}\max_{0\le t\le T_2}||u_1(\cdot,t) - u_2(\cdot,t)||_p^p \tag{2.28}$$

for any $u_1, u_2 \in B(M, T_2)$.

Proof. We take $T \in (0, T_1]$ and $u_i \in B(M,T)$ for $i = 1,2$. Let U_i and v_i be the solutions to (2.3) for $u = u_i$. Then, it holds that

$$v_2(\cdot,t) - v_1(\cdot,t) = \int_0^t e^{(s-t)(\Delta_N - 1)}(u_1(\cdot,s) - u_2(\cdot,s))\,ds \tag{2.29}$$

and we have

$$||\nabla(v_2(\cdot,t) - v_1(\cdot,t))||_p \le C_{21}\sqrt{t}\sup_{0\le s\le t}||u_2(\cdot,s) - u_1(\cdot,s)||_p. \tag{2.30}$$

Next, we note

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}||U_2(\cdot,t) - U_1(\cdot,t)||_p^p &= \int_\Omega (U_{2t} - U_{1t})\,|U_2 - U_1|^{p-2}(U_2 - U_1)\,dx \\
&= \int_\Omega \{\nabla\cdot[(A(u_2) - A(u_1))\nabla U_2] + \nabla\cdot[A(u_1)\nabla(U_2 - U_1)] \\
&\qquad - \nabla\cdot[(U_2 - U_1)\nabla v_2] - \nabla\cdot[U_1\nabla(v_2 - v_1)]\}\,|U_2 - U_1|^{p-2}(U_2 - U_1)\,dx \\
&= -(p-1)\int_\Omega (A(u_2) - A(u_1))[\nabla U_2\cdot\nabla(U_2 - U_1)]\,|U_2 - U_1|^{p-2}dx \\
&\quad - (p-1)\int_\Omega A(u_1)\,|\nabla(U_2 - U_1)|^2\,|U_2 - U_1|^{p-2}dx \\
&\quad + (p-1)\int_\Omega [\nabla v_2\cdot\nabla(U_2 - U_1)]\,|U_2 - U_1|^{p-2}(U_2 - U_1)\,dx \\
&\quad + (p-1)\int_\Omega U_1[\nabla(v_2 - v_1)\cdot\nabla(U_2 - U_1)]\,|U_2 - U_1|^{p-2}dx \\
&= -IX - X + XI + XII.
\end{aligned} \tag{2.31}$$

Those terms are estimated by Lemmas 2.1 and 2.2 in the following way:

$$
\begin{aligned}
|IX| \le{} & (p-1)\|A'\|_{L^\infty([0,C_6])}\|u_2-u_1\|_p\|\nabla U_2\|_\infty \\
& \cdot\left\{\int_\Omega |\nabla(U_2-U_1)|^2|U_2-U_1|^{p-2}dx\right\}^{1/2}\|U_2-U_1\|_p^{(p-2)/2} \\
\le{} & \frac{(p-1)\beta}{4}\int_\Omega |\nabla(U_2-U_1)|^2|U_2-U_1|^{p-2}dx \\
& + C_{22}\left(\|U_2-U_1\|_p^p+\|u_2-u_1\|_p^p\right), \\
& -X \le -\beta(p-1)\int_\Omega |\nabla(U_2-U_1)|^2|U_2-U_1|^{p-2}dx, \\
|XI| \le{} & (p-1)\|\nabla v_2\|_\infty\left\{\int_\Omega |\nabla(U_2-U_1)|^2|U_2-U_1|^{p-2}dx\right\}^{1/2}\cdot\|U_2-U_1\|_p^{p/2} \\
\le{} & \frac{\beta(p-1)}{4}\int_\Omega |\nabla(U_2-U_1)|^2|U_2-U_1|^{p-2}dx + C_{23}\|U_2-U_1\|_p^p, \\
|XII| \le{} & (p-1)\|U_1\|_\infty\left\{\int_\Omega |\nabla(U_2-U_1)|^2|U_2-U_1|^{p-2}dx\right\}^{1/2} \\
& \cdot\|U_2-U_1\|_p^{(p-2)/2}\|\nabla(v_2-v_1)\|_p \\
\le{} & \frac{(p-1)\beta}{4}\int_\Omega |\nabla(U_2-U_1)|^2|U_2-U_1|^{p}dx \\
& + C_{24}\left(\|U_2-U_1\|_p^p+\|\nabla(v_2-v_1)\|_p^p\right).
\end{aligned}
\tag{2.32}
$$

Thus, we obtain

$$
\frac{1}{p}\frac{d}{dt}\|U_2-U_1\|_p^p \le C_{25}\left(\|U_2-U_1\|_p^p+\|u_2-u_1\|_p^p+\|\nabla(v_2-v_1)\|_p^p\right), \tag{2.33}
$$

and hence it follows from (2.30) that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|U_2(\cdot,t)-U_1(\cdot,t)\|_p^p \\
&\quad\le pC_{25}T\left(\sup_{0\le t\le T}\|U_2(\cdot,t)-U_1(\cdot,t)\|_p^p+\sup_{0\le t\le T}\|u_2(\cdot,t)-u_1(\cdot,t)\|_p^p\right) \\
&\qquad + T^{(p/2)+1}pC_{25}C_{21}^p\sup_{0\le t\le T}\|u_2(\cdot,t)-u_1(\cdot,t)\|_p^p.
\end{aligned}
\tag{2.34}
$$

Therefore, for $T_2>0$ in

$$
T_2\le T_1,\quad pC_{25}T_2+T_2^{(p/2)+1}pC_{25}C_{21}^p\le\frac{1}{4}, \tag{2.35}
$$

it holds that

$$\sup_{0\le t\le T_2} \|U_2(\cdot,t)-U_1(\cdot,t)\|_p^p \le \frac{1}{2}\sup_{0\le t\le T_2}\|u_2(\cdot,t)-u_1(\cdot,t)\|_p^p, \tag{2.36}$$

and the proof is complete. □

Now, we can give the following.

Proof of Theorem 1.1. By Lemma 2.2 (with $T = T_2$) and Lemma 2.3, we have a positive constant C_{26} such that

$$\begin{aligned}\|U_t\|_p &\le \|A(u)\Delta U\|_p + \|A'(u)\nabla u\cdot\nabla U\|_p \\ &\quad + \|\nabla U\cdot\nabla v\|_p + \|U\Delta v\|_p \le C_{26}\end{aligned} \tag{2.37}$$

for $u \in B(M,T_2)$. Then [10, Lemma 3.3] guarantees the existence of $\ell \in (0,1)$ such that

$$|\mathcal{F}u|_{Q_{T_2}}^{(\ell)} + |\nabla\mathcal{F}u|_{Q_{T_2}}^{(\ell)} \le C_{27} \tag{2.38}$$

for $u \in B(M,T_2)$. Because the set

$$\mathcal{K} \equiv \left\{u\in B(M,T_2) \mid |\mathcal{F}u|_{Q_{T_2}}^{(\ell)} + |\nabla\mathcal{F}u|_{Q_{T_2}}^{(\ell)} \le C_{27}\right\}, \tag{2.39}$$

is compact in $X = C([0,T_2];L^p(\Omega))$, we can apply the standard contraction mapping principle and get a unique fix point of $\mathcal{F}$ on $B = B(M,T_2) \subset X$. This fixed point induces the classical solution $(u,v) \in H^{2+\ell,1+\ell/2}(\overline{Q_{T_2}})^2$ to (1.1) on Q_{T_2}. Conversely, if (u,v) is a classical solution to (1.1) and if $T_2 > 0$ is taken smaller, then, we can regard (u,v) as a fixed point of $\mathcal{F}$ on $B(M,T_2)$. This implies the uniqueness of the classical solution to (1.1) in Q_{T_2}, and the proof is complete. □

3. Time global solution

To prove Theorem 1.2, we take

$$W(u,v) = \int_\Omega \left\{B(u) - uv + \frac{1}{2}\left(|\nabla v|^2 + v^2\right)\right\}dx \tag{3.1}$$

for

$$B(u) = \int_1^u b(s)ds \quad \text{with } b(u) = \int_1^u \frac{A(s)}{s}ds \tag{3.2}$$

and show the following.

Lemma 3.1. *If (u,v) is the classical solution to (1.1), then it holds that*

$$\frac{d}{dt}W(u,v) + \int_\Omega \left(|v_t|^2 + u|\nabla(b(u)-v)|^2\right)dx = 0. \tag{3.3}$$

Proof. From the first equation of (1.1) we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (B(u)-uv)\,dx+\int_\Omega uv_t\,dx &= \int_\Omega u_t(b(u)-v)\,dx\\ &= -\int_\Omega u\,|\nabla(b(u)-v)|^2dx = 0,\end{aligned} \tag{3.4}$$

and the second equation implies that

$$\int_\Omega uv_t\,dx=\int_\Omega (v_t-\Delta v+v)v_t\,dx=\int_\Omega |v_t|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla v|^2+v^2)\,dx. \tag{3.5}$$

Thus, the conclusion follows from (3.4) and (3.5). □

We have

$$\begin{aligned} b(u) &\geq \frac{\alpha}{\gamma}(u^\gamma-1)+\beta\log u \quad (u\geq 1)\\ b(u) &\leq \frac{\alpha}{\gamma}(u^\gamma-1)+\beta\log u \quad (0<u\leq 1)\end{aligned} \tag{3.6}$$

for α, β, γ given in (1.5), and hence it follows that

$$B(u)\geq B_0(u)\equiv\frac{\alpha}{\gamma(1+\gamma)}(u^{1+\gamma}-(1+\gamma)u+\gamma)+\beta(u\log u-u+1) \tag{3.7}$$

for $u\in(0,\infty)$. Therefore, we have $W_0(u,v)\leq W(u,v)$ for $W_0=W_0(u,v)$ defined by

$$W_0(u,v)\equiv\int_\Omega\left\{B_0(u)-uv+\frac{1}{2}(|\nabla v|^2+v^2)\right\}dx. \tag{3.8}$$

Henceforth, $T_{\max}$ denotes the maximal existence time of the classical solution (u,v) to (1.1), and C_i $(i=28,29,30,\ldots,40)$ are positive constants independent of $T_{\max}$. We have

$$\|u(\cdot,t)\|_1=\|u_0\|_1 \tag{3.9}$$

and the following lemma is a consequence of the estimate on the fundamental solution to the heat equation, described in [21]. The proof is immediate and is omitted.

Lemma 3.2. *If p is in $[1,N/(N-2))$ and $[1,\infty)$ for $N\geq 3$ and $N=1,2$, respectively, then it follows that*

$$\sup_{0\leq t<T_{\max}}\|v(\cdot,t)\|_p\leq C_{28}. \tag{3.10}$$

Now, we show the following.

Lemma 3.3. *If (u,v) is the classical solution to (1.1), then it holds that*

$$\begin{aligned}&\sup_{0\leq t<T_{\max}} W_0(u(\cdot,t),v(\cdot,t))\leq C_{29}\\ &\sup_{0\leq t<T_{\max}}\left(\|u(\cdot,t)\|_{1+\gamma}^{1+\gamma}+\|v(\cdot,t)\|_{1,2}^2\right)\leq C_{29}.\end{aligned} \tag{3.11}$$

Proof. First, we have

$$\sup_{0\leq t<T_{\max}} W_0(u(\cdot,t),v(\cdot,t)) \leq \sup_{0\leq t<T_{\max}} W(u(\cdot,t),v(\cdot,t)) \leq W(u_0,v_0) \tag{3.12}$$

with

$$\int_\Omega uv\,dx \leq \|u\|_{1+\gamma}\|v\|_{(1+\gamma)/\gamma}. \tag{3.13}$$

Then, Gagliardo-Nirenberg's inequality is indicated as

$$\|w\|_q \leq K_{p,q}\left(\|\nabla w\|_2^2+\|w\|_2^2\right)^{\theta/2}\|w\|_p^{1-\theta} \tag{3.14}$$

for $w\in H^1(\Omega)$, where p, $q\in[1,\infty]$, and $\theta\in(0,1)$ are in

$$-\frac{N}{q}=\theta\{1-(N/2)\}-(1-\theta)\frac{N}{p}. \tag{3.15}$$

In the case of $N\geq 3$, we take $q=(\gamma+1)/\gamma$, $\theta=2\gamma/(\gamma+1)$ and $p=N(1-\gamma)/(2\gamma)$ to adjust (3.15). We have $p\in[1,N/(N-2))$ from the assumption, and (3.14) assures $C_\varepsilon>0$ for any $\varepsilon>0$ satisfying

$$\varepsilon\|\nabla v\|_2^2 \geq \|v\|_{(\gamma+1)/\gamma}^{(\gamma+1)/\gamma}-C_\varepsilon \tag{3.16}$$

by Lemma 3.2. The case $N=1,2$ is easier to guarantee (3.16). Therefore, we have from (3.12) that

$$W(u_0,v_0)\geq\int_\Omega\left\{\frac{\alpha}{2\gamma(\gamma+1)}u^{1+\gamma}+\frac{1}{4}\left(|\nabla v|^2+v^2\right)\right\}dx-C_{30}. \tag{3.17}$$

This implies the second inequality, and the proof is complete. □

Lemma 3.4. *If $q\geq(3+\gamma)/2$ and*

$$\sup_{0\leq t<T_{\max}}||u(\cdot,t)||_{q+(\gamma-1)/2}\leq C_{32}, \tag{3.18}$$

then it holds that

$$\sup_{0\leq t<T_{\max}}||u(\cdot,t)||_{2q-1}\leq C_{33}. \tag{3.19}$$

Proof. From the first equation of (1.1) we have

$$\begin{aligned}\int_\Omega u_t u^{2q-2}dx &= \int_\Omega \nabla\cdot(A(u)\nabla u-u\nabla v)u^{2q-2}dx\\ &\leq -(2q-2)\int_\Omega(\alpha u^\gamma+\beta)u^{2q-3}|\nabla u|^2dx+(2q-2)\int_\Omega u^{2q-2}\nabla u\cdot\nabla v\,dx\\ &= -(2q-2)\int_\Omega(\alpha u^\gamma+\beta)u^{2q-3}|\nabla u|^2dx-\frac{2q-2}{2q-1}\int_\Omega u^{2q-1}\Delta v\,dx.\end{aligned} \tag{3.20}$$

On the other hand, the second equation of (1.1) implies

$$(e^{t/(4q)}v)_t = \left(\Delta - 1 + \frac{1}{4q}\right)(e^{t/(4q)}v) + e^{t/(4q)}u \tag{3.21}$$

and the maximal regularity theorem [6] guarantees

$$\begin{aligned}
&\int_0^t \|(e^{s/(4q)}v)_s\|_{2q}^{2q}ds + \int_0^t \left\|\left(\Delta - 1 + \frac{1}{4q}\right)(e^{s/(4q)}v)\right\|_{2q}^{2q}ds \\
&\quad \le C_{34}\left\{\int_0^t \left\|\left(\Delta - 1 + \frac{1}{4q}\right)e^{\{\Delta-1+(1/(4q))\}s}v_0\right\|_{2q}^{2q}ds + \int_0^t \|e^{s/(4q)}u\|_{2q}^{2q}ds\right\}
\end{aligned} \tag{3.22}$$

with C_{34} independent of $t \ge 0$. This implies

$$\int_0^t e^{s/2}\|\Delta v\|_{2q}^{2q}ds \le C_{35}\left\{t\|v_0\|_{2,2q}^{2q} + \int_0^t e^{s/2}\|u\|_{2q}^{2q}ds\right\}. \tag{3.23}$$

We can apply (3.14) for $q = r$ and $p = 1$ if $\theta = N(1-(1/r))/(1+(N/2)) \in (0,1)$, or equivalently $1 < r < 2N/(N-2)$. Putting

$$k = (2q+\gamma-1)/2, \qquad r = 2q/k, \tag{3.24}$$

we have

$$r = \frac{4q}{2q+\gamma-1} \in \left[2, 2+\frac{1-\gamma}{1+\gamma}\right], \tag{3.25}$$

and therefore, that inequality is applicable to $w = u^k$, $q = r$ and $p = 1$:

$$\|u^k\|_r \le K_r\left(\|\nabla u^k\|_2^2 + \|u^k\|_2^2\right)^{\theta/2}\|u^k\|_1^{1-\theta}. \tag{3.26}$$

Here, we have

$$K = \max_{2\le r\le 2+(1-\gamma)/(1+\gamma)} K_r < +\infty, \qquad r\theta = \frac{2N}{N+2}(r-1) < 2, \tag{3.27}$$

and any $\varepsilon > 0$ admits $C_\varepsilon > 0$ such that

$$\begin{aligned}
\|u^k\|_r^r &\le K^r\left(\|\nabla u^k\|_2^2 + \|u^k\|_2^2\right)^{r\theta/2}\|u^k\|_1^{r(1-\theta)} \\
&\le \varepsilon\left(\|\nabla u^k\|_2^2 + \|u^k\|_2^2\right) + C_\varepsilon\|u^k\|_1^{r(1-\theta)(2/(r\theta))'},
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}\left(\frac{2}{r\theta}\right)' &= \frac{2/(r\theta)}{(2/(r\theta))-1} = \frac{N+2}{(N+2)-N(r-1)} \\ &\le \begin{cases} 3(1+\gamma)/(1+3\gamma) & \text{if } N=1 \\ (1+\gamma)/\gamma & \text{if } N=2 \\ (N+2)(N-1)/(N-2) & \text{if } N\ge 3 \end{cases} \\ &= \kappa \end{aligned} \tag{3.29}$$

with

$$\Lambda = \Lambda(q) = r(1-\theta)\left(\frac{2}{r\theta}\right)' = \frac{2N-(N-2)r}{2N+2-Nr}. \tag{3.30}$$

Thus, we have

$$\begin{aligned} C_{35}\int_0^t e^{s/2}\|u\|_{2q}^{2q}ds \le{}& (q-1)(2q-1)\alpha\int_0^t\int_\Omega e^{s/2}u^{2q+\gamma-3}|\nabla u|^2dx\,ds \\ &+ C_{36}\int_0^t e^{s/2}\left(\|u\|_{q+(\gamma-1)/2}^{q+(\gamma-1)/2}+1\right)^{\Lambda}ds. \end{aligned} \tag{3.31}$$

On the other hand, by (3.26), (3.29) and $\|u^k\|_r^r = \|u\|_{2q}^{2q}$, we have

$$\begin{aligned} &-(q-1)(2q-1)\int_\Omega \alpha u^{2q+\gamma-3}|\nabla u|^2dx + (2q-2)\int_\Omega u^{2q-1}|\Delta v|dx \\ &\quad\le -\frac{(q-1)(2q-1)}{k^2}\alpha\|\nabla u^k\|_2^2 + (2q-2)\|u\|_{2q}^{2q-1}\|\Delta v\|_{2q} \\ &\quad\le -\frac{1}{2}\|u\|_{2q-1}^{2q-1} + \|\Delta v\|_{2q}^{2q} + C_{37}\left(\|u\|_{q+(\gamma-1)/2}^{q+(\gamma-1)/2}+1\right)^{\Lambda}. \end{aligned} \tag{3.32}$$

From (3.32) and (3.20) we obtain

$$\begin{aligned} &\frac{d}{dt}\|u\|_{2q-1}^{2q-1} + (q-1)(2q-1)\alpha\int_\Omega u^{2q+\gamma-3}|\nabla u|^2dx \\ &\quad\le -(q-1)(2q-1)\alpha\int_\Omega u^{2q+\gamma-3}|\nabla u|^2dx + (2q-2)\int_\Omega u^{2q-1}|\Delta v|dx \\ &\quad\le -\frac{1}{2}\|u\|_{2q-1}^{2q-1} + \|\Delta v\|_{2q}^{2q} + C_{37}\left(\|u\|_{q+(\gamma-1)/2}^{q+(\gamma-1)/2}+1\right)^{\Lambda}. \end{aligned} \tag{3.33}$$

Multiplying $e^{t/2}$ for the above inequality, integrating over $[0,t]$ and using (3.23) and (3.31), we now get that

$$\begin{aligned} &e^{t/2}\|u\|_{2q-1}^{2q-1} + (q-1)(2q-1)\alpha\int_0^t e^{s/2}\int_\Omega u^{2q+\gamma-3}|\nabla u|^2dx\,ds \\ &\quad\le \int_0^t e^{s/2}\|\Delta v\|_{2q}^{2q}ds + C_{38}e^{t/2}\left(L_{q+(\gamma-1)/2}+1\right)^{\Lambda} \end{aligned} \tag{3.34}$$

or

$$||u(\cdot,t)||_{2q-1}^{2q-1} \le C_{39}\left(L_{q+(\gamma-1)/2}+1\right)^{\Lambda}, \tag{3.35}$$

where

$$L_{q+(\gamma-1)/2} = \sup_{0\le t<T_{\max}} \|u\|_{q+(\gamma-1)/2}^{q+(\gamma-1)/2}. \tag{3.36}$$

Thus, the proof is complete. □

Now, we show the uniform boundedness of the solution.

Proposition 3.5. *It holds that*

$$\sup_{0\le t<T_{\max}} \left(||u(\cdot,t)||_\infty + ||v(\cdot,t)||_\infty\right) \le C_{40}. \tag{3.37}$$

Proof. By means of Lemmas 3.3 and 3.4, we have

$$\sup_{0\le t<T_{\max}} ||u(\cdot,t)||_q < +\infty \quad \text{for any } q \ge 1+\gamma. \tag{3.38}$$

We have $D((-\Delta_{N,2N+1}+1)^{3/4}) \subset W^{1,\infty}(\Omega)$, and hence it holds that

$$\begin{aligned}
||\nabla v(\cdot,t)||_\infty &\le C_{41}||(-\Delta_N+1)^{3/4}v(\cdot,t)||_{2N+1} \\
&\le C_{41}\Big(||(-\Delta_N+1)^{3/4}e^{-t(\Delta_N-1)}v_0||_{2N+1} \\
&\qquad + \int_0^t ||(-\Delta_N+1)^{3/4}e^{-(t-s)(\Delta_N-1)}u(\cdot,s)||_{2N+1}ds\Big) \\
&\le C_{42}\left(||(-\Delta_N+1)^{3/4}v_0||_{2N+1} + \int_0^t (t-s)^{-3/4}e^{-(t-s)}||u(\cdot,t)||_{2N+1}ds\right) \\
&\le C_{43}\left(||v_0||_{2,2N+1} + \sup_{0\le t<T_{\max}} ||u(\cdot,t)||_{2N+1}\right) \le C_{44}.
\end{aligned} \tag{3.39}$$

Taking $q \ge 1+\gamma$ and recalling (3.24), we have

$$\begin{aligned}
&\frac{1}{2q-1}\frac{d}{dt}\int_\Omega u^{2q-1}dx + (2q-2)\int_\Omega (\alpha u^\gamma+\beta)u^{2q-3}|\nabla u|^2 dx \\
&\quad\le (2q-2)\int_\Omega u^{2q-2}\nabla u\cdot\nabla v\,dx \\
&\quad\le (2q-2)\|\nabla v\|_\infty\left(\int_\Omega u^{2q+\gamma-3}|\nabla u|^2dx\right)^{1/2}\cdot\left(\int_\Omega u^{2q-\gamma-1}\right)^{1/2} \\
&\quad\le \alpha(q-1)\int_\Omega u^{2q+\gamma-3}|\nabla u|^2dx + \frac{q-1}{\alpha}\|\nabla v\|_\infty^2\int_\Omega u^{2q-\gamma-1}dx \\
&\quad\le \alpha(q-1)\int_\Omega u^{2q+\gamma-3}|\nabla u|^2dx + \frac{q-1}{\alpha}\|\nabla v\|_\infty^2\left(\|u\|_{2q}^{2q}+|\Omega|\right),
\end{aligned} \tag{3.40}$$

or

$$\frac{d}{dt}\int_\Omega u^{2q-1}dx + \frac{\alpha(q-1)(2q-1)}{k^2}\int_\Omega |\nabla u^k|^2 dx \leq \frac{(q-1)(2q-1)}{\alpha}\|\nabla v\|_\infty^2 \left(\|u\|_{2q}^{2q} + |\Omega|\right). \tag{3.41}$$

Here, we have from (3.24), (3.26) and (3.39) that

$$\begin{aligned} &\left(\frac{(q-1)(2q-1)}{\alpha}\|\nabla v\|_\infty^2 + 1\right)\left(\|u\|_{2q}^{2q} + |\Omega|\right) \\ &\quad\leq C_{44}q^2\left\{\int_\Omega |\nabla u^k|^2 dx + \int_\Omega u^{2k}dx\right\}^{r\theta/2} \cdot \left\{\int_\Omega u^k dx\right\}^{r(1-\theta)} + C_{44}q^2 \\ &\quad\leq \frac{\alpha(q-1)(2q-1)}{k^2}\left\{\int_\Omega |\nabla u^k|^2 dx + \int_\Omega u^{2k}dx\right\} \\ &\qquad + C_{45}q^{2(2/(r\theta))'}\left\{\int_\Omega u^k dx\right\}^{r(1-\theta)(2/(r\theta))'} + C_{44}q^2. \end{aligned} \tag{3.42}$$

In other words,

$$\begin{aligned} &\left(\frac{(q-1)(2q-1)}{\alpha}\|\nabla v\|_\infty^2 + 1\right)\left(\|u\|_{2q}^{2q} + |\Omega|\right) \\ &\quad\leq \frac{\alpha(q-1)(2q-1)}{k^2}\left\{\int_\Omega |\nabla u^k|^2 dx + \|u\|_{2q+\gamma-1}^{2q+\gamma-1}\right\} \\ &\qquad + C_{45}q^{2(2/(r\theta))'}\left\{\int_\Omega u^k dx\right\}^{r(1-\theta)(2/(r\theta))'} + C_{44}q^2. \end{aligned} \tag{3.43}$$

Here and henceforce, C_i $(i = 41, 42, 43, \ldots, 50)$ denote positive constants independent of $q \geq 1+\gamma$ and $T_{\max}$.

Combining this with (3.30) and (3.41), we get that

$$\frac{d}{dt}\|u\|_{2q-1}^{2q-1} + \|u\|_{2q}^{2q} \leq C_{46}q^{2\kappa}\left\{\int_\Omega u^{q+((\gamma-1)/2)}dx + 1\right\}^{\Lambda}. \tag{3.44}$$

Because of $\gamma \in (0,1)$ and (3.29), we have

$$\left(\sup_{0\leq t<T_{\max}} ||u(\cdot,t)||_{2q-1}^{2q-1} + 1\right) \leq C_{47}q^{2\kappa}\left(\sup_{0\leq t<T_{\max}} ||u(\cdot,t)||_q^q + 1\right)^{\Lambda}. \tag{3.45}$$

For $q = 2^{j-1}+1$, we put

$$\alpha_j = \sup_{0\leq t<T_{\max}} ||u(\cdot,t)||_{2q-1}^{2q-1} + 1, \qquad \Lambda_j = \max\left(2, \widetilde{\Lambda}_j\right), \tag{3.46}$$

where $\widetilde{\Lambda}_j = \Lambda(q) = \Lambda(2^{j-1}+1)$.

Here, we have $\Lambda_j \le 2(1+C_{48}2^{-j})$. In use of $\Lambda_j \ge 2$, we apply (3.45) as $q = 2^{j-1}+1$ for $j = 0,1,2,\dots$. Then, we get that

$$\begin{aligned}\log\alpha_j &\le \log C_{47} + 2\kappa\log\left(2^{j-1}+1\right) + \Lambda_j\log\alpha_{j-1}\\ &\le \log C_{47} + 2\kappa j\log 2 + \Lambda_j\log\alpha_{j-1}\end{aligned} \tag{3.47}$$

or

$$\log\alpha_j + 2\kappa(j+2)\log 2 + \log C_{47} \le \Lambda_j\left(\log\alpha_{j-1} + 2\kappa(j+1)\log 2 + \log C_{47}\right), \tag{3.48}$$

which implies

$$\log\alpha_j + 2\kappa(j+2)\log 2 + \log C_{47} \le \left(\prod_{i=1}^{j}\Lambda_i\right)\left(\log\alpha_0 + 4\kappa\log 2 + \log C_{47}\right), \tag{3.49}$$

or

$$\alpha_j \le \left(C_{47}2^{4\kappa}\alpha_0\right)^{\prod_{i=1}^{j}\Lambda_i}\left(C_{47}2^{2\kappa(j+2)}\right)^{-1}. \tag{3.50}$$

Hence, we obtain

$$\sup_{0\le t<T_{\max}}\|u(\cdot,t)\|_{2^j+1} \le \left(C_{47}2^{4\kappa}\alpha_0\right)^{(\prod_{i=1}^{j}\Lambda_i)/(2^j+1)}\left(C_{47}2^{2\kappa(j+3)}\right)^{-1/(2^j+1)}. \tag{3.51}$$

Here, we apply

$$\prod_{i=1}^{j}\Lambda_i \le 2^j\exp\left(C_{48}\sum_{i=1}^{j}2^{-i}\right) \le 2^j e^{C_{48}} = 2^j C_{49}, \tag{3.52}$$

and get that

$$\sup_{0\le t<T_{\max}}\|u(\cdot,t)\|_{\infty} \le C_{50}\left(\sup_{0\le t<T_{\max}}\|u(\cdot,t)\|_2^2 + 1\right)^{C_{49}}. \tag{3.53}$$

The desired conclusion follows from this inequality and (3.38) with $q = 2$. □

Let us confirm that Proposition 3.5 says that $\sup_{0\le t<T_{\max}}\|u(\cdot,t)\|_\infty$ and $\sup_{0\le t<T_{\max}}\|v(\cdot,t)\|_\infty$ are estimated from above by constants independent of $T_{\max}$. We recall $p > N+2$ and $\ell \in (0, 1-(N+2)/p)$. Then, (3.23) with $2q = p$ means that

$$\int_0^t e^{s/2}\|\Delta v\|_p^p ds \le C_{51}\left(t\|v_0\|_{2,p}^p + \int_0^t e^{s/2}\|u\|_p^p ds\right) \tag{3.54}$$

and hence we obtain

$$\int_0^t e^{(s-t)/2}\|\Delta v\|_p^p ds \le C_{52}\left(te^{-t/2}\|v_0\|_{2,p}^p + \sup_{0\le t<T_{\max}}\|u(\cdot,t)\|_p^p\right). \tag{3.55}$$

Here, C_i $(i = 51, 52)$ are independent of $t \in [0, T_{\max})$.

We take $\tau \in (0,(1/2)\min(T_{\max},1))$ in $1 < T_{\max} - \tau$ without loss of generality. Moreover, we take $T \in [\tau, T_{\max} - 1)$. Henceforce, C_i $(i = 53, 54, 55, \ldots, 58)$ denote positive constants independent of $T \in [\tau, T_{\max})$ and $T_{\max}$. In use of the reflection with respect to $\partial\Omega$, we have a domain $\widetilde{\Omega} \supset \overline{\Omega}$ with smooth boundary and the extension $(\widetilde{u},\widetilde{v})$ of (u,v) defined on $\widetilde{\Omega}$. Then, we can apply [10, Theorem 3.1] for $(\widetilde{u},\widetilde{v})$, and find $\theta \in (0,1)$ such that

$$|\widetilde{u}|^{(\theta)}_{\Omega\times(T,T+1)} \leq C_{53}. \tag{3.56}$$

In fact, $\int_T^{T+1} \|\Delta v\|_p^p dt$ is estimated from above independent of T and $T_{\max}$ by (3.55), and this implies (3.56). Then, it holds that

$$|u|^{(\theta)}_{\Omega\times(T,T+1)} \leq C_{53}, \tag{3.57}$$

and the parabolic regularity guarantees that

$$|u|^{(\theta)}_{Q_{T_{\max}}} \leq C_{54}. \tag{3.58}$$

Then,

$$|v|^{(2+\theta)}_{Q_{T_{\max}}} \leq C_{55} \tag{3.59}$$

follows from [10, Theorem 5.3].

We take $\zeta \in C^\infty(\mathbf{R})$ in $0 \leq \zeta \leq 1$ and

$$\zeta(s) = \begin{cases} 1 & \text{if } \tau \leq s, \\ 0 & \text{if } s \leq 0, \end{cases} \tag{3.60}$$

and set $\zeta_T(s) = \zeta(s - T)$. The function $W = \mathscr{A}(u)$ defined by

$$\mathscr{A}(s) = \int_0^s A(s')ds' \tag{3.61}$$

satisfies that

$$W_t = A\Delta W - (\nabla v \cdot \nabla W + Au\Delta v) \tag{3.62}$$

and hence we can deduce for $W\zeta_T$ that

$$\begin{aligned} (W\zeta_T)_t &= A\Delta(W\zeta_T) - (\nabla v \cdot \nabla(W\zeta_T)) \\ &\quad + (W\zeta_T' - Au\Delta v\zeta_T) \quad \text{in } \Omega \times [T, T+1], \\ \frac{\partial}{\partial\nu}(W\zeta_T) &= 0 \quad \text{on } \partial\Omega \times [T, T+1], \\ W(\cdot, T)\zeta_T(T) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{3.63}$$

Here, we regard u and v as coefficients, and apply [10, Theorem 5.3] by (3.58). Then, we obtain

$$|u|^{(2+\theta)}_{\Omega\times(T+\tau,T+1)} \leq \left|u\zeta_T\right|^{(2+\theta)}_{\Omega\times(T,T+1)} \leq C_{56}, \tag{3.64}$$

or

$$|u|^{(2+\theta)}_{\Omega\times(\tau,T_{\max})} \le C_{57}. \tag{3.65}$$

On the other hand we have the parabolic regularity locally in time, so that we obtain

$$|u|^{(2+\theta)}_{Q_{T_{\max}}}, |v|^{(2+\theta)}_{Q_{T_{\max}}} \le C_{58}. \tag{3.66}$$

Finally, if $T_{\max} < \infty$, the solution (u,v) is extended after $T_{\max}$, a contradiction. Hence it holds that $T_{\max} = \infty$. We have the uniformly bounded solution globally in time, and the proof of Theorem 1.1 is complete.

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