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Research Article

Homogenization of Elliptic Differential Equations in One-Dimensional Spaces

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Linear elliptic differential equations with periodic coefficients in one-dimensional domains are considered. The approximation properties of the homogenized system are investigated. For H^{-1}-data, it turns out that the order of approximation is strongly related to the decay of the Fourier coefficients of the L^2-functions involved.

1. Introduction

We consider the linear elliptic differential equation

$$\frac{d}{dx}\left(a_\epsilon(x)\frac{d}{dx}u_\epsilon(x)\right) = f(x) \tag{1.1}$$

on the interval $x \in (0,1)$ with Dirichlet boundary conditions

$$u_\epsilon(0) = u_\epsilon(1) = 0. \tag{1.2}$$

Here, $\epsilon > 0$ is a small parameter and $a_\epsilon(x) = a(x/\epsilon)$. Under natural suppositions on the coefficient function a and the data f given below, the boundary problem given above does possess a unique solution $u_\epsilon \in H_0^1((0,1);\mathbb{R})$.

Assumption 1.1. The space-dependent coefficient is of the form $a_\epsilon(x) = a(x/\epsilon)$, where the mapping $a : \mathbb{R} \to \mathbb{R}$ is measurable, essentially bounded, and periodic. In particular, there are constants $0 < \alpha \leq \beta < \infty$ and $Y > 0$ such that $0 < \alpha \leq a(y) \leq \beta < \infty$ and $a(y) = a(y+Y)$ for a.a. $y \in \mathbb{R}$.

Assumption 1.2. As for the data, we assume that $f \in H^{-1}((0,1);\mathbb{R}) = (H_0^1((0,1);\mathbb{R}))^*$.

Accordingly, f is of the form $f = f_0 + (d/dx)f_1$, with $f_0, f_1 \in L^2((0,1);\mathbb{R})$, where d/dx denotes the distributional derivative. Note that f_0, f_1 are not uniquely determined by f.

For both numerical and theoretical considerations, it is quite advantageous to simplify the differential equation. Averaging leads to the so-called homogenized differential equation

$$\frac{d}{dx}\left(a_0 \frac{d}{dx} u_0(x)\right) = f(x), \tag{1.3}$$

where the constant coefficient is given by

$$\frac{1}{a_0} = \frac{1}{Y}\int_0^Y \frac{1}{a(y)}dy. \tag{1.4}$$

Naturally, the corresponding Dirichlet boundary problem is uniquely solvable in $H_0^1((0,1);\mathbb{R})$ as well. The relation between the original and the homogenized solutions can be described as follows, see [1, Theorem 6.1].

Theorem 1.3. *Let Assumptions 1.1, 1.2 be satisfied. Then one has the convergence relation* $u_\epsilon \to u_0$ *weak in* $H_0^1((0,1);\mathbb{R})$*, as* $\epsilon \to 0$.

There are several methods available to prove the homogenization result above. We mention Tatar's method displayed in [2] and the two-scale convergence method elaborated in [3]. Both methods are applicable to differential equations in higher dimensional domains. It is well known, see [4], that the approximation is linear in $\epsilon > 0$ for L^2-data, but it seems that the order of approximation has not yet been investigated for H^{-1}-data. In the present paper, we assume that the Fourier coefficients given by the H^{-1}-data have a sufficiently fast decay and obtain approximation orders (with respect to the uniform convergence) in dependence of the order of the decay of the Fourier coefficients. Here, the usual linear order for L^2-data appears as a limit, as the H^{-1}-data approaches an L^2-function. Note that the method of proof has nothing in common with the Fourier homogenization method, see, for instance, [5], since we do not use a Fourier analysis for the rapidly varying coefficients.

2. Order of approximation

In order to obtain an order of approximation we have to suppose a sufficient fast decay of the Fourier coefficients of f_1. Here we set

$$\begin{gathered} c_0 := \int_0^1 f_1(x)dx, \\ c_k := \int_0^1 f_1(x)\frac{\cos(k2\pi x)}{\sqrt{2}}dx, \qquad s_k := \int_0^1 f_1(x)\frac{\sin(k2\pi x)}{\sqrt{2}}dx, \end{gathered} \tag{2.1}$$

for $k = 1, 2, 3, \ldots$.

Theorem 2.1. *Let Assumptions 1.1, 1.2 be satisfied. Moreover, assume that the Fourier coefficients of f_1 fulfill the estimation*

$$c_k = O(k^{-\gamma}), \quad s_k = O(k^{-\gamma}), \quad \text{as } k \longrightarrow \infty, \tag{2.2}$$

for a $\gamma > 1/2$. Then the solutions u_0 and u_ϵ are continuous on $[0,1]$ and, as $\epsilon \to 0$, one can estimate

$$\|u_\epsilon - u_0\|_\infty = \begin{cases} O(\epsilon^{\gamma-1/2}), & \text{if } \dfrac{1}{2} < \gamma < \dfrac{3}{2}, \\ O(\epsilon\,|\log(\epsilon)|), & \text{if } \gamma = \dfrac{3}{2}, \\ O(\epsilon), & \text{if } \gamma > \dfrac{3}{2}. \end{cases} \tag{2.3}$$

Proof. The differential equation (1.1) is equivalent to

$$\frac{d}{dx}\xi_\epsilon(x) = f_0(x) + \frac{d}{dx}f_1(x), \qquad \frac{d}{dx}u_\epsilon(x) = \frac{\xi_\epsilon(x)}{a_\epsilon(x)}. \tag{2.4}$$

Hence, with an appropriate constant $C_\epsilon \in \mathbb{R}$, we can write

$$\xi_\epsilon(x) = \int_0^x f_0(w)dw + f_1(x) + C_\epsilon, \qquad u_\epsilon(x) = \int_0^x \frac{\int_0^z f_0(w)dw + f_1(z) + C_\epsilon}{a_\epsilon(z)}dz, \tag{2.5}$$

for all $\epsilon > 0$. The homogenized differential equation (1.3) is equivalent to

$$\frac{d}{dx}\xi_0(x) = f_0(x) + \frac{d}{dx}f_1(x), \qquad \frac{d}{dx}u_0(x) = \frac{\xi_0(x)}{a_0}. \tag{2.6}$$

Hence, with an appropriate constant $C_0 \in \mathbb{R}$, we can write

$$\xi_0(x) = \int_0^x f_0(w)dw + f_1(x) + C_0, \qquad u_0(x) = \int_0^x \frac{\int_0^z f_0(w)dw + f_1(z) + C_0}{a_0}dz. \tag{2.7}$$

Integrating by parts, we conclude that

$$\begin{aligned} u_\epsilon(x) - u_0(x) &= \int_0^x \frac{C_\epsilon + F_0(z) + f_1(z)}{a_\epsilon(z)}dz - \int_0^x \frac{C_0 + F_0(z) + f_1(z)}{a_0}dz \\ &= \int_0^x \frac{C_\epsilon - C_0}{a_\epsilon(z)}dz + \int_0^x C_0 g_\epsilon(z)dz + \int_0^x F_0(z)g_\epsilon(z)dz + \int_0^x f_1(z)g_\epsilon(z)dz \\ &= \int_0^x \frac{C_\epsilon - C_0}{a_\epsilon(z)}dz + C_0\epsilon A\left(\frac{x}{\epsilon}\right) - \int_0^x f_0(z)\epsilon A\left(\frac{z}{\epsilon}\right)dz + F_0(x)\epsilon A\left(\frac{x}{\epsilon}\right) + \int_0^x f_1(z)g_\epsilon(z)dz, \end{aligned} \tag{2.8}$$

where

$$g_\epsilon(z) := \frac{1}{a_\epsilon(z)} - \frac{1}{a_0}, \qquad F_0(z) := \int_0^z f_0(w)dw, \qquad A(y) := \int_0^y \left(\frac{1}{a(q)} - \frac{1}{a_0}\right)dq. \tag{2.9}$$

By the periodicity of $a : \mathbb{R} \to \mathbb{R}$ and $A(0) = A(Y) = 0$, we easily obtain the estimation

$$\max_{y\in\mathbb{R}} |A(y)| = \max_{0\leq y\leq Y} |A(y)| \leq Y\left(\frac{1}{\alpha} - \frac{1}{\beta}\right). \tag{2.10}$$

Furthermore, we obviously have

$$\max_{z\in\mathbb{R}} |g_\epsilon(z)| \leq \frac{1}{\alpha} - \frac{1}{\beta}. \tag{2.11}$$

The boundary condition $0 = u_\epsilon(1) = u_0(1)$ implies that

$$\int_0^1 \frac{C_\epsilon - C_0}{a_\epsilon(z)}dz = -C_0\epsilon A\left(\frac{1}{\epsilon}\right) + \int_0^1 f_0(z)\epsilon A\left(\frac{z}{\epsilon}\right)dz - F_0(1)\epsilon A\left(\frac{1}{\epsilon}\right) - \int_0^1 f_1(z)g_\epsilon(z)dz \tag{2.12}$$

and that

$$\begin{aligned} |C_\epsilon - C_0| \leq \beta\Bigg(&\left|C_0\epsilon A\left(\frac{1}{\epsilon}\right)\right| + \left|\int_0^1 f_0(z)\epsilon A\left(\frac{z}{\epsilon}\right)dz\right| + \left|F_0(1)\epsilon A\left(\frac{1}{\epsilon}\right)\right| \\ &+ \left|\int_0^1 f_1(z)g_\epsilon(z)dz\right|\Bigg). \end{aligned} \tag{2.13}$$

Overall, we obtain

$$\begin{aligned} |u_\epsilon(x) - u_0(x)| \leq &\left(\frac{\beta}{\alpha}+1\right)\epsilon Y\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\left(|C_0| + 2\int_0^1 |f_0(z)|dz\right) \\ &+ \left(\frac{\beta}{\alpha}+1\right)\max_{0\leq x\leq 1}\left|\int_0^x f_1(z)g_\epsilon(z)dz\right|. \end{aligned} \tag{2.14}$$

Let us note that the boundary condition $u_0(1) = 0$ gives us

$$|C_0| \leq \|f_0\|_{L^2} + \|f_1\|_{L^2}, \tag{2.15}$$

which yields

$$\|u_\epsilon - u_0\|_\infty \leq \frac{\beta^2-\alpha^2}{\alpha^2\beta}\epsilon Y\left(3\|f_0\|_{L^2} + \|f_1\|_{L^2}\right) + \left(\frac{\beta}{\alpha}+1\right)\max_{0\leq x\leq 1}\left|\int_0^x f_1(z)g_\epsilon(z)dz\right|. \tag{2.16}$$

In case that $\gamma > 3/2$, the distributional derivative $(d/dx)f_1$ is square integrable, that is, $f \in L^2((0,1);\mathbb{R})$ and we can assume that $f_1 = 0$. By (2.16), we obtain the linear approximation. In case that $1/2 < \gamma \leq 3/2$, it only remains to obtain an estimation for the last summand in (2.16). To this end, we write with $K_\epsilon \in \mathbb{N}$ (to be specified later)

$$f_1(x) = c_0 + \sum_{k=1}^{K_\epsilon}\left(c_k\frac{\cos(2\pi kx)}{\sqrt{2}} + s_k\frac{\sin(2\pi kx)}{\sqrt{2}}\right) + \sum_{k=K_\epsilon+1}^{\infty}\left(c_k\frac{\cos(2\pi kx)}{\sqrt{2}} + s_k\frac{\sin(2\pi kx)}{\sqrt{2}}\right). \tag{2.17}$$

As $K_\epsilon \to \infty$, we can estimate

$$\begin{aligned}
&\left\|\frac{d}{dx}\sum_{k=1}^{K_\epsilon}\left(c_k\frac{\cos(2\pi k\cdot)}{\sqrt{2}} + s_k\frac{\sin(2\pi k\cdot)}{\sqrt{2}}\right)\right\|_{L^2}^2 \\
&\qquad = (2\pi)^2\sum_{k=1}^{K_\epsilon}\left(\left(kc_k\right)^2 + \left(ks_k\right)^2\right) = \begin{cases} O\left(K_\epsilon^{3-2\gamma}\right), & \text{if } \dfrac{1}{2} < \gamma < \dfrac{3}{2}, \\ O\left(\log\left(K_\epsilon\right)\right), & \text{if } \gamma = \dfrac{3}{2}, \end{cases} \\
&\left\|c_0 + \sum_{k=1}^{K_\epsilon}\left(c_k\frac{\cos(2\pi k\cdot)}{\sqrt{2}} + s_k\frac{\sin(2\pi k\cdot)}{\sqrt{2}}\right)\right\|_{\infty} \\
&\qquad \le |c_0| + \frac{1}{\sqrt{2}}\sum_{k=1}^{K_\epsilon}\left(|c_k| + |s_k|\right) = \begin{cases} O\left(K_\epsilon^{3/2-\gamma}\right), & \text{if } \dfrac{1}{2} < \gamma < \dfrac{3}{2}, \\ O(1), & \text{if } \gamma = \dfrac{3}{2}, \end{cases} \\
&\left\|\sum_{k=K_\epsilon+1}^{\infty}\left(c_k\frac{\cos(2\pi k\cdot)}{\sqrt{2}} + s_k\frac{\sin(2\pi k\cdot)}{\sqrt{2}}\right)\right\|_{L^2}^2 \\
&\qquad = \sum_{k=K_\epsilon+1}^{\infty}\left(c_k^2 + s_k^2\right) = \begin{cases} O\left(K_\epsilon^{1-2\gamma}\right), & \text{if } \dfrac{1}{2} < \gamma < \dfrac{3}{2}, \\ O\left(K_\epsilon^{-2}\right), & \text{if } \gamma = \dfrac{3}{2}. \end{cases}
\end{aligned} \tag{2.18}$$

Integrating by parts, we can estimate

$$\begin{aligned}
\max_{0\le x\le 1}\left|\int_0^x f_1(z)g_\epsilon(z)dz\right| &\le \max_{0\le x\le 1}\left|\int_0^x\left(c_0 + \sum_{k=1}^{K_\epsilon}\left(c_k\frac{\cos(2\pi kz)}{\sqrt{2}} + s_k\frac{\sin(2\pi kz)}{\sqrt{2}}\right)\right)g_\epsilon(z)dz\right| \\
&\quad + \max_{0\le x\le 1}\left|\int_0^x\sum_{k=K_\epsilon+1}^{\infty}\left(c_k\frac{\cos(2\pi kz)}{\sqrt{2}} + s_k\frac{\sin(2\pi kz)}{\sqrt{2}}\right)g_\epsilon(z)dz\right| \\
&\le \left\|\frac{d}{dx}\sum_{k=1}^{K_\epsilon}\left(c_k\frac{\cos(2\pi k\cdot)}{\sqrt{2}} + s_k\frac{\sin(2\pi k\cdot)}{\sqrt{2}}\right)\right\|_{L^2}\left\|\epsilon A\left(\frac{\cdot}{\epsilon}\right)\right\|_{L^2} \\
&\quad + \left\|c_0 + \sum_{k=1}^{K_\epsilon}\left(c_k\frac{\cos(2\pi k\cdot)}{\sqrt{2}} + s_k\frac{\sin(2\pi k\cdot)}{\sqrt{2}}\right)\right\|_{\infty}\left\|\epsilon A\left(\frac{\cdot}{\epsilon}\right)\right\|_{\infty} \\
&\quad + \left\|\sum_{k=K_\epsilon+1}^{\infty}\left(c_k\frac{\cos(2\pi k\cdot)}{\sqrt{2}} + s_k\frac{\sin(2\pi k\cdot)}{\sqrt{2}}\right)\right\|_{L^2}\left\|g_\epsilon(\cdot)\right\|_{L^2}.
\end{aligned} \tag{2.19}$$

For $1/2 < \gamma < 3/2$, the boundedness of A and g yields

$$\max_{0\le x\le 1}\left|\int_0^x f_1(z)g_\epsilon(z)dz\right| = \epsilon O\left(K_\epsilon^{3/2-\gamma}\right)+\epsilon O\left(K_\epsilon^{3/2-\gamma}\right)+O\left(K_\epsilon^{1/2-\gamma}\right). \tag{2.20}$$

Choosing $K_\epsilon \in \mathbb{N}$ such that

$$K_\epsilon - 1 \le \frac{1}{\epsilon} \le K_\epsilon \tag{2.21}$$

yields the required estimation.

For $\gamma = 3/2$, the boundedness of A and g yields

$$\max_{0\le x\le 1}\left|\int_0^x f_1(z)g_\epsilon(z)dz\right| = \epsilon O\left(\sqrt{\log(K_\epsilon)}\right)+\epsilon O(1)+O(K_\epsilon^{-1}). \tag{2.22}$$

Choosing $K_\epsilon \in \mathbb{N}$ such that

$$K_\epsilon - 1 \le \frac{1}{\epsilon}^{|\log(\epsilon)|} \le K_\epsilon \tag{2.23}$$

yields the required estimation. □

Remark 2.2. The suppositions of Theorem 2.1 can be verified for $f_1 \in C([0,1];\mathbb{R})$. Let the modulus of continuity ω of f_1 satisfy $\omega(\delta) = O(\delta^\gamma)$, as $\delta \to 0$, for a $\gamma \in (1/2, 1]$. Then the Fourier coefficients fulfill

$$c_k = O(k^{-\gamma}),\quad s_k = O(k^{-\gamma}),\quad \text{as } k \longrightarrow \infty, \tag{2.24}$$

see [6, Theorem 4.6].

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