Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 314656, 18 pages
doi:10.1155/2009/314656

*Research Article*

# The Existence of Positive Solution to Three-Point Singular Boundary Value Problem of Fractional Differential Equation

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Received 13 May 2009; Accepted 23 June 2009

Recommended by Yong Zhou

We investigate the existence of positive solution to nonlinear fractional differential equation three-point singular boundary value problem: $D^q u(t) + f(t, u(t)) = 0$, $0 < t < 1$, $u(0) = 0$, $u(1) = \alpha D^{(q-1)/2} u(t)|_{t=\xi}$, where $1 < q \le 2$ is a real number, $\xi \in (0, 1/2]$, $\alpha \in (0, +\infty)$ and $\alpha\Gamma(q)\xi^{(q-1)/2} < \Gamma((q+1)/2), D^q$ is the standard Riemann-Liouville fractional derivative, and $f \in C((0,1] \times [0, +\infty), [0, +\infty)), \lim_{t \to +0} f(t, \cdot) = +\infty$ (i.e., $f$ is singular at $t = 0$). By using the fixed-point index theory, the existence result of positive solutions is obtained.



## 1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, and engineering. For details, see [1–10] and the reference therein. However, up to our knowledge, most of those papers have studied the existence and multiplicity of solution (or positive solution) to the initial value problem of nonlinear fractional differential equations; see [1, 4, 10, 11].

Recently, there are a few paper considering the Dirichlet-type boundary value problem for nonlinear ordinary differential equations of fractional order; see [12–14]. Delbosco [13] has investigated the nonlinear Dirichlet-type problem

$$\begin{gathered} u^{q-1} D^q u(t) = f(u(t)), \quad 0 < t < 1,\ 1 < q < 2, \\ u(0) = u(1) = 0. \end{gathered} \tag{1.1}$$

He has proved that if $f(u)$ is Lipschitizan function, then the problem has at least one solution $u(t)$ in a certain subspace of $C[0,1]$ in which fractional derivative has a Holder property. When $f(t,u)$ is continuous on $[0,1] \times [0,+\infty)$, by the use of some fixed-point theorem on cones, Bai and Lü [12] and Zhang [14] have given the existence of positive solutions to the equation

$$D^q u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \tag{1.2}$$

with boundary condition

$$\begin{gathered} u(0) = u(1) = 0, \\ u(0) + u'(0) = u(1) + u'(1), \end{gathered} \tag{1.3}$$

respectively.

This paper is to study the existence of positive solution for the three-point singular boundary value problem of nonlinear fractional differential equation

$$\begin{gathered} D^q u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \alpha D^{(q-1)/2} u(t)\Big|_{t=\xi}. \end{gathered} \tag{1.4}$$

By using the fixed-point index theory, where $1 < q \leq 2$ is a real number, $\xi \in (0,1/2]$, $\alpha \in (0,+\infty)$ satisfy that $\alpha\Gamma(q)\xi^{(q-1)/2} < \Gamma((q+1)/2)$, $D^q$ is the standard Riemann-Liouville fractional derivative, and the function $f$ satisfies the following condition:

$(H_1)$ $f \in C((0,1] \times [0,+\infty),[0,+\infty))$, $\lim_{t\to+0} f(t,\cdot) = +\infty$, there exists a constant $b : 0 < b < 1$ such that $t^b f(t,u(t))$ is continuous function on $[0,1] \times [0,+\infty)$.

The organization of this paper is as follows. In Section 2, we present some necessary definitions and Preliminary results that will be used to prove our main results. The proof of our main result is given in Section 3. In Section 4, we will give an example to ensure our main result.

## 2. Preliminaries

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions from fractional calculus theory and preliminary results.

*Definition 2.1.* The fractional integral of order $q > 0$ of a function $x : (0,+\infty) \to R$ is given by

$$I^q x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s)\,ds, \tag{2.1}$$

provided that the right side is pointwise defined on $(0,\infty)$.

*Definition 2.2.* The fractional derivative of order $q > 0$ of a continuous function $x : (0, +\infty) \to R$ is given by

$$D^q x(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{x(s)}{(t-s)^{q-n+1}} ds, \tag{2.2}$$

where $n = [q] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

**Lemma 2.3** (see [7]). (1) *If $x \in L(0,1)$, $\rho > \sigma > 0$, then $D^\sigma I^\rho x(t) = I^{\rho-\sigma} x(t)$.*

(2) *If $\rho > 0$, $\lambda > 0$, then $D^\rho t^{\lambda-1} = (\Gamma(\lambda)/\Gamma(\lambda-\rho)) t^{\lambda-\rho-1}$.*

**Lemma 2.4** (see [12]). *Assume that $x \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $q > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^q D^q x(t) = x(t) + A_1 t^{q-1} + A_2 t^{q-2} + \cdots + A_N t^{q-N}, \tag{2.3}$$

*$A_i \in R$, $i = 1, 2, \ldots, N$, where $N$ is the smallest integer greater than or equal to $q$.*

**Lemma 2.5.** *If $y \in C(0,1) \cap L(0,1)$ and $1 < q \le 2$, $\xi \in (0,1)$, $\alpha \in \mathbb{R}$ satisfy that $\alpha\Gamma(q)\xi^{(q-1)/2} \neq \Gamma((q+1)/2)$, then the problems*

$$\begin{gathered} D^q u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \alpha D^{(q-1)/2} u(t)\Big|_{t=\xi} \end{gathered} \tag{2.4}$$

*have the unique solution*

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s) ds \\ &+ \frac{t^{q-1}\Gamma((q+1)/2)}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left\{ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s) ds - \alpha \int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)} y(s) ds \right\}. \end{aligned} \tag{2.5}$$

*Proof.* By applying Lemma 2.4, we may reduce (2.4) to an equivalent integral equation

$$u(t) = -I^q y(t) + A_1 t^{q-1} + A_2 t^{q-2} \tag{2.6}$$

for some $A_1, A_2 \in R$. Consequently the general solution of (2.4) is

$$u(t) = -\int_0^t \frac{(t-s)^{q-1} y(s)}{\Gamma(q)} ds + A_1 t^{q-1} + A_2 t^{q-2}. \tag{2.7}$$

Note that $u(0)=0$, we have $A_2=0$ and

$$u(1) = -\int_0^1 \frac{(1-s)^{q-1}y(s)}{\Gamma(q)}ds + A_1. \tag{2.8}$$

On the other hand, by (2.6) and Lemma 2.3, we have

$$\begin{aligned} D^{(q-1)/2}u(t) &= -D^{(q-1)/2}I^q y(t) + A_1 D^{(q-1)/2}t^{q-1} \\ &= -I^{(q+1)/2}y(t) + A_1\frac{\Gamma(q)}{\Gamma((q+1)/2)}t^{(q-1)/2}. \end{aligned} \tag{2.9}$$

Therefore

$$D^{(q-1)/2}u(t)\Big|_{t=\xi} = -\int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)}y(s)ds + A_1\frac{\Gamma(q)}{\Gamma((q+1)/2)}\xi^{(q-1)/2}. \tag{2.10}$$

By $u(1)=\alpha D^p u(t)|_{t=\xi}$, combine with (2.8) and (2.10), we obtain

$$A_1 = \frac{\Gamma((q+1)/2)}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}\left\{\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}y(s)ds - \alpha\int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)}y(s)ds\right\}. \tag{2.11}$$

So, the unique solution of problem (2.4) is

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)ds \\ &+ \frac{t^{q-1}\Gamma((q+1)/2)}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}\left\{\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}y(s)ds - \alpha\int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)}y(s)ds\right\}. \end{aligned} \tag{2.12}$$

The proof is completed. □

**Lemma 2.6.** *If* $y \in C((0,1),[0,+\infty)) \cap L(0,1)$ *and* $1 < q \leq 2$, $\xi \in (0,1/2]$, $\alpha \in (0,+\infty)$ *satisfy that* $\alpha\Gamma(q)\xi^{(q-1)/2} < \Gamma((q+1)/2)$, *then the unique solution of the problem* (2.4)

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds \\ &\quad + \frac{t^{q-1}\Gamma((q+1)/2)}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left\{ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s)ds - \alpha \int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)} y(s)ds \right\} \\ &= \int_0^1 G(t,s)y(s)ds + \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \\ &\quad \times \left\{ \int_0^\xi \left( (1-s)^{q-1}\xi^{(q-1)/2} - (\xi-s)^{(q-1)/2} \right) y(s)ds + \int_\xi^1 (1-s)^{q-1}\xi^{(q-1)/2} y(s)ds \right\}. \end{aligned} \tag{2.13}$$

*is nonnegative on* $[0,1]$, *where*

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{q-1} - (t-s)^{q-1}}{\Gamma(q)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{[t(1-s)]^{q-1}}{\Gamma(q)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.14}$$

*Proof.* By Lemma 2.5, the unique solution of problem (2.4) is

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds \\ &\quad + \frac{t^{q-1}\Gamma((q+1)/2)}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left\{ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s)ds - \alpha \int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)} y(s)ds \right\} \\ &= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds \\ &\quad + \frac{t^{q-1}\Gamma((q+1)/2)}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s)ds \\ &\quad - \frac{\alpha t^{q-1}\Gamma((q+1)/2)}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)} y(s)ds \end{aligned}$$

$$
\begin{aligned}
&= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds \\
&\quad + \left(1 + \frac{\alpha\Gamma(q)\xi^{(q-1)/2}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}}\right) t^{q-1} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s)ds \\
&\quad - \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \int_0^\xi (\xi - s)^{(q-1)/2} y(s)ds \\
&= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds + \int_0^1 \frac{(1-s)^{q-1} t^{q-1}}{\Gamma(q)} y(s)ds \\
&\quad + \frac{\alpha t^{q-1}\xi^{(q-1)/2}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \int_0^1 (1-s)^{q-1} y(s)ds \\
&\quad - \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \int_0^\xi (\xi - s)^{(q-1)/2} y(s)ds \\
&= \int_0^t \frac{[t(1-s)]^{q-1} - (t-s)^{q-1}}{\Gamma(q)} y(s)ds + \int_t^1 \frac{[t(1-s)]^{q-1}}{\Gamma(q)} y(s)ds \\
&\quad + \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left\{ \int_0^1 (1-s)^{q-1}\xi^{(q-1)/2} y(s)ds - \int_0^\xi (\xi - s)^{(q-1)/2} y(s)ds \right\} \\
&= \int_0^1 G(t,s) y(s)ds + \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \\
&\quad \times \left\{ \int_0^\xi \left((1-s)^{q-1}\xi^{(q-1)/2} - (\xi - s)^{(q-1)/2}\right) y(s)ds + \int_\xi^1 (1-s)^{q-1}\xi^{(q-1)/2} y(s)ds \right\}.
\end{aligned}
\tag{2.15}
$$

Observing the expression of $G(t,s)$, it is clear that $G(t,s) > 0$ for $s,t \in (0,1)$. On the other hand, by $\xi \in (0,1/2]$, we have

$$
(1-s)^2 \geq \left(1 - \frac{s}{\xi}\right), \quad \forall s \in [0,\xi]. \tag{2.16}
$$

Hence

$$
(1-s)^{q-1} \geq \left(1 - \frac{s}{\xi}\right)^{(q-1)/2}. \tag{2.17}
$$

It implies that

$$
(1-s)^{q-1}\xi^{(q-1)/2} \geq (\xi - s)^{(q-1)/2}. \tag{2.18}
$$

Therefore $u(t)$ is nonnegative on $[0,1]$. □

**Lemma 2.7.** *Let* $1 < q \leq 2$, $\xi \in (0, 1/2]$, $\alpha \in (0, +\infty)$ *satisfy that* $\alpha\Gamma(q)\xi^{(q-1)/2} < \Gamma((q+1)/2)$ *and* $y : (0, 1] \to [0, +\infty)$ *is continuous, and* $\lim_{t\to+0} y(t) = +\infty$. *Suppose that there exists a constant* $b : 0 < b < 1$ *such that* $t^b y(t)$ *is continuous function on* $[0, 1]$. *Then the unique solution of* (2.4)

$$u(t) = -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds + \frac{t^{q-1}\Gamma((q+1)/2)}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left\{ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s)ds - \alpha \int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)} y(s)ds \right\}. \tag{2.19}$$

*is continuous on* $[0, 1]$.

*Proof.* Since $t^b y(t)$ is continuous in $[0, 1]$, thus there exists a constant $L > 0$ such that $|t^b y(t)| \leq L$ for all $t \in [0, 1]$ and

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{-b} \cdot s^b y(s)ds \\ &+ \frac{t^{q-1}\Gamma((q+1)/2)}{[\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}]\Gamma(q)} \int_0^1 (1-s)^{q-1} s^{-b} \cdot s^b y(s)ds \\ &- \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \int_0^\xi (\xi-s)^{(q-1)/2} s^{-b} \cdot s^b y(s)ds. \end{aligned} \tag{2.20}$$

For any $t_0 \in [0, 1]$, we will prove $u(t) \to u(t_0)$ $(t \to t_0, t \in [0, 1])$. For the convenience, the proof is divided into three cases.

*Case 1* $(t_0 = 0)$. It is easy to know that $u(0) = 0$. For all $t \in (0, 1]$, we have

$$\begin{aligned} |u(t) - u(t_0)| \leq &\left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{-b} \cdot s^b y(s)ds \right| \\ &+ \left| \frac{t^{q-1}\Gamma((q+1)/2)}{[\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}]\Gamma(q)} \int_0^1 (1-s)^{q-1} s^{-b} \cdot s^b y(s)ds \right| \\ &+ \left| \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \int_0^\xi (\xi-s)^{(q-1)/2} s^{-b} \cdot s^b y(s)ds \right| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{L}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-b}ds \\
&\quad + \frac{Lt^{q-1}\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)}\int_0^1 (1-s)^{q-1}s^{-b}ds \\
&\quad + \frac{\alpha L t^{q-1}}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}\int_0^\xi (\xi-s)^{(q-1)/2}s^{-b}ds \\
&= \frac{Lt^{q-b}}{\Gamma(q)}B(1-b,q) + \frac{Lt^{q-1}\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)}B(1-b,q) \\
&\quad + \frac{\alpha L\xi^{(1-b+(q-1)/2)}t^{q-1}}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}B(1-b,(q+1)/2) \\
&= \frac{L\Gamma(1-b)}{\Gamma(1-b+q)}t^{q-b} + \frac{L\Gamma(1-b)\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+q)}t^{q-1} \\
&\quad + \frac{\alpha L\Gamma(1-b)\Gamma((q+1)/2)\xi^{(1-b+(q-1)/2)}}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+(q+1)/2)}t^{q-1} \longrightarrow 0 \quad (t\longrightarrow 0),
\end{aligned}
\tag{2.21}
$$

where $B$ denotes beta function.

*Case 2* ($t_0 \in (0,1)$, for all $t \in (t_0, 1]$). We have

$$
\begin{aligned}
&|u(t)-u(t_0)| \\
&\quad = \Bigg| -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds + \frac{1}{\Gamma(q)}\int_0^{t_0}(t_0-s)^{q-1}y(s)ds \\
&\qquad + \frac{\left(t^{q-1}-t_0^{q-1}\right)\Gamma((q+1)/2)}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}\left\{\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}y(s)ds - \alpha\int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)}y(s)ds\right\}\Bigg| \\
&\quad \le \left|\frac{1}{\Gamma(q)}\int_0^{t_0}\left((t-s)^{q-1}-(t_0-s)^{q-1}\right)s^{-b}\cdot s^b y(s)ds + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}s^{-b}\cdot s^b y(s)ds\right| \\
&\qquad + \left|\frac{\left(t^{q-1}-t_0^{q-1}\right)\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)}\int_0^1 (1-s)^{q-1}s^{-b}\cdot s^b y(s)ds\right| \\
&\qquad + \left|\frac{\alpha\left(t^{q-1}-t_0^{q-1}\right)}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}\int_0^\xi (\xi-s)^{(q-1)/2}s^{-b}\cdot s^b y(s)ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{L}{\Gamma(q)}\int_0^{t_0}\left((t-s)^{q-1}-(t_0-s)^{q-1}\right)s^{-b}ds+\frac{L}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}s^{-b}ds \\
&\quad+\frac{\left(t^{q-1}-t_0^{q-1}\right)\Gamma((q+1)/2)L}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)}\int_0^1(1-s)^{q-1}s^{-b}ds \\
&\quad+\frac{\alpha\left(t^{q-1}-t_0^{q-1}\right)L}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}\int_0^{\xi}(\xi-s)^{(q-1)/2}s^{-b}ds \\
&=\frac{L}{\Gamma(q)}\left(t^{q-b}-t_0^{q-b}\right)B(1-b,q)+\frac{\left(t^{q-1}-t_0^{q-1}\right)\Gamma((q+1)/2)L}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)}B(1-b,q) \\
&\quad+\frac{\alpha\left(t^{q-1}-t_0^{q-1}\right)L\xi^{(1-b+(q-1)/2)}}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}B\left(1-b,\frac{q+1}{2}\right) \\
&=\frac{L\Gamma(1-b)}{\Gamma(1-b+q)}\left(t^{q-b}-t_0^{q-b}\right)+\frac{L\Gamma(1-b)\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+q)}\left(t^{q-1}-t_0^{q-1}\right) \\
&\quad+\frac{\alpha L\Gamma(1-b)\Gamma((q+1)/2)\xi^{(1-b+(q-1)/2)}}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+(q+1)/2)}\left(t^{q-1}-t_0^{q-1}\right)\longrightarrow 0\quad(t\longrightarrow t_0).
\end{aligned}
\tag{2.22}
$$

*Case 3* ($t_0 \in (0,1]$, for all $t \in [0,t_0)$). The proof is similar to the step 2. Here, we omit it. □

The main tool of this paper is the following well-known fixed-point index theorem (see [15]).

**Lemma 2.8.** *Let $E$ be a Banach space, $P \subset E$ is a cone. For $r > 0$, define $\Omega_r = \{u \in P \mid \|u\| < r\}$. Assume that $T : \overline{\Omega}_r \longrightarrow P$ is a completely continuous such that $Tu \neq u$ for $u \in \partial\Omega_r = \{u \in P \mid \|u\| = r\}$.*

(1) *If $\|Tu\| \geq \|u\|$ for $u \in \partial\Omega_r$, then $i(T,\Omega_r,P) = 0$.*

(2) *If $\|Tu\| \leq \|u\|$ for $u \in \partial\Omega_r$, then $i(T,\Omega_r,P) = 1$.*

## 3. Main Results

For the convenience we introduce the following notations:

$$
C_1 = \frac{\Gamma(1-b)}{\Gamma(1-b+q)},
$$

$$C_2 = \frac{\Gamma(1-b)\Gamma((q+1)/2)}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left( \frac{1}{\Gamma(1-b+q)} + \frac{\alpha\xi^{(1-b+(q-1)/2)}}{\Gamma(1-b+(q+1)/2)} \right),$$
$$C_3 = \frac{\alpha\Gamma(1-b)\xi^{(q-1)/2}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left( \frac{\Gamma(q)}{\Gamma(1-b+q)} - \frac{\Gamma((q+1)/2)\xi^{1-b}}{\Gamma(1-b+(q+1)/2)} \right). \tag{3.1}$$

*Remark 3.1.* If $1 < q \le 2$, $0 < b < 1$, $0 < \xi \le 1/2$, $\alpha > 0$, and $\Gamma((q+1)/2) > \alpha\Gamma(q)\xi^{(q-1)/2}$, then

$$\begin{aligned} C_3 &= \frac{\alpha}{\left[\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}\right]} \left\{ \int_0^\xi \left( (1-s)^{q-1}\xi^{(q-1)/2} - (\xi-s)^{(q-1)/2} \right) \cdot s^{-b} ds \right. \\ &\qquad \left. + \int_\xi^1 (1-s)^{q-1}\xi^{(q-1)/2} \cdot s^{-b} ds \right\} \\ &= \frac{\alpha\Gamma(1-b)\xi^{(q-1)/2}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left( \frac{\Gamma(q)}{\Gamma(1-b+q)} - \frac{\Gamma((q+1)/2)\xi^{1-b}}{\Gamma(1-b+(q+1)/2)} \right) > 0. \end{aligned} \tag{3.2}$$

Let $E = C[0,1]$ be a Banach spaces with the maximum norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Define the cone $P \subset E$ by

$$P = \{u \in E \mid u(t) \ge 0,\ 0 \le t \le 1\}. \tag{3.3}$$

The positive solution which we consider in this paper is the form $u(0) = 0$, $u(t) > 0$, $0 < t \le 1$, $u \in E$.

Define an operator $T : P \to E$ by

$$\begin{aligned} Tu(t) &= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds \\ &\quad + \frac{t^{q-1}\Gamma((q+1)/2)}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left\{ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds \right. \\ &\qquad \left. -\alpha \int_0^\xi \frac{(\xi-s)^{(q-1)/2}}{\Gamma((q+1)/2)} f(s,u(s))ds \right\}. \end{aligned} \tag{3.4}$$

**Lemma 3.2.** *Assume that condition $(H_1)$ holds. Then the operator $T : P \to P$ is completely continuous.*

*Proof.* If condition $(H_1)$ holds, by Lemmas 2.6 and 2.7, we have $T(P) \subset P$. Let $u_0 \in P$ and $\|u_0\| = a_0$; if $u \in P$ and $\|u - u_0\| < 1$, then $\|u\| < 1 + a_0 := a$. By the continuous of $t^b f(t,x)$, we know that $t^b f(t,x)$ is uniformly continuous on $[0,1] \times [0,a]$. Thus, for all $\varepsilon > 0$, there exists a $\delta > 0$ $(\delta < 1)$ such that

$$\left| t^b f(t,x_1) - t^b f(t,x_2) \right| < \frac{\varepsilon}{C_1 + C_2} \tag{3.5}$$

for all $t \in [0,1]$ and $x_1, x_2 \in [0,a]$ with $|x_1 - x_2| < \delta$. Obviously, if $\|u - u_0\| \le \delta$, then $u_0(t), u(t) \in [0,a]$ and $|u(t) - u_0(t)| < \delta$ for each $t \in [0,1]$. Hence, we have

$$\left| t^b f(t, u(t)) - t^b f(t, u_0(t)) \right| < \frac{\varepsilon}{C_1 + C_2} \tag{3.6}$$

for all $t \in [0,1], u \in P$ with $\|u - u_0\| < \delta$. It follows from (3.6) that

$$\begin{aligned}
&\|Tu - Tu_0\| \\
&\quad = \max_{0 \le t \le 1} |Tu(t) - Tu_0(t)| \\
&\quad \le \max_{0 \le t \le 1} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{-b} \left| s^b f(s, u(s)) - s^b f(s, u_0(s)) \right| ds \right. \\
&\qquad\quad + \frac{t^{q-1} \Gamma((q+1)/2)}{\left[\Gamma((q+1)/2) - \alpha \Gamma(q) \xi^{(q-1)/2}\right] \Gamma(q)} \\
&\qquad\quad \times \int_0^1 (1-s)^{q-1} s^{-b} \left| s^b f(s, u(s)) - s^b f(s, u_0(s)) \right| ds \\
&\qquad\quad + \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha \Gamma(q) \xi^{(q-1)/2}} \\
&\qquad\quad \left. \times \int_0^{\xi} (\xi - s)^{(q-1)/2} s^{-b} \left| s^b f(s, u(s)) - s^b f(s, u_0(s)) \right| ds \right\} \\
&\quad < \max_{0 \le t \le 1} \left\{ \frac{\varepsilon}{\Gamma(q)(C_1 + C_2)} \int_0^t (t-s)^{q-1} s^{-b} ds \right. \\
&\qquad\quad + \frac{\varepsilon t^{q-1} \Gamma((q+1)/2)}{\left[\Gamma((q+1)/2) - \alpha \Gamma(q) \xi^{(q-1)/2}\right] (C_1 + C_2) \Gamma(q)} \int_0^1 (1-s)^{q-1} s^{-b} ds \\
&\qquad\quad \left. + \frac{\alpha \varepsilon t^{q-1}}{\left[\Gamma((q+1)/2) - \alpha \Gamma(q) \xi^{(q-1)/2}\right] (C_1 + C_2)} \int_0^{\xi} (\xi - s)^{(q-1)/2} s^{-b} ds \right\} \\
&\quad = \max_{0 \le t \le 1} \left\{ \frac{\varepsilon}{\Gamma(q)(C_1 + C_2)} t^{q-b} B(1-b, q) + \frac{\varepsilon t^{q-1} \Gamma((q+1)/2) B(1-b, q)}{\left[\Gamma((q+1)/2) - \alpha \Gamma(q) \xi^{(q-1)/2}\right] (C_1 + C_2) \Gamma(q)} \right. \\
&\qquad\quad \left. + \frac{\alpha \varepsilon t^{q-1} \xi^{(1-b+(q-1)/2)}}{\left[\Gamma((q+1)/2) - \alpha \Gamma(q) \xi^{(q-1)/2}\right] (C_1 + C_2)} B\left(1-b, \frac{q+1}{2}\right) \right\} \\
&\quad \le \left\{ \frac{\Gamma(1-b)}{\Gamma(1-b+q)} + \frac{\Gamma((q+1)/2) \Gamma(1-b)}{\left[\Gamma((q+1)/2) - \alpha \Gamma(q) \xi^{(q-1)/2}\right] \Gamma(1-b+q)} \right. \\
&\qquad\quad \left. + \frac{\alpha \Gamma(1-b) \Gamma((q+1)/2) \xi^{(1-b+(q-1)/2)}}{\left[\Gamma((q+1)/2) - \alpha \Gamma(q) \xi^{(q-1)/2}\right] \Gamma(1-b+(q+1)/2)} \right\} \frac{\varepsilon}{(C_1 + C_2)} = \varepsilon.
\end{aligned} \tag{3.7}$$

By the arbitrariness of $u_0$, we have that $T : P \to P$ is continuous.

Let $\Omega \subset P$ be bounded; that is, there exists a positive constant $M$ such that $\Omega \in \{u \in P \mid \|u\| \leq M\}$. Since $t^b f(t,u(t))$ is continuous on $[0,1]\times[0,+\infty)$, we let

$$N = \max_{(t,u)\in[0,1]\times[0,M]} t^b f(t,u(t)) + 1. \tag{3.8}$$

For all $u \in \Omega$, we have

$$\begin{aligned}
|Tu(t)| &\leq \left|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-b}\cdot s^b f(t,u(t))ds\right| \\
&\quad+\left|\frac{t^{q-1}\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)}\int_0^1 (1-s)^{q-1}s^{-b}\cdot s^b f(s,u(s))ds\right| \\
&\quad+\left|\frac{\alpha t^{q-1}}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}\int_0^{\xi} (\xi-s)^{(q-1)/2}s^{-b}\cdot s^b f(s,u(s))ds\right| \\
&\leq \frac{N}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-b}ds + \frac{Nt^{q-1}\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)}\int_0^1 (1-s)^{q-1}s^{-b}ds \\
&\quad+\frac{\alpha N t^{q-1}}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}\int_0^{\xi}(\xi-s)^{(q-1)/2}s^{-b}ds \\
&= \frac{N}{\Gamma(q)}t^{q-b}B(1-b,q) + \frac{Nt^{q-1}\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)}B(1-b,q) \\
&\quad+\frac{\alpha N\xi^{(1-b+(q-1)/2)}t^{q-1}}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}}B\left(1-b,\frac{q+1}{2}\right) \\
&\leq \frac{N\Gamma(1-b)}{\Gamma(1-b+q)} + \frac{N\Gamma((q+1)/2)\Gamma(1-b)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+q)} \\
&\quad+\frac{\alpha N\Gamma(1-b)\Gamma((q+1)/2)\xi^{(1-b+(q-1)/2)}}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+(q+1)/2)} \\
&= (C_1+C_2)N.
\end{aligned} \tag{3.9}$$

Hence $T(\Omega)$ is bounded.

On the other hand, given $\varepsilon > 0$, set

$$\delta = \min\left\{\frac{1}{2}\left(\frac{\varepsilon}{4NC_1}\right)^{1/(q-b)}, \frac{1}{2}\left(\frac{\varepsilon}{2NC_2}\right)^{1/(q-1)}, \frac{\varepsilon}{4NC_1}\right\}. \tag{3.10}$$

For each $u \in \Omega$, we will prove that if $t_1, t_2 \in [0,1]$ and $0 < t_2 - t_1 < \delta$, then

$$|Tu(t_2) - Tu(t_1)| < \varepsilon. \tag{3.11}$$

In fact, similar to the proof of Lemma 2.7, we have

$$\begin{aligned}
|Tu(t_2)-Tu(t_1)| \le{}& \frac{N\Gamma(1-b)}{\Gamma(1-b+q)}\left(t_2^{q-b}-t_1^{q-b}\right)\\
&+\frac{N\Gamma((q+1)/2)\Gamma(1-b)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+q)}\left(t_2^{q-1}-t_1^{q-1}\right)\\
&+\frac{\alpha N\Gamma(1-b)\Gamma((q+1)/2)\xi^{(1-b+(q-1)/2)}}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+(q+1)/2)}\left(t_2^{q-1}-t_1^{q-1}\right)\\
={}& NC_1\left(t_2^{q-b}-t_1^{q-b}\right)+NC_2\left(t_2^{q-1}-t_1^{q-1}\right).
\end{aligned}\tag{3.12}$$

In the following, the proof is divided into three cases.

*Case 1* ($\delta \le t_1 < t_2 < 1,\ q-b-1 \le 0$).

$$\begin{aligned}
|Tu(t_2)-Tu(t_1)| &\le NC_1\left(t_2^{q-b}-t_1^{q-b}\right)+NC_2\left(t_2^{q-1}-t_1^{q-1}\right)\\
&\le NC_1(q-b)\delta^{q-b-1}(t_2-t_1)+NC_2(q-1)\delta^{q-2}(t_2-t_1)\\
&< 2NC_1\delta^{q-b}+NC_2\delta^{q-1}\\
&\le 2NC_1\cdot\frac{\varepsilon}{4NC_1 2^{q-b}}+NC_2\cdot\frac{\varepsilon}{2NC_2 2^{q-1}}\\
&= \frac{\varepsilon}{2\times 2^{q-b}}+\frac{\varepsilon}{2\times 2^{q-1}}\\
&< \varepsilon.
\end{aligned}\tag{3.13}$$

*Case 2* ($\delta \le t_1 < t_2 < 1,\ q-b-1 > 0$).

$$\begin{aligned}
|Tu(t_2)-Tu(t_1)| &\le NC_1\left(t_2^{q-b}-t_1^{q-b}\right)+NC_2\left(t_2^{q-1}-t_1^{q-1}\right)\\
&\le NC_1(q-b)(t_2-t_1)+NC_2(q-1)\delta^{q-2}(t_2-t_1)\\
&< 2NC_1\delta+NC_2\delta^{q-1}\\
&\le 2NC_1\cdot\frac{\varepsilon}{4NC_1}+NC_2\cdot\frac{\varepsilon}{2NC_2 2^{q-1}}\\
&= \frac{\varepsilon}{2}+\frac{\varepsilon}{2\times 2^{q-1}}\\
&< \varepsilon.
\end{aligned}\tag{3.14}$$

*Case 3* ($0 \leq t_1 < \delta, t_2 < 2\delta$).

$$
\begin{aligned}
|Tu(t_2) - Tu(t_1)| &\leq NC_1\left(t_2^{q-b} - t_1^{q-b}\right) + NC_2\left(t_2^{q-1} - t_1^{q-1}\right) \\
&\leq NC_1 t_2^{q-b} + NC_2 t_2^{q-1} \\
&\leq NC_1 (2\delta)^{q-b} + NC_2 (2\delta)^{q-1} \\
&\leq NC_1 \cdot \frac{\varepsilon}{4NC_1} + NC_2 \cdot \frac{\varepsilon}{2NC_2} \\
&= \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}
\tag{3.15}
$$

Therefore, $T(\Omega)$ is equicontinuous. The Arzela-Ascoli Theorem implies that $T(\Omega)$ is compact. Thus, the operator $T : P \to P$ is completely continuous. □

We obtain the following existence results of the positive solution for problem (1.4).

**Theorem 3.3.** *If condition* $(H_1)$ *holds and assume further that there exist two positive constants* $R > r > 0$ *such that*

$(H_2)$ $t^b f(t,u) > r/C_3$, *for* $(t,u) \in [0,1] \times [0,r]$;

$(H_3)$ $t^b f(t,u) < R/(C_1 + C_2)$, *for* $(t,u) \in [0,1] \times [0,R]$,

*then problem* (1.4) *has at least one positive solution* $u$ *such that* $r < \|u\| < R$.

*Proof.* Problem (1.4) has a solution $u = u(t)$ if and only if $u$ is a solution of the operator equation $u = Tu$. In order to apply Lemma 2.8, we separate the proof into the following two steps.

*Step 1.* Let $\Omega_r := \{u \in P \mid \|u\| < r\}$. For any $u \in \partial\Omega_r$, we have $\|u\| = r$ and $0 \leq u(t) \leq r$ for all $t \in [0,1]$. Observing the expression of $G(t,s)$ (see (2.14)), it is clear that $G(1,s) = 0$. By assumption $(H_2)$, we have

$$
\begin{aligned}
Tu(1) = &\int_0^1 G(1,s) f(s,u(s))ds \\
&+ \frac{\alpha}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left\{ \int_0^{\xi} \left( (1-s)^{q-1}\xi^{(q-1)/2} - (\xi - s)^{(q-1)/2} \right) f(s,u(s))ds \right. \\
&\left. + \int_{\xi}^1 (1-s)^{q-1}\xi^{(q-1)/2} f(s,u(s))ds \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left\{ \int_0^{\xi} \left( (1-s)^{q-1}\xi^{(q-1)/2} - (\xi - s)^{(q-1)/2} \right) t^{-b} \cdot t^b f(t, u(t)) ds \right. \\
&\qquad \left. + \int_{\xi}^{1} (1-s)^{q-1}\xi^{(q-1)/2} \cdot t^{-b} \cdot t^b f(t, u(t)) ds \right\} \\
&> \frac{\alpha r}{\left[\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}\right] C_3} \\
&\quad \times \left\{ \int_0^{\xi} \left( (1-s)^{q-1}\xi^{(q-1)/2} - (\xi - s)^{(q-1)/2} \right) \cdot s^{-b} ds + \int_{\xi}^{1} (1-s)^{q-1}\xi^{(q-1)/2} \cdot s^{-b} ds \right\} \\
&= \frac{\alpha r}{\left[\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}\right] C_3} \left\{ \xi^{(q-1)/2} \int_0^1 (1-s)^{q-1} s^{-b} ds - \int_0^{\xi} (\xi - s)^{(q-1)/2} s^{-b} ds \right\} \\
&= \frac{\alpha r}{\left[\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}\right] C_3} \left\{ \xi^{(q-1)/2} B(1-b, q) - \xi^{1-b+(q-1)/2} B\left(1-b, \frac{q+1}{2}\right) \right\} \\
&= \frac{\alpha\Gamma(1-b)\xi^{(q-1)/2}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \left( \frac{\Gamma(q)}{\Gamma(1-b+q)} - \frac{\Gamma((q+1)/2)\xi^{1-b}}{\Gamma(1-b+(q+1)/2)} \right) \cdot \frac{r}{C_3} = r.
\end{aligned}
\tag{3.16}
$$

So

$$
\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_r. \tag{3.17}
$$

By Lemma 2.8, we have

$$
i(T, \Omega_r, P) = 0. \tag{3.18}
$$

*Step 2.* Let $\Omega_R := \{u \in P \mid \|u\| < R\}$. For any $u \in \partial\Omega_R$, we have $\|u\| = R$ and $0 \leq u(t) \leq R$ for all $t \in [0,1]$. By assumption $(H_3)$, for $t \in [0,1]$, we get

$$
\begin{aligned}
|Tu(t)| \leq &\left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{-b} \cdot s^b f(t, u(t)) ds \right| \\
&+ \left| \frac{t^{q-1}\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)} \int_0^1 (1-s)^{q-1} s^{-b} \cdot s^b f(s, u(s)) ds \right| \\
&+ \left| \frac{\alpha t^{q-1}}{\Gamma((q+1)/2) - \alpha\Gamma(q)\xi^{(q-1)/2}} \int_0^{\xi} (\xi - s)^{(q-1)/2} s^{-b} \cdot s^b f(s, u(s)) ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{R}{\Gamma(q)(C_1+C_2)}\int_0^t (t-s)^{q-1}s^{-b}ds \\
&\quad + \frac{Rt^{q-1}\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)(C_1+C_2)}\int_0^1 (1-s)^{q-1}s^{-b}ds \\
&\quad + \frac{\alpha Rt^{q-1}}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}(C_1+C_2)}\int_0^{\xi} (\xi-s)^{(q-1)/2}s^{-b}ds \\
&= \frac{R}{\Gamma(q)(C_1+C_2)}t^{q-b}B(1-b,q) \\
&\quad + \frac{Rt^{q-1}\Gamma((q+1)/2)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(q)(C_1+C_2)}B(1-b,q) \\
&\quad + \frac{\alpha N\xi^{(1-b+(q-1)/2)}t^{q-1}}{\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}(C_1+C_2)}B\left(1-b,\frac{q+1}{2}\right) \\
&\le \left\{\frac{\Gamma(1-b)}{\Gamma(1-b+q)} + \frac{\Gamma((q+1)/2)\Gamma(1-b)}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+q)}\right. \\
&\qquad \left. + \frac{\alpha\Gamma(1-b)\Gamma((q+1)/2)\xi^{(1-b+(q-1)/2)}}{\left[\Gamma((q+1)/2)-\alpha\Gamma(q)\xi^{(q-1)/2}\right]\Gamma(1-b+(q+1)/2)}\right\}\cdot\frac{R}{C_1+C_2} = R.
\end{aligned}
\tag{3.19}
$$

Therefore

$$
\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega_R. \tag{3.20}
$$

By Lemma 2.8, we have

$$
i(T,\Omega_R,P) = 1. \tag{3.21}
$$

Combine with (3.18) and (3.21), we have

$$
i\left(T,\Omega_R\setminus\overline{\Omega}_r,P\right) = i(T,\Omega_R,P) - i(T,\Omega_r,P) = 1-0 = 1. \tag{3.22}
$$

Therefore, $T$ has a fixed point $u \in \Omega_R\setminus\overline{\Omega}_r$. Then problem (1.4) has at least one positive solution $u$ such that $r < \|u\| < R$.

□

## 4. Example

Let $q = 3/2$, $\alpha = \xi = 1/2$. We consider the following boundary value problem:

$$D^{3/2}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = \frac{1}{2} D^{1/4} u(t)\Big|_{t=1/2}, \tag{4.1}$$

where

$$f(t, u) = \frac{1}{4} t^{-1/2} \left( \frac{1}{18} u^2 + t^4 + 1 \right). \tag{4.2}$$

Let $b = 1/2$. By simple computation, we have

$$C_1 \approx 1.7724, \qquad C_2 \approx 3.9824, \qquad C_3 \approx 0.2635. \tag{4.3}$$

Choosing $R = 6$, $r = 1/16$, we have

$$t^{1/2} f(t, u) = \frac{1}{4} \left( \frac{1}{18} u^2 + t^4 + 1 \right) \le 1 < \frac{R}{C_1 + C_2} \approx 1.0426, \quad (t, u) \in [0, 1] \times [0, 6],$$
$$t^{1/2} f(t, u) = \frac{1}{4} \left( \frac{1}{18} u^2 + t^4 + 1 \right) \ge \frac{1}{4} > \frac{r}{C_3} \approx \frac{1}{4.216}, \quad (t, u) \in [0, 1] \times \left[0, \frac{1}{16}\right]. \tag{4.4}$$

By Theorem 3.3, problem (4.1) has at least one solution $u$ such that $1/16 < \|u\| < 6$.

## Acknowledgments

This work was supported by the Scientific Research Foundation of Hunan Provincial Education Department (05A057, 08C826) was also supported by the Aid Program for Science and Technology Innovative Research Team in Higher Educational Institutions of Hunan Province, and the Construct Program of the Key Discipline in Hunan Province.

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