Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 350762, 26 pages
doi:10.1155/2009/350762

Research Article

Smooth Approximations of Global in Time Solutions to Scalar Conservation Laws

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Received 16 August 2008; Accepted 15 January 2009

Recommended by Samuel Shen

We construct global smooth approximate solution to a scalar conservation law with arbitrary smooth monotonic initial data. Different kinds of singularities interactions which arise during the evolution of the initial data are described as well. In order to solve the problem, we use and further develop the weak asymptotic method, recently introduced technique for investigating nonlinear waves interactions.

1. Introduction

In the current paper we present an approach for constructing uniform and global in time approximate solutions to a Cauchy problem for one dimensional scalar conservation law with arbitrary smooth nonlinear convex flux. This approximating solution (see Definition 2.1) is continuous, piecewise smooth for $\varepsilon \geq 0$ (ε is a regularization parameter) and tends to an admissible weak solution [1] of the corresponding Cauchy problem. More precisely, we consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad f'' > 0, \tag{1.1}$$

$$u|_{t=0} = v(x) \in C(\mathbf{R}), \tag{1.2}$$

where we also assume that v is piecewise monotonic. It is well known that, since $f'' > 0$, in the interval where the initial data $v(x)$ decreases, the shock wave will arise sooner or later (see Figure 3). In order to be more precise in our considerations, we will assume that the initial data are decreasing everywhere. The same procedure can be applied if the initial data are continuous piecewise monotonic functions.

Also, for the initial data we will assume that they are such that for every fixed $t \in \mathbf{R}^+$ there exists at most finite number of points of the gradient catastrophe (for more precise explanation see beginning of the next section).

Note that a smooth approximating solution to the problem under consideration was firstly constructed by Il'in [2] (with initial data which are such that there exists exactly one point of the gradient catastrophe along entire time axis; see (3.4), (3.5), (3.6) and corresponding assumptions). The starting point of his construction is the viscosity regularization of the considered conservation law. Using this regularization, the author in [3] constructs a global approximating solution via a set of functional series which are defined in appropriate domains in $\mathbf{R}^+ \times \mathbf{R}$. Then, he shows that every two such series match in the domains where they are both defined. His method is known as the matching method.

We mention also two very famous methods for a construction of approximate solution to conservation laws based on the piecewise constant approximations—Glimm scheme [4] and the wave front tracking [5].

Here, we use different technique, so-called the weak asymptotic method [6–14]. In the framework of our approach, the process of shock wave formation is considered as interaction of weak discontinuities, that is, nonlinear waves whose derivatives are the Heaviside type functions.

We stress that the formulas which we will give here are much simpler than the ones obtained by using the matching method. More precisely, our approximate solution is found almost in the same way as the classical solution of a Cauchy problem, by using the method of characteristics.

However, unlike the standard characteristics, we use so-called "new characteristics" which do not intersect in the moment of bifurcation, but they remain on the distance $\mathcal{O}(\varepsilon)$, where ε is a parameter of approximation.

The difference between standard and new characteristics is plotted on Figure 2 (one of possible scenarios). As one can see from there, the new characteristics do not intersect each other except ones emanating from the intervals $(a_1, a_1+\sigma)$ and $(a_2-\sigma, a_2)$. More precisely, we will allow the intersection to happen only when characteristics bear the same information. In the case plotted on Figure 2, this means that initial data are constant on the intervals $(a_1, a_1 + \sigma)$ and $(a_2 - \sigma, a_2)$.

Therefore, we can go back along the trajectory given by a "new characteristics" and thus obtain value of the approximate solution at every point $(t, x) \in \mathbf{R}^+ \times \mathbf{R}^d$. Of course, the main problem is how to find analytic expression for the new characteristics (see [6, 7, 10–12]).

Still, unlike the situation that we had in [6, 7, 10–12] with simple initial data providing the shock wave to be the same once it is formed (more precisely, to be of constant strength; see Figure 1), here we have more complicated situation. Namely, in the case of initial data (1.2), the shock wave increases the strength after its formation. The increase will not start immediately since in the left neighborhood of a_2 and right neighborhood of a_1 the initial data are constant. Therefore, unlike the situation from [6, 7, 10–12], we also need to modify characteristics emanating out of the interval $(a_2 - \sigma, a_1 + \sigma)$, and this issue is far from being trivial.

Next important moment of the approach presented here is the following. In one-dimensional case, the solution u for quasilinear equation (1.1) is an impulse for the corresponding Hamilton-Jacobi equation (see Example 5.2) and the shock wave formation (or the gradient catastrophe) denotes appearance of singularity for the projection mapping of the

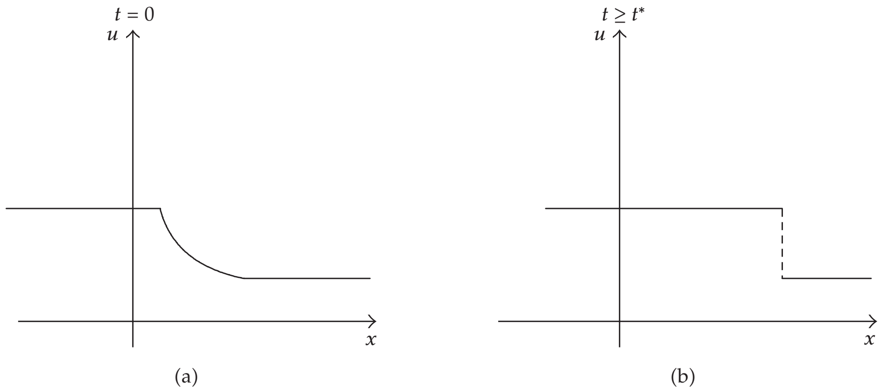

Figure 1: Evolution of the initial data considered in the previous works.

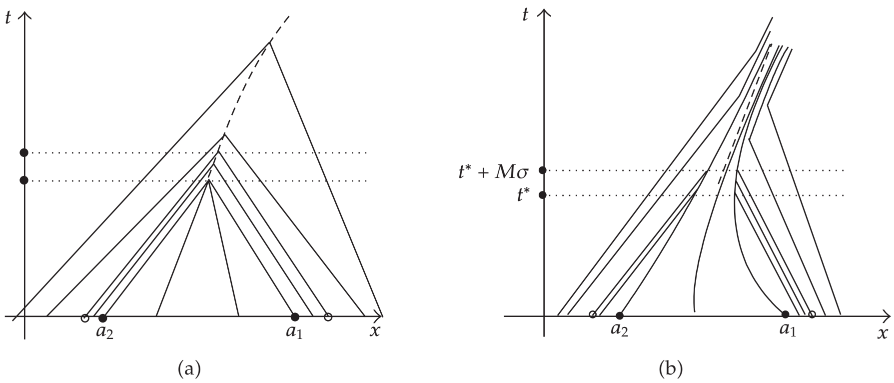

Figure 2: Standard characteristics are plotted on the left coordinate system. The new characteristics are plotted on the right coordinate system. The discontinuity line of an admissible weak solution is dashed on both plots. The points $a_2 - \sigma < a_1 + \sigma$ are hollowed on the x-axis.

corresponding Lagrange manifold on the x-axis. In the linear hyperbolic case this situation is described by using Maslov's canonical operator (integral Fourier operator) [15–18] and entire Lagrange manifold.

In the linear parabolic case (corresponding to the flux $f(u) = u^2$), it is also possible to apply the method of tunnel canonical Maslov's operator [15–17] and for the construction of the solution only "essential" part of the Lagrange manifold (not the entire one) is used (see [15–18]). This essential part exactly coincides with the shock wave profile in one-dimensional case. Approximative analytic description of that "essential" part of the Lagrange manifold is given in this paper.

In the sequel we will often use the notion of smooth description in $t \in \mathbf{R}^+$ of a process. Therefore, we give formal definition of the notion.

Definition 1.1. By smooth in $t \in \mathbf{R}^+$ description of a process one implies function which is smooth in $t \in \mathbf{R}^+$ and approximately (in the weak sense) solves an equation that governs the process.

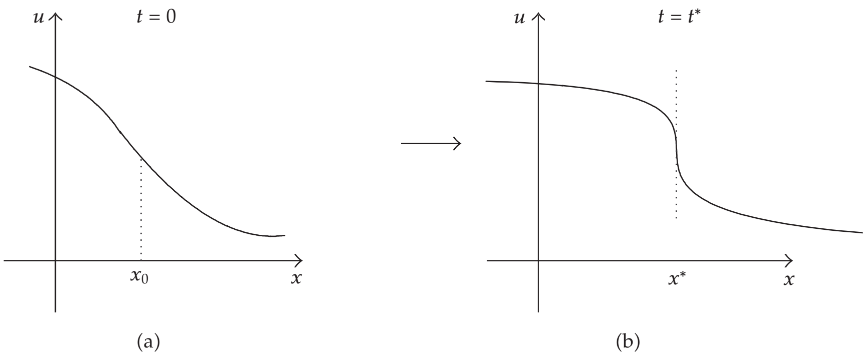

Figure 3: Evolution of a decreasing initial data.

The paper is organized as follows.

In Section 2, we give basic notions and notations of the weak asymptotic method and describe two problematic situations arising when solving the problem. In Section 3, we resolve the first problematic situation that arises in the construction of global approximating solution to the considered problem—we describe smoothly in $t \in \mathbf{R}^+$ the shock wave formation process from continuous initial data. If we would assume that we have only one shock wave formed from initial data like in [2] (i.e., if we would have situation similar to one plotted in Figure 2, this section would be enough). However, since we can have two shock waves formed on different places on x axis, we need to describe smoothly in $t \in \mathbf{R}^+$ interaction of the two shock waves. This is done in Section 4. In Section 5, we use results from previous two sections to describe global approximating solution to the considered problem.

2. Notions, Notations, and Further Explanations

We give basic definitions and fundamental theorem of the method that we are going to use—the weak asymptotic method.

Definition 2.1. By $O_{\mathfrak{D}'}(\varepsilon^\alpha) \subset \mathfrak{D}'(\mathbf{R})$, $\alpha \in \mathbf{R}$, one denotes the family of distributions depending on $\varepsilon \in (0,1)$ and $t \in \mathbf{R}^+$ such that for any test function $\eta(x) \in C_0^1(\mathbf{R})$, the estimate

$$\langle O_{\mathfrak{D}'}(\varepsilon^\alpha), \eta(x) \rangle = O(\varepsilon^\alpha), \quad \varepsilon \longrightarrow 0, \tag{2.1}$$

holds, where the estimate on the right-hand side is understood in the usual Landau sense and locally uniformly in t, that is, $|O(\varepsilon^\alpha)| \le C_T \varepsilon^\alpha$ for $t \in [0,T]$.

Definition 2.2. The family of functions $(u_\varepsilon)_\varepsilon = (u_\varepsilon(x,t))_\varepsilon$ is called a weak asymptotic solution of (1.1) and (1.2) if there exists $\alpha > 0$ such that in the space $C^1(\mathbf{R}^+; \mathfrak{D}'(\mathbf{R}))$, one has

$$u_{\varepsilon t} + \left(f(u_\varepsilon)\right)_x = \mathcal{O}_{\mathfrak{D}'}(\varepsilon^\alpha), \quad u_\varepsilon|_{t=0} - v = \mathcal{O}_{\mathfrak{D}'}(\varepsilon^\alpha), \quad \varepsilon \longrightarrow 0. \tag{2.2}$$

The following theorem is the basic theorem of the method. We also call it the nonlinear superposition law.

Theorem 2.3 (see [10]). *Suppose that the functions $\omega_i \in C^\infty(\mathbf{R})$, $i = 1, 2$, satisfy $\lim_{z\to+\infty}\omega_i(z) = 1$, $\lim_{z\to-\infty}\omega_i(z) = 0$ and $d\omega_i(z)/dz \in \mathcal{S}(\mathbf{R})$ where $\mathcal{S}(\mathbf{R})$ is the Schwartz space of rapidly decreasing functions. For the bounded functions a, b, c defined on $\mathbf{R}^+ \times \mathbf{R}$, one has*

$$\begin{aligned} f\left(a + b\omega_1\left(\frac{\varphi_1 - x}{\varepsilon}\right) + c\omega_2\left(\frac{\varphi_2 - x}{\varepsilon}\right)\right) \\ &= f(a) + \theta(\varphi_1 - x)\big(f(a+b+c)B_1 + f(a+b)B_2 - f(a+c)B_1 - f(a)B_2\big) \\ &\quad + \theta(\varphi_2 - x)\big(f(a+b+c)B_2 - f(a+b)B_2 + f(a+c)B_1 - f(a)B_1\big) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \end{aligned} \tag{2.3}$$

where θ is the Heaviside function and $B_i = B_i((\varphi_2 - \varphi_1)/\varepsilon)$, $i = 1, 2$, and for $\rho \in \mathbf{R}$ one has

$$\begin{gathered} B_1(\rho) = \int \dot{\omega}_1(z)\omega_2(z+\rho)dz, \qquad B_2(\rho) = \int \dot{\omega}_2(z)\omega_1(z-\rho)dz, \\ B_1(\rho) + B_2(\rho) = 1. \end{gathered} \tag{2.4}$$

At the beginning we make some remarks. Consider the point $x_0 \in \mathbf{R}$ such that

$$t^* = \min_{x\in\mathbf{R}} - \frac{1}{f''(u_0(x))u_0'(x)} = -\frac{1}{f''(u_0(x_0))u_0'(x_0)}. \tag{2.5}$$

Assume for the simplicity that such $x_0 \in \mathbf{R}$ is unique. In that case the shock wave will firstly arise on the characteristics along which the point x_0 moves. More precisely, blow up of the classical solution happens on the height $u_0(x_0)$ in the moment t^* and in the point x^* (i.e., initial data changes as plotted on Figure 3). The point (x^*, t^*) we call the point of the gradient catastrophe.

In order to describe phenomenon of the gradient catastrophe smoothly in $t \in \mathbf{R}^+$, on the first step we define the function $u_1(x)$ such that

$$f'(u_1(x)) = -Kx + b, \qquad x \in \mathbf{R}, \tag{2.6}$$

where K and b are constants determined from the conditions

$$u_1(x_0 - \varepsilon^\mu) = v(x_0 - 2\varepsilon^\mu), \quad u_1(x_0 + \varepsilon^\mu) = v(x_0 + 2\varepsilon^\mu), \quad 0 < \mu < 1. \tag{2.7}$$

After that, we replace the piece of original initial data $v(x)$ in a small neighborhood $(x_0 - 2\varepsilon^\mu, x_0 + 2\varepsilon^\mu)$ of the point x_0 by the function $u_1(x)$, $x \in (x_0 - \varepsilon^\mu, x_0 + \varepsilon^\mu)$ and by constants in the intervals $(x_0 - 2\varepsilon^\mu, x_0 - \varepsilon^\mu)$ and $(x_0 + \varepsilon^\mu, x_0 + 2\varepsilon^\mu)$ so that the continuity is preserved (see Figure 4). This change of initial condition provides the shock wave to arise from the interval $(x_0 - \varepsilon^\mu, x_0 + \varepsilon^\mu)$ in the moment $t^* = 1/K$. The amplitude of the shock wave is going to be $|v(x_0 - 2\varepsilon) - v(x_0 + 2\varepsilon)|$. On the other hand, for $t < t^*$, the solution u_ε to our problem (with transformed initial data) will be continuous function (see Figure 4 again).

As we will see, with such new initial data it is much easier to find global smooth approximating solution. Namely, it appears that the most difficult issue here is to describe evolution of the initial data in the moment of blowing up of the classical solution, and that

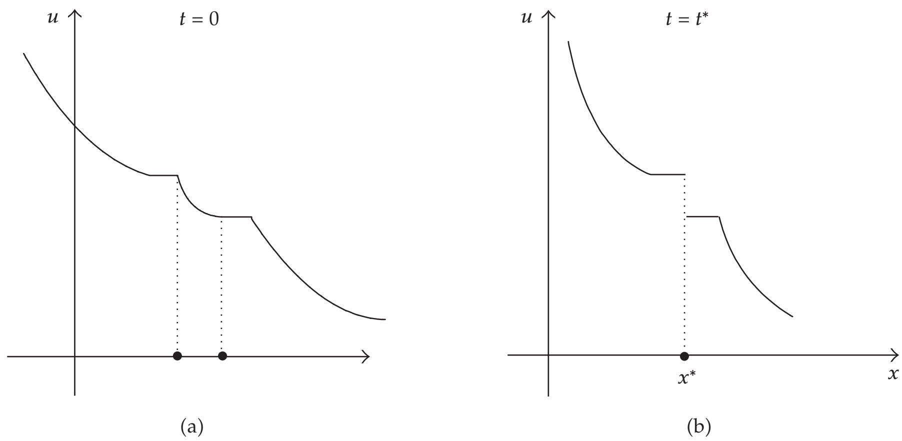

Figure 4: Evolution of the approximated initial data. The points $x_0 - \varepsilon^\mu$ and $x_0 + \varepsilon^\mu$ are dotted on the first plot.

it is much easier to do it if we have "line" of gradient catastrophe (in this case it is the line $(x_0 - \varepsilon^\mu, x_0 + \varepsilon^\mu)$; see Figure 4) than the point of gradient catastrophe (in the case plotted in Figure 3 it is the point x_0). We address the reader to [2, 3] in order to understand difficulties which are caused if we have only a point of gradient catastrophe.

The second problematic situation which we meet is shock waves interaction. This is simpler case and we can describe this process smoothly in $t \in \mathbf{R}^+$ by direct substitution of an ansatz into the equation, and then applying the weak asymptotic formulas (see Section 4).

3. Formation of the Shock Wave with Nonconstant Amplitude

We return to (1.1) and (1.2). Before we begin, we introduce the notations that we will use. First of all, we imply everywhere $t \in \mathbb{R}^+$, $x \in \mathbb{R}$. Next,

$$
\begin{gathered}
u_1 = u_1(x,t,\varepsilon), \qquad U = U(x,t), \qquad u_0 = u_0(x,t), \\
B_i = B_i(\rho), \quad \phi_i = \phi_i(t,\varepsilon), \quad i = 1,2, \\
x_{0,R} = x_{0,R}(t,\varepsilon), \qquad x_{0,L} = x_{0,L}(t,\varepsilon), \\
\theta_1 = \theta(\phi_1 + x_{0,R} - x), \qquad \theta_2 = \theta(\phi_2 - x_{0,L} - x), \\
\delta_i = -(\theta_i)'_x, \quad i = 1,2, \\
t^* = \frac{a_1 - a_2}{f'(U^0) - f'(u_0^0)} = \frac{1}{K}, \\
x^* = f'(U^0)t^* + a_2 = f'(u_0^0)t^* + a_1 = \frac{f'(U^0)a_1 - f'(u_0^0)a_2}{f'(U^0) - f'(u_0^0)},
\end{gathered}
\tag{3.1}
$$

where θ is the Heaviside function and B_i, $i = 1,2$, are from Theorem 2.3. The remaining functions will be defined in what follows.

On the first step, we assume that the function $v(x)$ from (1.2) has the form (see Figure 4 with $\sigma = \varepsilon^{\mu}$)

$$v(x) = \begin{cases} U_0(x), & x < a_2 - \sigma, \\ U_0(a_2 - \sigma), & a_2 - \sigma < x < a_2, \\ u_{10}(x), & a_2 < x < a_1, \\ u_{00}(a_1 + \sigma), & a_1 < x < a_1 + \sigma, \\ u_{00}(x), & x > a_1 + \sigma, \end{cases} \tag{3.2}$$

where σ is a positive constant. Furthermore, the functions u_{00}, u_{10}, and U_0 are nonincreasing and smooth, and they satisfy

$$\begin{gathered} u_{00}(x) = u_{00}(a_1 + \sigma), \quad x > a_1 + \sigma, \\ U_0(x) = U_0(a_2 - \sigma), \quad x < a_2 - \sigma, \\ f'(u_{10}(x)) = -Kx + b, \quad x \in [a_2, a_1], \\ u_{10}(a_2) = u_{00}(a_2 + \sigma) = u_0^0, \quad u_{10}(a_1) = U_0(a_1 - \sigma) = U^0, \quad a_1 > a_2, \\ -\frac{1}{f''(u_{00}(x))u_{00}'(x_0)} \geq C > \frac{1}{K}, \quad x < a_2, \\ -\frac{1}{f''(U_0(x))U_0'(x_0)} \geq C > \frac{1}{K}, \quad x > a_1. \end{gathered} \tag{3.3}$$

This assumption is here in order to obtain the situation such that on characteristics emanating from the intervals $(-\infty, a_2)$ and (a_1, ∞) we can have gradient catastrophe only for $t \geq C > 1/K$ (i.e., after the straightening the curve connecting the points $(a_1, v(a_1))$ and $(a_2, v(a_2))$).

In order to clarify as much as possible the presentation, in this section we will assume more then this. Namely, it is well known that for $t > t^* = 1/K$ the solution of (1.1) and (3.2) will admit the shock wave moving along the line $x = c(t)$ given by the Cauchy problem defining the Rankine-Hugoniot conditions

$$\begin{gathered} c'(t) = \frac{f(U(c(t),t)) - f(u_0(c(t),t))}{U(c(t),t) - u_0(c(t),t)}, \quad t \geq t^*, \\ c(t^*) = x^* = f'(U^0)t^* + a_2 = f'(u_0^0)t^* + a_1, \quad t \leq t^*. \end{gathered} \tag{3.4}$$

Here, U and u_0 are classical solutions to the following Cauchy problems, respectively (in a subdomain of $\mathbf{R}^+ \times \mathbf{R}$ where they exist)

$$\partial_t u_0 + \partial_x f(u_0) = 0, \qquad u|_{t=0} = u_{00}(x), \tag{3.5}$$

$$\partial_t U + \partial_x f(U) = 0, \qquad u|_{t=0} = U_0(x). \tag{3.6}$$

So, in the sequel, we will assume that the functions u_0, U, and c satisfying (3.4), (3.5), and (3.6) are well defined (the same is done in [2]).

Before we globally define the "new characteristics" (denoted $x(x_0, t, \varepsilon)$), we need to define extremal "new characteristics," that is, the ones emanating from $a_2 + \mathcal{O}(\varepsilon)$ and $a_1 + \mathcal{O}(\varepsilon)$. The proof is relatively simple and it relies on the basic ODE theory.

Lemma 3.1. *The curves $\phi_2 = x(x_{02}, t, \varepsilon)$ and $\phi_1 = x(x_{01}, t, \varepsilon)$ for $x_{02} = a_2 - \varepsilon((a_1 - a_2)/2)$ and $x_{01} = a_1 + \varepsilon((a_1 - a_2)/2)$ are given by the following Cauchy problems:*

$$\begin{aligned}
\frac{d\phi_1}{dt} &= (B_2(\rho) - B_1(\rho)) f'(u_0(\phi_1, t)) + c(t) B_1(\rho), \qquad \phi_1(0, \varepsilon) = a_1 + \varepsilon A \frac{a_1 - a_2}{2}, \\
\frac{d\phi_2}{dt} &= (B_2(\rho) - B_1(\rho)) f'(U(\phi_2, t)) + c(t) B_1(\rho), \qquad \phi_2(0, \varepsilon) = a_2 - \varepsilon A \frac{a_1 - a_2}{2}.
\end{aligned} \tag{3.7}$$

Introduce the function

$$\tau = \frac{f'(U^0)t + a_2 - f'(u_0^0)t - a_1}{\varepsilon} = \frac{\psi_0(t)}{\varepsilon}, \tag{3.8}$$

describing the relation between standard characteristics emanating from a_2 and a_1, respectively.

The function ρ given by

$$\rho(t, \tau) = \frac{\phi_2(t, \varepsilon) - \phi_1(t, \varepsilon)}{\varepsilon}, \tag{3.9}$$

describing the distance between two nonintersecting curves ϕ_2 and ϕ_1, satisfies the following Cauchy problem

$$\dot{\rho} = 1 - 2B_1(\rho), \qquad \lim_{\tau \to -\infty} \frac{\rho}{\tau} = 1, \tag{3.10}$$

where $\dot{\rho} = \partial_\tau \rho$ and it tends to a constant ρ_0 as $\tau \to \infty$.

Proof. The proof follows from the definition of the curves $x(x_{01}, t)$ and $x(x_{02}, t)$, and standard ODE theory (see Figure 5 and put $F(\rho, \cdot) = 1 - 2B_1(\rho)$). For details one can consult [11]. □

After inessential changes caused by replacing the constant c by the function $c(t)$, the proof of the following lemma can be found in the frame of [11, Theorem 5].

Lemma 3.2. *Consider the set of solutions $x = x(x_0, t)$, $x_0 \in [a_2, a_1]$, to the following Cauchy problem*

$$\begin{gathered}
\dot{x} = (B_2(\rho) - B_1(\rho)) f'(u_1) + c(t) B_1(\rho), \qquad \dot{u}_1 = 0, \\
u_1(0) = u_{10}(x_0), \quad x(0) = x_0 + \varepsilon A\left(x_0 - \frac{a_1 + a_2}{2}\right), \quad x_0 \in [a_2, a_1],
\end{gathered} \tag{3.11}$$

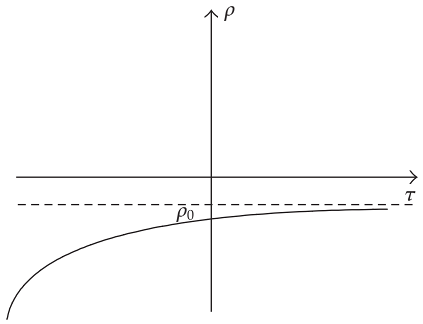

Figure 5: The curve represents solution of (4.17). Dot on the ρ axis, denoted by ρ_0, is the minimal zero of the function $F(\rho, \cdot)$.

where A is a constant large enough (which provides global solvability of the implicit equation $x = x(x_0, t, \varepsilon)$, $t > 0$, $\varepsilon > 0$; see [11, Theorem 5]). The curves $x(x_0, t, \varepsilon)$, $x_0 \in [a_2, a_1]$ one calls the "new characteristics" emanating from the interval $(a_2 - \varepsilon A((a_1 - a_2)/2), a_1 + \varepsilon A((a_1 - a_2)/2))$.

Then, for arbitrary two $x_{01}, x_{02} \in [a_2, a_1]$, the curves $x(x_{01}, t, \varepsilon)$ and $x(x_{02}, t, \varepsilon)$ are nonintersecting.

Note that we extended a little bit the interval $[a_2, a_1]$. This is necessary in order to prove that the "new characteristics" do not mutually intersect (see [11]). Also note that this does not affect the weak asymptotic solution of the problem since we perturbed initial function for $\mathcal{O}(\varepsilon)$.

The previous lemma gives a formula for determining the "new characteristics" emanating from the interval $[a_2, a_1]$, and states that they do not intersect each other along entire time axis. Remark that the "new characteristics" emanate from the interval $(a_2 - \varepsilon A((a_1 - a_2)/2), a_1 + \varepsilon A((a_1 - a_2)/2))$ and that we state that only them are non-intersecting. The characteristics emanating out of the interval $(a_2 - \varepsilon A((a_1 - a_2)/2), a_1 + \varepsilon A((a_1 - a_2)/2))$ are standard and they can intersect each other. Still, it will happen "late" enough (see (3.4)–(3.6)).

In order to define the new characteristics along entire x-axis, we introduce the following notations

$$\begin{aligned} \Psi_L(x_0, t, \varepsilon) &= f'(U_0(x_0))t + x_0 + \varepsilon\left(x_0 - \frac{a_1 - a_2}{2}\right), \quad x_0 < a_2 - \sigma, \\ \Psi_R(x_0, t, \varepsilon) &= f'(u_{00}(x_0))t + x_0 + \varepsilon\left(x_0 - \frac{a_1 - a_2}{2}\right), \quad x_0 > a_1 + \sigma. \end{aligned} \tag{3.12}$$

Note that Ψ_L and Ψ_R are standard characteristics emanating from the intervals $(a_2 - \sigma, a_2)$ and $(a_1, a_1 + \sigma)$, respectively. Now, we can define the functions representing the "new characteristics" emanating out of $(a_2 - \sigma, a_1 + \sigma)$.

For $x_0 < a_2 - \sigma$ we put

$$x_L(t, x_0, \varepsilon) = \begin{cases} \Psi_L(x_0, t, \varepsilon), & \phi_2(t, x_0, \varepsilon) - \Psi_L(t, x_0, \varepsilon) > \varepsilon(a_2 - \sigma - x_0), \\ \phi_2(t, x_0, \varepsilon) - \varepsilon(a_2 - x_0), & \phi_2(t, x_0, \varepsilon) - \Psi_L(t, x_0, \varepsilon) \le \varepsilon(a_2 - \sigma - x_0). \end{cases} \tag{3.13}$$

Similarly, for $x_0 > a_1 + \sigma$,

$$x_R(t, x_0, \varepsilon) = \begin{cases} \Psi_R(x_0, t, \varepsilon), & \Psi_R(t, x_0, \varepsilon) - \phi_1(t, x_0, \varepsilon) > \varepsilon(x_0 - a_1 - \sigma), \\ \phi_1(t, x_0, \varepsilon) - \varepsilon(a_2 - x_0), & \Psi_R(t, x_0, \varepsilon) - \phi_1(t, x_0, \varepsilon) \le \varepsilon(x_0 - a_1 - \sigma). \end{cases} \tag{3.14}$$

For better understanding of the previous formulas see Figure 2. Actually, x_L as well as x_R are equal to the standard characteristics before they come "close" to the shock curve. After that, they are parallel to the shock curve.

Finally, we can write formula for the "new characteristics" which holds for $x_0 \in \mathbf{R} \setminus ((a_2 - \sigma, a_2) \cup (a_1, a_1 + \sigma))$ (and this is set for which we need the "new characteristics"):

$$X(t, x_0, \varepsilon) = \begin{cases} x_R(t, x_0, \varepsilon), & x_0 \le a_2 - \sigma, \\ x(t, x_0, \varepsilon), & a_2 \le x_0 \le a_1, \\ x_L(t, x_0, \varepsilon), & x_0 > a_1, \end{cases} \tag{3.15}$$

where $x(x_0, t, \varepsilon)$ is defined in Lemma 3.2. Denote by $\tilde{x}_0 = \tilde{x}_0(x, t, \varepsilon)$ the inverse function to the function $x = X(x_0, t, \varepsilon)$, $t > 0$, $\varepsilon > 0$, of the "new characteristics" defined by (3.15). Clearly, the function $\tilde{x}_0$ is not defined in the regions

$$\begin{aligned} \Gamma_L &= \{(x, t) : 0 < t < t^* + M\sigma,\ f'(U_0^0)t + a_2 - \sigma < x < \phi_2\}, \\ \Gamma_R &= \{(x, t) : 0 < t < t^* + M\sigma,\ f'(u_{00}^0)t + a_1 + \sigma > x > \phi_1\}. \end{aligned} \tag{3.16}$$

Therefore, we introduce the following extension of $\tilde{x}_0$:

$$x_0(x, t, \varepsilon) = \begin{cases} \tilde{x}_0(x, t, \varepsilon), & (x, t) \notin \Gamma_L \cup \Gamma_R, \\ f'(U_0^0)t + a_2 - \sigma, & (x, t) \in \Gamma_L, \\ f'(u_{00}^0)t + a_1 + \sigma, & (x, t) \in \Gamma_R. \end{cases} \tag{3.17}$$

To continue, assume that the increase of strength of the shock wave starts in the moment $t^* + M\sigma$ for a constant $M > 0$ (actually, this is the moment when standard characteristics emanating out of the interval $[a_2 - \sigma, a_1 + \sigma]$ start to intersect with the shock curve). Then, we introduce the functions $x_{0,L}(t, \varepsilon)$ and $x_{0,R}(t, \varepsilon)$ so that they are equal to zero for $t < t^* + M\sigma$ and defined for $t \ge t^* + M\sigma$ as follows:

(i) $x_{0,L}(t, \varepsilon) = \varepsilon(a_2 - x_0)$ for $x_0 < a_2 - \sigma$ such that $\phi_2(t, x_0, \varepsilon) - \Psi_L(t, x_0, \varepsilon) = \varepsilon(a_2 - x_0)$,

(ii) $x_{0,R}(t, \varepsilon) = \varepsilon(x_0 - a_1)$ for $x_0 > a_1 + \sigma$ such that $\Psi_R(t, x_0, \varepsilon) - \phi_1(t, x_0, \varepsilon) = \varepsilon(x_0 - a_1)$.

Note that the functions $x_{0,L}$ and $x_{0,R}$ have the same regularity as the functions U_0 and u_{00}, respectively. Therefore,

$$\frac{\partial x_{0,R}}{\partial t} = \frac{\partial x_{0,L}}{\partial t} = \mathcal{O}(\varepsilon). \tag{3.18}$$

The following theorem is a slight modification of Theorem 5 from [11]. The motivation for the procedure can be found in [11].

Theorem 3.3. *The weak asymptotic solution of problem* (1.1) *and* (1.2) *has the form*

$$\begin{aligned} u_\varepsilon(x,t) &= u_0(x,t) + \left(u_1(x,t,\varepsilon) - u_0(x,t)\right)\omega_1\left(\frac{\phi_1(t,\varepsilon) + x_{0,R}(t,\varepsilon) - x}{\varepsilon}\right) \\ &\quad + \left(U(x,t) - u_1(x,t,\varepsilon)\right)\omega_2\left(\frac{\phi_2(t,\varepsilon) - x_{0,L}(t,\varepsilon) - x}{\varepsilon}\right), \end{aligned} \tag{3.19}$$

where ω_i, $i = 1,2$, satisfy the conditions from Theorem 2.3 and the functions $\phi_i = \phi_i(t,\varepsilon)$, $t \in \mathbf{R}^+$, $i = 1,2$, are given in Lemma 3.1.

The function $u_1(x,t,\varepsilon)$ is given by

$$u_1(x,t,\varepsilon) = \begin{cases} U(x,t), & x < \phi_2 - x_{0,L}(t,\varepsilon), \\ u_0(x_0(x,t,\varepsilon)), & \phi_2 - x_{0,L}(t,\varepsilon) \le x \le \phi_1 + x_{0,R}(t,\varepsilon), \\ u_0(x,t), & x > \phi_1 + x_{0,R}(t,\varepsilon), \end{cases} \tag{3.20}$$

where $x_0(x,t,\varepsilon)$ is defined by (3.17).

The functions U and u_0 are classical solutions to Cauchy problems (3.5) *and* (3.6), *respectively.*

Proof. The proof completely follows the construction from [11]:

(a) we substitute (3.19) in (1.1);

(b) we use formula (2.3);

(c) we divide the real line on three intervals $(-\infty, \phi_2 - x_{0,L}]$, $(\phi_2 - x_{0,L}, \phi_1 + x_{0,R})$, and $[\phi_1 + x_{0,R}, +\infty)$. In that way we get three equations in each of the intervals, and we solve them separately in each of them.

We substitute the function $u_\varepsilon(x,t)$ from (3.19) in (1.1). Using Theorem 2.3 we get

$$\begin{aligned} &\left[\frac{\partial u_0}{\partial t} + \frac{\partial f(u_0)}{\partial x}\right](1-\theta_1) + \left(\frac{\partial u_1}{\partial t} + \left[(B_2 - B_1)f'(u_1) + 2c'(t)B_1\right]\frac{\partial u_1}{\partial x}\right)(\theta_1 - \theta_2) \\ &\quad + B_1\left[\frac{d}{dx}\left(f\left(U + u_0^0 - u_1\right) + f(u_1) - 2c'(t)u_1\right)\right](\theta_1 - \theta_2) + \left[\frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x}\right]\theta_2 \\ &\quad + \left((u_1 - u_0)\phi_{1t} - B_2\left(f(u_1) - f(u_0)\right) - B_1\left(f(U) - f(U + u_0 - u_1)\right)\right)\delta_1 \\ &\quad + \left((U - u_1)\varphi_{2t} - B_2\left(f(U) - f(u_1)\right) - B_1\left(f(U + u_0 - u_1) - f(u_0)\right)\right)\delta_2 = \mathcal{O}_{\mathcal{D}'}(\varepsilon). \end{aligned} \tag{3.21}$$

For more detailed computation see [11].

Considering the previous expression for $x \in (-\infty, \phi_2 - x_{0,L})$ and $x \in (\phi_1 + x_{0,R}, +\infty)$ we get the following equations:

$$\begin{aligned} \frac{\partial U}{\partial t} + f'(U)\frac{\partial U}{\partial x} &= \mathcal{O}_{\mathcal{D}'}(\varepsilon), \quad x \in (-\infty, \phi_2 - x_{0,L}), \\ \frac{\partial u_0}{\partial t} + f'(u_0)\frac{\partial u_0}{\partial x} &= \mathcal{O}_{\mathcal{D}'}(\varepsilon), \quad x \in (\phi_1 + x_{0,R}, \infty), \end{aligned} \tag{3.22}$$

which is true by definition of the functions U and u_0 (see (3.5) and (3.6)). Thus, (3.21) reduces to

$$\begin{aligned} &\left(\frac{\partial u_1}{\partial t} + \left[(B_2 - B_1)f'(u_1) + 2c'(t)B_1\right]\frac{\partial u_1}{\partial x}\right)(\theta_1 - \theta_2) \\ &\quad + B_1\left[\frac{d}{dx}\left(f(U + u_0 - u_1) + f(u_1) - 2c'(t)u_1\right)\right](\theta_1 - \theta_2) \\ &\quad + \left((u_1 - u_0)\partial_t(\phi_1 + x_{0,L}) - B_2(f(u_1) - f(u_0)) - B_1(f(U) - f(U + u_0 - u_1))\right)\delta_1 \\ &\quad + \left((U - u_1)\partial_t(\phi_2 - x_{0,R}) - B_2(f(U) - f(u_1)) - B_1(f(U + u_0 - u_1) - f(u_0))\right)\delta_2 = \mathcal{O}_{\mathcal{D}'}(\varepsilon). \end{aligned} \tag{3.23}$$

We pass to the interval $(\phi_2 - x_{0,L}, \phi_1 + x_{0,R})$.

Notice that for $t < t^* + \sigma M$ we have $x_{0,R}(t,\varepsilon) = x_{0,L}(t,\varepsilon) = 0$. Therefore, the situation is the same as in [11] (see proof of Theorem 5 from [11]).

Consider the interval $(t^* + \sigma M/2, T)$ for a $T > t^* + \sigma M/2$. In that interval we have (see again [10, 11])

$$B_2 - B_1 = \mathcal{O}(\varepsilon^N), \quad N \in \mathbf{N}, \tag{3.24}$$

and from here, since $B_2 + B_1 = 1$, we also have

$$B_i = \frac{1}{2} + \mathcal{O}(\varepsilon^N), \quad i = 1,2, \ N \in \mathbf{N}. \tag{3.25}$$

Having this in mind, from (3.23) we get

$$\begin{aligned} &\left(\frac{\partial u_1}{\partial t} + c'(t)\frac{\partial u_1}{\partial x}\right)(\theta_1 - \theta_2) \\ &\quad + \frac{1}{2}\left[\frac{d}{dx}\left(f(U + u_0 - u_1) + f(u_1) - 2c'(t)u_1\right)\right](\theta_1 - \theta_2) \\ &\quad + \left((u_1 - u_0)\partial_t(\phi_1 + x_{0,R}) - \frac{1}{2}(f(u_1) - f(u_0)) - \frac{1}{2}(f(U) - f(U + u_0 - u_L))\right)\delta_1 \\ &\quad + \left((U - u_1)\partial_t(\phi_2 - x_{0,L}) - \frac{1}{2}(f(U) - f(u_1)) - \frac{1}{2}(f(U + u_0 - u_1) - f(u_0))\right)\delta_2 = \mathcal{O}_{\mathcal{D}'}(\varepsilon). \end{aligned} \tag{3.26}$$

Multiplying the last relation by a test function $\eta \in C_0^1(\mathbf{R})$ and integrating over $\mathbf{R}$ we get (recall that t is fixed)

$$\begin{aligned}
&\big(u_1(\phi_1 + x_{0,R}, t, \varepsilon)(\partial_t\phi_1 + \partial_t x_{0,L})\eta(\phi_1 + x_{0,R}, t) - u_1(\phi_2 - x_{0,L}, t, \varepsilon)(\partial_t\phi_2 - \partial_t x_{0,R})\eta(\phi_2 - x_{0,L}, t)\\
&\quad - c'(t)u_1(\phi_1 + x_{0,R}, t, \varepsilon)\eta(\phi_1 + x_{0,R}, t) + c'(t)u_1(\phi_2 - x_{0,L}, t, \varepsilon)\eta(\phi_2 - x_{0,L}, t)\big)\\
&\quad + \int_{\phi_2 - x_{0,L}}^{\phi_1 + x_{0,R}} \frac{1}{2}\left[\frac{d}{dx}(f(U + u_0 - u_1) + f(u_1) - 2c'(t)u_1)\right]\eta(x)dx\\
&\quad + \int_{\mathbf{R}} ((u_1 - u_0)\partial_t(\phi_1 + x_{0,R}) - \frac{1}{2}(f(u_1) - f(u_0)) + \frac{1}{2}(f(u_1) + f(u_0) - 2c'(t)u_1))\delta_1 \cdot \eta(x)dx\\
&\quad + \int_{\mathbf{R}} ((U - u_1)\partial_t(\phi_2 - x_{0,L}) - \frac{1}{2}(f(U) - f(u_1)) + \frac{1}{2}(f(u_1) + f(u) - 2c'(t)u_1))\delta_2 \cdot \eta(x)dx = \mathcal{O}(\varepsilon).
\end{aligned} \tag{3.27}$$

Since for $t > t^* + M\sigma/2$ we have

$$\begin{gathered}
u_1(\phi_1 + x_{0,R}, t, \varepsilon) = u_0(x_0(\phi_1 + x_{0,R}, t, \varepsilon)),\\
u_1(\phi_2 - x_{0,L}, t, \varepsilon) = U(x_0(\phi_2 - x_{0,L}, t, \varepsilon)),\\
\phi_1 + x_{0,R} - \phi_2 + x_{0,L} = \mathcal{O}(\varepsilon),\\
\phi_1 + x_{0,R} - c(t) = \phi_2 + x_{0,L} - c(t) = \mathcal{O}(\varepsilon),\\
\partial_t(\phi_1 + x_{0,L}) - c'(t) = \partial_t(\phi_2 - x_{0,R}) - c'(t) = \mathcal{O}(\varepsilon),\\
\partial_t x_{0,L} = \partial_t x_{0,R} + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon),
\end{gathered} \tag{3.28}$$

and since the functions U and u_0 are continuous, from the definition of δ distribution we see that (3.27) is fulfilled.

Details of the construction for $t < t^* + M\sigma$ can be found in [11].

Since u_ε given by (3.2) is the weak asymptotic solution in the intervals $I_1 = [0, t^* + \sigma M)$ and $I_2 = (t^* + \sigma M/2, T)$ and common part of the intervals is large enough (more precisely, it is enough to be $|I_1 \cap I_2| > \mathcal{O}(\varepsilon^\alpha)$ for an $\alpha < 1$), we see that u_ε is the weak asymptotic solution to (1.1) and (3.2) along entire time axis. □

4. Interaction of Shock Waves with a Nonconstant Amplitude

In this section we construct the weak asymptotic solution to equation (1.1) with the following initial condition:

$$u(x,0) = u_0(x) = u_{10}(x) + (u_{20}(x) - u_{10}(x))\theta(a_1 - x) + (u_{30}(x) - u_{20}(x))\theta(a_2 - x), \tag{4.1}$$

where $u_i(x)$, $x \in \mathbf{R}$, are continuous decreasing functions. Furthermore, we assume that u_{0i}, $i = 1, 2, 3$, satisfy

$$u_1(a_1) < u_2(a_1), \qquad u_2(a_2) < u_3(a_2). \tag{4.2}$$

Clearly, the function $u_0(x)$ has two admissible jumps at the points a_1 and a_2. Those jumps start to move according to the Rankine-Hugoniot conditions until they merge at the moment t^*. Furthermore, since we want to investigate interaction of shock waves appearing in the initial data, we assume that

$$-\frac{1}{f''(u_i(x))u_i'(x)} \geq T > t^*, \quad i = 1,2,3, \; x \in \mathbf{R}. \tag{4.3}$$

Such conditions provide that the gradient catastrophe will not happen before the two shock waves collide.

By $u_i(x,t)$, $i = 1,2,3$, $(x,t) \in \mathbf{R} \times \mathbf{R}^+$, respectively, we denote classical solutions of the following Cauchy problems (in the regions of $\mathbf{R}^+ \times \mathbf{R}$ where they exist)

$$\begin{aligned} u_t + (f(u))_x &= 0, \\ u|_{t=0} &= u_{i0}(x), \quad i = 1,2,3, \end{aligned} \tag{4.4}$$

where the flux f is the same as the one from (1.1). Since the initial functions $u_{i0}(x)$, $i = 1,2,3$, are decreasing, the solutions to (4.4) will be also decreasing.

By $\phi_{i0}(t)$, $i = 1,2$, $t \in [0,t^*)$, and $\varphi(t)$, $t \in [t^*,T]$, we denote the solutions to the following Cauchy problems

$$\begin{aligned} \phi_{i0t}(t) &= \frac{f(u_{i+1}(\phi_{i0}(t),t)) - f(u_i(\phi_{i0}(t),t)}{u_{i+1}(\phi_{i0}(t),t) - u_i(\phi_{i0}(t),t)}, \quad \phi_{i0}(0) = a_i, \\ \varphi_t(t) &= \frac{f(u_3(\varphi(t))) - f(u_1(\varphi(t)))}{u_3(\varphi(t)) - u_1(\varphi(t))}, \qquad \varphi(t^*) = \phi_{10}(t^*) = \phi_{20}(t^*). \end{aligned} \tag{4.5}$$

Note that continuous solutions to those Cauchy problems always exist since those are actually Rankine-Hugoniot conditions for the admissible shock placed at ϕ_{i0}, $i = 1,2$, and corresponding to Cauchy problem (1.1) with initial conditions, respectively (below we imply $u_{i,0} = u_{i0}$)

$$\begin{aligned} u|_{t=0} &= u_{i0}(x) + (u_{i+1,0}(x) - u_{i0}(x))\theta(a_i - x), \quad i = 1,2, \\ u|_{t=t^*} &= u_1(x,t^*) + (u_3(x,t^*) - u_1(x,t^*))\theta(\varphi(t^*) - x). \end{aligned} \tag{4.6}$$

Furthermore, since the function $u_0(x)$ decreases and f is convex it is clear that $(\phi_{20t}(t) - \phi_{10t}(t))' > 0$.

We will prove the following theorem.

Theorem 4.1. *The weak asymptotic solution to Cauchy problem* (1.1) *and* (4.1) *one has in the form*

$$u_\varepsilon(x,t) = u_1(x,t) + (u_2(x,t) - u_1(x,t))\omega_1\left(\frac{\phi_1(\tau,t) - x}{\varepsilon}\right) + (u_3(x,t) - u_2(x,t))\omega_2\left(\frac{\phi_2(\tau,t) - x}{\varepsilon}\right), \tag{4.7}$$

where $x \in \mathbf{R}$, $t \in [0,t^* + \sigma) \subset [0,T)$, and the functions ω_i, $i = 1,2$, are the ones from Theorem 2.3.

Here,

$$\tau = \frac{\phi_{20}(t) - \phi_{10}(t)}{\varepsilon} = \frac{\psi_0(t)}{\varepsilon}. \tag{4.8}$$

The function $\rho = \rho(\tau, t)$ *is given by the relation*

$$\rho(\tau, t) = \frac{\phi_2(\tau, t) - \phi_1(\tau, t)}{\varepsilon}, \tag{4.9}$$

and the functions $\phi_i(t,\tau)$, $i = 1, 2$, *are given by the formulas*

$$\phi_1(t,\tau) = \phi_{10}(t) + \psi_0(t)\Omega_1(t,\tau)\phi_{11}^+(t), \tag{4.10}$$

$$\phi_2(t,\tau) = \phi_{20}(t) + \psi_0(t)\Omega_2(t,\tau)\phi_{21}^+(t), \tag{4.11}$$

where

$$\phi_{i1}^+(t) = \frac{\varphi(t) - \phi_{i0}(t)}{\psi_0(t)}, \quad i = 1, 2, \tag{4.12}$$

and Ω_i, $i = 1, 2$, *are given by* (4.26).

Proof. In the sequel we use the following notation.

By θ and δ the Heaviside and Dirac distributions, respectively, and

$$\begin{gathered}\delta_i = \delta(x - \phi_i(t)), \quad \theta_i = \theta(x - \phi_i(t)), \quad i = 1, 2, \\ u_j(\phi_i) = u_j(\phi_i(t), t), \quad i = 1, 2, \ j = 1, 2, 3.\end{gathered} \tag{4.13}$$

We start by substituting (4.7) in (1.1) and applying Theorem 2.3. We obtain

$$\begin{aligned}u_{1t} &+ (u_{2t} - u_{1t})\theta_1 + (u_{3t} - u_{2t})\theta_2 + \phi_{1t}(u_2(\phi_1) - u_1(\phi_1))\delta_1 \\ &+ \phi_{2t}(u_3(\phi_2) - u_2(\phi_2))\delta_2 + u_{1x}f'(u_1) \\ &+ \theta_1\left[(u_{3x}f'(u_{3x}) - f'(u_1+u_3-u_2)(u_{1x}+u_{3x}-u_{2x}))B_1 + (f'(u_2)u_{2x} - f'(u_1)u_{1x})B_2\right] \\ &+ \theta_2\left[(u_{3x}f'(u_3) - u_{2x}f'(u_2))B_2 + (f'(u_1+u_3-u_2)(u_{1x}+u_{3x}-u_2) - f'(u_1)u_{1x})B_1\right] \\ &- \left[B_1(f(u_3(\phi_1)) - f(u_1+u_3-u_2)(\phi_1))) + B_2(f(u_2(\phi_1)) - f(u_1(\phi_1)))\right]\delta_1 \\ &- \left[B_2(f(u_3(\phi_2)) - f(u_2(\phi_2))) + B_1(f((u_1+u_3-u_2)(\phi_2)) - f(u_1(\phi_2)))\right]\delta_2 = \mathcal{O}_{\mathcal{D}'}(\varepsilon),\end{aligned} \tag{4.14}$$

where, as usual, $B_i = B_i((\phi_2 - \phi_1)/\varepsilon)$, $i = 1, 2$. If we equalize by zero the sum of the coefficients multiplying δ_1 and δ_2, respectively, we get

$$\begin{aligned}\phi_{1t}(u_2(\phi_1) - u_1(\phi_1)) - B_1(f(u_3(\phi_1)) - f(u_1 + u_3 - u_2)(\phi_1)) \\ - B_2(f(u_2(\phi_1)) - f(u_1(\phi_1))) = 0,\end{aligned} \tag{4.15}$$

$$\begin{aligned}\phi_{2t}(u_3(\phi_2) - u_2(\phi_2)) - B_2(f(u_3(\phi_2)) - f(u_2(\phi_2))) \\ + B_1(f((u_1 + u_3 - u_2)(\phi_2)) - f(u_1(\phi_2))) = 0.\end{aligned} \tag{4.16}$$

Now, we subtract (4.15) and (4.16). Using (4.10) and (4.11), and passing from the variable t to the variable τ, we obtain the following Cauchy problem (see also [19, pages 108–110])

$$\begin{gathered}\frac{d\rho}{d\tau} = \dot{\psi}_0 - b_1\left(\frac{\alpha}{e_1}\bigg|_{x=\phi_1} + \frac{\alpha}{e_2}\bigg|_{x=\phi_2}\right) = f(\rho, t), \\ \frac{\rho}{\tau}\bigg|_{\tau\to-\infty} = 1,\end{gathered} \tag{4.17}$$

where

$$\begin{gathered}\alpha = f(u_3) - f(u_2) - f(u_1 + u_3 - u_2) + f(u_1), \\ e_1 = u_2 - u_1, \\ e_2 = u_3 - u_2.\end{gathered} \tag{4.18}$$

We can rewrite the function $F(\rho, t)$ in the following manner:

$$f(\rho, t) = \left[\frac{f(u_3) - f(u_1 + u_3 - u_2)}{u_2 - u_1}\bigg|_{x=\phi_1} - \frac{f(u_1 + u_3 - u_2) - f(u_1)}{u_3 - u_2}\bigg|_{x=\phi_2}\right] + \left(\frac{\alpha}{e_1}\bigg|_{x=\phi_1} + \frac{\alpha}{e_2}\bigg|_{x=\phi_1}\right). \tag{4.19}$$

Now, we return to (4.17). From the standard ODE theory we see that the solution ρ of (4.17) tends to the stationary solution ρ_0 of (4.17). Clearly, $\rho_0 = \rho_0(t)$ is the minimal root to the equation $F(\rho, t) = 0$ in ρ (Cauchy theorem for existence and uniqueness of the solution to an ODE with an initial condition; see Figure 5). Since $\rho = (\phi_2 - \phi_1)/\varepsilon$ after the interaction we have $\phi_1 = \phi_2 + \mathcal{O}(\varepsilon)$.

It remained to determine the functions Ω_i, $i = 1, 2$. In the sequel, we let

$$\dot{\Omega}_i = \frac{\partial \Omega_i}{\partial \tau}, \quad i = 1, 2. \tag{4.20}$$

Substituting expressions (4.10) and (4.11) in (4.15) and (4.16), respectively, we obtain the equations

$$\frac{\partial \Omega_i}{\partial t}\frac{\psi_0\phi_{i1}^+}{\tau}+\dot{\Omega}_i+\Omega_i\frac{(\psi_0\phi_{i1}^+)_t}{\tau\phi_{i1}^+}=(-1)^i\frac{B_1}{\tau}\frac{f(u_3)-f(u_2)-f(u_1+u_3-u_2)+f(u_1)}{u_{i+1}-u_i}\bigg|_{x=\phi_i},\quad i=1,2. \tag{4.21}$$

Furthermore, notice that if $|\partial \Omega_i/\partial t|<\infty$, we have

$$\frac{\partial \Omega_i}{\partial t}\frac{\psi_0\phi_{i1}^+}{\tau}=\varepsilon\frac{\partial \Omega_i}{\partial t}\phi_{i1}^+=\mathcal{O}(\varepsilon). \tag{4.22}$$

Therefore, we will determine Ω_i, $i=1,2$, so that they satisfy the following differential equations in $\tau\in IR$ (compare with (4.21))

$$\dot{\Omega}_i+\Omega_i\frac{(\psi_0\phi_{i1}^+)_t}{\tau\phi_{i1}^+}=(-1)^i\frac{B_1}{\tau}\frac{f(u_3)-f(u_2)-f(u_1+u_3-u_2)+f(u_1)}{u_2-u_1}\bigg|_{x=\phi_i},\quad i=1,2, \tag{4.23}$$

and then we will prove that

$$\left|\frac{\partial \Omega_i}{\partial t}\right|<\infty. \tag{4.24}$$

Notice that differential equation (4.23) is defined only on the interval $[0,t^*]$. Therefore, it is necessary to extend continuously the functions $F_i(\rho,t)$ after the moment $t=t^*$. Put for $t>t^*$,

$$F_i(t)=\frac{f(u_3)-f(u_2)-f(u_1+u_3-u_2)+f(u_1)}{u_{i+1}-u_i}\bigg|_{x=\varphi(t)},\quad i=1,2. \tag{4.25}$$

Since for $t=t^*$ we have $\phi_1(t^*)=\phi_2(t^*)=\varphi(t^*)$, it follows that the extension is well defined. It is not difficult to find the solution to (4.23)

$$\Omega_i(t,\tau)=\frac{1}{\tau^{(\psi_0\phi_{i1}^+)_t/\phi_{i1}^+}}\int_0^\tau(-1)^iB_1(\rho)F_i(t)\tilde{\tau}^{(\psi_0\phi_{i1}^+)_t/\phi_{i1}^+}d\tilde{\tau}. \tag{4.26}$$

Using the l'Hospital theorem for limits we get from here

$$\left|\lim_{\tau\to\pm\infty}\Omega_i(\tau,t)\right|<\infty,\qquad \left|\lim_{\tau\to\pm\infty}\frac{\partial\Omega_i(\tau,t)}{\partial t}\right|<\infty, \tag{4.27}$$

proving (4.24) and thus (4.22). This, in turn, proves that Ω_i, $i=1,2$, are approximate solutions to (4.21), and thus, (4.10) and (4.11) are fulfilled.

In that way, we have eliminated the coefficients multiplying δ distribution in (4.14). Now, we have to annulate the coefficients multiplying θ distribution in (4.14), more precisely, we need to get

$$\begin{aligned} &u_{1t} + u_{1x} f'(u_1) + \left[u_{3t} + f(u_3)u_{3x} - u_{1t} - f(u_1)u_{1x}\right]\theta_1 \\ &\quad + \left[u_{3t} - u_{2t} + u_{3x} f(u_3) - u_{2x} f(u_2) + B_1\left(-u_{3x} f'(u_{3x}) + u_{2x} f(u_2)\right.\right. \\ &\qquad \left.\left. - f'(u_1 + u_3 - u_2)(u_{1x} + u_{3x} - u_{2x}) + f(u_1)u_{1x}\right)\right](\theta_2 - \theta_1) = \mathcal{O}_{\mathcal{D}'}(\varepsilon). \end{aligned} \tag{4.28}$$

Since the functions u_i, $i = 1, 2, 3$ satisfy (4.4), we can rewrite the previous expression in the form

$$B_1\left[-u_{3x} f'(u_{3x}) + u_{2x} f(u_2) - f'(u_1 + u_3 - u_2)(u_{1x} + u_{3x} - u_{2x}) + f(u_1)u_{1x}\right](\theta_2 - \theta_1) = \mathcal{O}_{\mathcal{D}'}(\varepsilon). \tag{4.29}$$

If we multiply this by $\eta \in C_0^1(\mathbf{R})$, we have

$$\begin{aligned} &\varepsilon\rho B_1(\rho)\frac{1}{\phi_1 - \phi_2}\int_{\phi_2}^{\phi_1}\left[\left(-u_{3x} f'(u_{3x}) + u_{2x} f(u_2) - \right.\right. \\ &\qquad\qquad \left.\left. f'(u_1 + u_3 - u_2)(u_{1x} + u_{3x} - u_{2x}) + f(u_1)u_{1x}\right)\right]\eta(x)dx = \mathcal{O}(\varepsilon), \end{aligned} \tag{4.30}$$

since $|\rho B_1(\rho)| < \infty$ (it is clear from the definition but one can check in [11]). In that way, we see that (4.14) is correct which proves the theorem.

Finally, notice that since u_ε represents the weak asymptotic solution to the considered problem we have $w - \lim_{\varepsilon \to 0} u_\varepsilon = u$ where u is the weak solution to the considered problem. Therefore, the functions ϕ_i, $i = 1, 2$, up to $\mathcal{O}(\varepsilon)$, satisfy the Rankine-Hugoniot conditions. In turn, from there it follows that for $t > t^*$ we have

$$\lim_{\tau \to +\infty} \Omega(\tau, t) = 1, \tag{4.31}$$

and for $t < t^*$ we have

$$\lim_{\tau \to -\infty} \Omega(\tau, t) = 0. \tag{4.32}$$

□

5. Scalar Conservation Law with Decreasing Initial Data

In this section, we consider (1.1) with the following initial condition:

$$u|_{t=0} = v(x) \in C(\mathbf{R}). \tag{5.1}$$

We assume that the function $v(x)$ decreases and that it has finite number of points in which the function

$$-\frac{1}{f''(v(x))v'(x)}, \quad x \in \mathbf{R}, \tag{5.2}$$

reaches maximum. Denote this set of points by $S = \{x_1, x_2, \ldots, x_n\}$. Assume also that $x_1 < x_2 < \cdots < x_n$.

Now, we continue in the following way. Around every $x_i \in S$, $i = 1, \ldots, n$, we allocate ε-neighborhoods of the form $(x_i - 2\varepsilon^\mu, x_i + 2\varepsilon^\mu)$, $\mu \in (0,1)$. Then, we transform the initial data v, replacing it by the function $u_i(x)$, $x \in (x_i - 2\varepsilon^\mu, x_i + 2\varepsilon^\mu)$, such that for every $i = 1, \ldots, n$, we have the following (compare Figures 3 and 4)

$$\begin{aligned}
&f'(u_i(x)) = -K_i x + b_i, \quad x \in (x_i - \varepsilon^\mu, x_i + \varepsilon^\mu), \\
&u_i(x) = v(x_i - 2\varepsilon^\mu), \quad x \in (x_i - 2\varepsilon^\mu, x_i - \varepsilon^\mu), \\
&u_i(x) = v(x_i - 2\varepsilon^\mu), \quad x \in (x_i + \varepsilon^\mu, x_i + 2\varepsilon^\mu),
\end{aligned} \tag{5.3}$$

where the constants K_i and b_i are determined by the conditions

$$u_i(x_i - \varepsilon^\mu) = v(x_i - 2\varepsilon^\mu), \quad u_i(x_i + \varepsilon^\mu) = v(x_i + 2\varepsilon^\mu), \quad i = 1, \ldots, n. \tag{5.4}$$

So, we have replaced the original initial data v by the function

$$\hat{v}(x) = \begin{cases} v(x), & x \in (-\infty, x_1 - 2\varepsilon^\mu) \cup (x_1 + 2\varepsilon^\mu, x_2 - 2\varepsilon^\mu) \cup \cdots \\ & \qquad \cup (x_{n-1} + 2\varepsilon^\mu, x_n - 2\varepsilon^\mu) \cup (x_n + 2\varepsilon^\mu, \infty), \\ u_i(x), & x \in (x_i - 2\varepsilon^\mu, x_i + 2\varepsilon^\mu), \quad i = 1, \ldots, n. \end{cases} \tag{5.5}$$

Obviously, we have the following estimate fulfilled $v(x) - \hat{v}(x) = \mathcal{O}_{\mathfrak{D}'}(\varepsilon^\mu)$. Beside that, since the function $f''(v(x))v'(x)$ reaches its maximum at the points $x_i \in S$, then there exist neighborhoods $\mathcal{U}(x_i)$, $i = 1, \ldots, n$, such that for every $x \in \mathcal{U}(x_i)$ we have

$$f''(v(x))v'(x) > f''(v(y))v'(y) \quad \text{where } y \notin \bigcup_{i=1}^{n} \mathcal{U}(x_i). \tag{5.6}$$

The moment of "straightening" of the curves connecting the points $(x_i - \varepsilon^\mu, v(x_i - \varepsilon^\mu))$ and $(x_i + \varepsilon^\mu, v(x_i + \varepsilon^\mu))$, according to Section 2 and the Lagrange mean value theorem is given by

$$\begin{aligned}
t_1^* &= \frac{2\varepsilon^\mu}{f'(v(x_i - \varepsilon^\mu)) - f'(v(x_i + \varepsilon^\mu))} \\
&= -\frac{1}{(f' \circ v)'(\hat{x}_i)} = -\frac{1}{(f''(v(\hat{x}_i))v'(\hat{x}_i)} \\
&< \max_{y \in U} -\frac{1}{f''(v(x))v'(x)}, \quad \hat{x}_i \in (x_i - \varepsilon^\mu, x_i + \varepsilon^\mu).
\end{aligned} \tag{5.7}$$

From here we see that for every $i = 1, \dots, n$ we have $t_1^* > -1/f''(v(y))v'(y)$, where $y \notin \bigcup_{i=1}^n \mathcal{U}(x_i)$. This actually means that a gradient catastrophe will not happen before straightening of the lines connecting the points $x_i - \varepsilon^\mu$ and $x_i + \varepsilon^\mu$, at least for ε small enough. Knowing that, we can describe behavior of the weak asymptotic solution of the considered problem relying on the previous sections.

We track evolution of the new initial data $\widehat{v}_\varepsilon(x)$ in (x, u) space. In the interval $[0, t_1^* - g(\varepsilon))$, for appropriate function $g(\varepsilon) = \mathcal{O}(\varepsilon^\mu)$, the solution will be continuous function. In the interval $[t_1^* - s(\varepsilon), t^* + s(\varepsilon)]$, for some function $s(\varepsilon) = \mathcal{O}(\varepsilon^\mu)$, the curves connecting the points $(x_i - \varepsilon^\mu, v(x_i - \varepsilon^\mu))$ and $(x_i + \varepsilon^\mu, v(x_i + \varepsilon^\mu))$, $i = 1, \dots, n$, will straighten in the shock waves of increasing amplitude. After that, two general cases as well as their combinations can happen:

(a) in the intervals between some pair of shock waves, the gradient catastrophe will happen in the moment $t_2^* = t_1^* + \xi$, but before the collision of the two shock waves, or

(b) two shock waves will collide in the moment $t_2^* = t_1^* + \xi$, but before a gradient catastrophe happens in the interval between them.

In case (a), we repeat the procedure from the beginning. More precisely, we take the weak limit in $x \in \mathbf{R}$ of the weak asymptotic solution in the moment t_1^*. Denote it by $\widetilde{v}(x, t_1^*)$. Then, we replace the part of the function $\widetilde{v}(x, t_1^*)$ around the point x_0 from which emanates the characteristics along which the gradient catastrophe will happen. We replace it completely analogically as we did for the initial condition v. Namely, we take a smooth function $r(x, t_1^*)$ such that $f'(r(x, t_1^*)) = -Kx + b$, for some constants K and b, in the interval $(x_0 - \varepsilon^\mu, x_0 + \varepsilon^\mu)$. Then, instead of the part of $\widetilde{v}(x, t_1^*)$ in the interval $(x_0 - 2\varepsilon^\mu, x_0 + 2\varepsilon^\mu)$ we put the function r. In that way we get the function $v^*(x)$. More precisely, we take

$$v(x, t_1^*) = \begin{cases} \widetilde{v}(x, t_1^*), & x \notin (x_0 - 2\varepsilon^\mu, x_0 + 2\varepsilon^\mu), \\ r(x, t_1^*), & x \in (x_0 - \varepsilon^\mu, x_0 + \varepsilon^\mu), \\ \widetilde{v}(x_0 - \varepsilon^\mu, t_1^*), & x \in (x_0 - 2\varepsilon^\mu, x_0 - \varepsilon^\mu), \\ \widetilde{v}(x_0 + \varepsilon^\mu, t_1^*), & x \in (x_0 + \varepsilon^\mu, x_0 + 2\varepsilon^\mu). \end{cases} \tag{5.8}$$

Then, using Sections 3 and 4, we find the weak asymptotic solution $u_\varepsilon^*(x, t)$ to (1.1) with initial data $v^*(x)$ in the strip $(t_1^*, t_2^* + (t_3^* - t_2^*)/2) \times \mathbf{R}$ where t_3^* is the moment of the next situation (a) or (b) or their combination. Then, using the partition of unity, we connect the weak asymptotic solutions in the intervals $[0, t_1^* + (t_2^* - t_1^*)/2)$ and $[t_1^*, t_2^* + (t_3^* - t_2^*)/2)$.

In case (b), we use the results of Section 4 on (1.1) with initial condition $\widetilde{v}(x, t_1^*)$. Then, like in case (a) we use the partition of unity to connect solutions on the intervals $[0, t_1^* + (t_2^* - t_1^*)/2)$ and $[t_1^*, t_2^* + (t_3^* - t_2^*)/2)$, where t_3^* is the moment of the next situation (a) or (b) or their combination.

To detail the previous analysis, we formulate the following theorem.

Theorem 5.1. *Fix arbitrary $T > 0$ and denote by t_i^*, $i = 1, \dots, n$, moments of nonlinear wave interactions in the interval $[0, T]$ (more precisely, situations (a) and/or (b)) corresponding to Cauchy problem* (1.1) *and* (4.1).

Global weak asymptotic solution $u_\varepsilon(x,t) \in C^1(\mathbf{R}^+ \times \mathbf{R})$ of (1.1) *and* (4.1) *has the form*

$$u_\varepsilon(x,t) = \eta_1(t)u_\varepsilon^{[1]}(x,t) + \eta_2(t)u_\varepsilon^{[2]}(x,t) + \cdots + \eta_n(t)u_\varepsilon^{[n]}(x,t), \tag{5.9}$$

where η_i, $i = 1,\dots,n$ is partition of unity of the interval $[0,T]$ such that (we take $t_0^* = 0$, $t_{-1}^* = -1$ and $t_{n+1}^* = T+1$)

$$\eta_i(t) \equiv 1, \quad t \in \left(t_{i-1}^*, t_i^*\right), \qquad \eta_i(t) \equiv 0, \quad t \notin \left(t_{i-2}^*, t_{i+1}^*\right). \tag{5.10}$$

The function $u_\varepsilon^{[1]}(x,t)$ is given by (5.11) *while the functions $u_\varepsilon^{[k]}(x,t)$, $k = 2,\dots,n$, are of form* (5.17) *(with difference in indexing depending on time and place of singularities interactions).*

Proof. Denote by $\{x_1,\dots,x_n\}$ the set such that for every $x_0 \in \{x_1,\dots,x_n\}$ we have (2.5) satisfied. According to the previous analysis, in the interval $[0, t_1^* + (t_2^* - t_1^*)/2)$ the weak asymptotic solution to (1.1) and (5.1) we have in the form

$$\begin{aligned}
u_\varepsilon^{[1]}(x,t) &= v_0(x,t) + \left(u_1\left(x_0^1(x,t,\varepsilon)\right) - v_0(x,t)\right)\omega_1\left(\frac{\phi_1^1(t,\tau^1) - x}{\varepsilon}\right) \\
&\quad + \left(v_1(x,t) - u_1\left(x_0^1(x,t,\varepsilon)\right)\right)\omega_2\left(\frac{\phi_1^2(t,\tau^1) - x}{\varepsilon}\right) \\
&\quad + \cdots + \left(u_n\left(x_0^n(x,t,\varepsilon),t\right) - v_{n-1}(x,t)\right)\omega_1\left(\frac{\phi_n^1(t,\tau^n) - x}{\varepsilon}\right) \\
&\quad + \left(v_n(x,t) - u_n\left(x_0^n(x,t,\varepsilon),t\right)\right)\omega_2\left(\frac{\phi_n^2(t,\tau^n) - x}{\varepsilon}\right),
\end{aligned} \tag{5.11}$$

where ω_i, $i = 1,2$, are the ones from Theorem 3.3.

Using the last two sections we can describe all unknown functions appearing in (5.11). We have the following

(i) The functions $x_0^i = x_0^i(x,t,\varepsilon)$, $i = 1,\dots,n-1$, are inverse functions to the "new characteristics". The "new characteristics" we obtain if in (3.15) we replace ρ by ρ_i, u_1 by u_i, and c by c_i. The functions c_i we obtain when in (3.4) we replace U by v_{i-1}, and u_0 by v_i.

(ii) The functions $\phi_i^j = \phi_i^j(t,\tau^i)$, $i = 0,1,\dots,n$, $j = 1,2$, are the "new characteristics" emanating from the points $x_i + (-1)^j\varepsilon^\mu$, and

$$\begin{aligned}
\tau^i &= \frac{\varphi_i^{10}(t) - \varphi_i^{20}(t)}{\varepsilon}, \qquad \rho_i = \rho_i(t,\tau_i) = \frac{\phi_i^1(t,\tau^i) - \phi_i^2(t,\tau^i)}{\varepsilon}, \\
&\left.\begin{aligned}\varphi_i^{20}(t) &= f'\left(v(x_i - \varepsilon^\mu)\right)t + x_i - \varepsilon^\mu, \\
\varphi_i^{10}(t) &= f'\left(v(x_i + \varepsilon^\mu)\right)t + x_i + \varepsilon^\mu,\end{aligned}\right\} \quad i = 1,\dots,n.
\end{aligned} \tag{5.12}$$

Also, notice that according to Theorem 3.3 the functions ρ_i are the solutions to the following Cauchy problems (analogical to Cauchy problem (3.10)):

$$\dot{\rho}_i = 1 - 2B_1(\rho_i), \qquad \lim_{\tau \to -\infty} \frac{\rho_i}{\tau_i} = 1, \tag{5.13}$$

where $\dot{\rho}_i = \partial_{\tau_i}\rho_i$.

The functions $v_i(x,t)$, $i = 1,\ldots,n$, are classical solutions to the following Cauchy problems in the regions $t \in [0, t^* + (t_2^* - t_1^*)/2)$, $x \in \mathbf{R}$,

$$v_{it} + (f(v_i)v_i)_x = 0, \qquad v_i(x,0) = v(x). \tag{5.14}$$

As we have already explained, in the moment $t_1^* + \xi = t_2^*$ we will have situation (a) or (b) or we will have situation (a) in an interval between two shock waves and, at the same time, situation (b) between two other shock waves.

Due to locality of the singularity formation process, without loosing on the generality, we can assume that exactly one collision of shock waves and exactly one gradient catastrophe happened at the moment t_2^* (that means that we have combination of situations (a) and (b)). It only remains to impose initial data at the moment $t = t_1^*$ for (1.1). We simply put $t = t_1^*$ in (1.1) and use the fact that up to $\mathcal{O}(\varepsilon)$, we have $\phi_i^1(t_1^*,0) = \phi_i^2(t_1^*,0)$, $i = 1,\ldots,n$. Thus, we obtain the following initial data:

$$\begin{aligned} v_{[2]}(x,t_1^*) &= v_0(x,t_1^*) + (v_1(x,t_1^*) - v_0(x,t_1^*))\omega_1\left(\frac{\phi_1^1(t_1^*,0) - x}{\varepsilon}\right) \\ &\quad + (v_1(x,t_1^*) - v_2(x,t_1^*))\omega_2\left(\frac{\phi_2^1(t_1^*,0) - x}{\varepsilon}\right) \\ &\quad + \cdots + (v_{n-1}(x,t_1^*) - v_{n-2}(x,t_1^*))\omega_1\left(\frac{\phi_{n-1}^1(t_1^*,0) - x}{\varepsilon}\right) \\ &\quad + (v_n(x,t_1^*) - v_{n-1}(x,t_1^*))\omega_2\left(\frac{\phi_n^1(t_1^*,0) - x}{\varepsilon}\right). \end{aligned} \tag{5.15}$$

Then, we can assume that case (a) happened at the point $x_0 = x_{[2]}$ placed between shock waves concentrated at the points $\phi_1^1(t_1^*,0)$ and $\phi_2^1(t_1^*,0)$, and, at the same time, case (b) happened between shock waves concentrated in the points $\phi_2^1(t_1^*)$ and $\phi_3^1(t_1^*)$.

For the part of the function $v_{[2]}(x,t_1^*)$ disposed between $\phi_1^1(t_1^*,0)$ and $\phi_2^1(t_1^*,0)$, we repeat the procedure from the beginning of the section. Namely, we approximate the function $v_1(x,t_1^*)$ in the interval $(x_{[2]} - 2\varepsilon^\mu, x_{[2]} + 2\varepsilon^\mu)$ by the function $u_{[2]}(x)$ such that

$$\begin{aligned} f'(u_{[2]}(x)) &= -Kx + b, \quad x \in (x_{[2]} - \varepsilon^\mu, x_{[2]} + \varepsilon^\mu), \\ u_{[2]}(x) &= v_1(x_{[2]} - 2\varepsilon^\mu, t_1^*), \quad x \in (x_{[2]} - 2\varepsilon^\mu, x_{[2]} - \varepsilon^\mu), \\ u_{[2]}(x) &= v_1(x_{[2]} + 2\varepsilon^\mu, t_1^*), \quad x \in (x_{[2]} + \varepsilon^\mu, x_{[2]} + 2\varepsilon^\mu), \end{aligned} \tag{5.16}$$

where K and b are appropriate constants.

Denote this new function by $\widetilde{v}_{[2]}(x)$. Now, we solve (1.1) with the function $\widetilde{v}_{[2]}$ as initial data but starting from $t = t_1^*$. According to Sections 3 and 4, the weak asymptotic solution to (1.1) with initial condition $u(x, t_1^*) = \widetilde{v}_{[2]}(x)$ in the interval $[t_1^*, t_2^* + (t_3^* - t_2^*)/2)$, where t_3^* is the moment of the third set of interactions (more precisely, the third set of gradient catastrophe or shock wave collisions), we have in the form

$$\begin{aligned}
u_\varepsilon^{[2]}(x,t) = v_0(x,t) &+ \left(v_{0[2]}(x,t) - v_0(x,t)\right)\omega\left(\frac{\phi_1(t,\tau^1) - x}{\varepsilon}\right) \\
&+ \left(u_{1[2]}\left(x_0^1(x,t,\varepsilon)\right) - v_{0[2]}(x,t)\right)\omega\left(\frac{\phi_{[2]}^1(t,\tau_{[2]}) - x}{\varepsilon}\right) \\
&+ \left(v_{1[2]}(x,t) - u_{1[2]}\left(x_0^1(x,t,\varepsilon)\right)\right)\omega\left(\frac{\phi_{[2]}^2(t,\tau_{[2]}) - x}{\varepsilon}\right) \\
&+ \left(v_2(x,t) - v_{1[2]}(x,t)\right)\omega\left(\frac{\phi_2(t,\tau^2) - x}{\varepsilon}\right) \\
&+ \left(v_3(x,t) - v_2(x,t,\varepsilon)\right)\omega\left(\frac{\phi_3(t,\tau^2) - x}{\varepsilon}\right) \\
&+ \cdots + \left(v_{n-1}(x,t) - v_{n-2}(x,t)\right)\omega\left(\frac{\phi_{n-1}(t,\tau^{n-1}) - x}{\varepsilon}\right) \\
&+ \left(v_n(x,t) - v_{n-1}(x,t)\right)\omega\left(\frac{\phi_n(t,\tau^n) - x}{\varepsilon}\right).
\end{aligned} \tag{5.17}$$

Here, the functions ϕ_i, $i = 1, 4, \ldots, n$, are independent on ε and they are given by the following Cauchy problems

$$\begin{aligned}
\phi_{1t} &= \frac{f\left(v_{0[2]}(\phi_1,t)\right) - f\left(v_0(\phi_1,t)\right)}{v_{0[2]}(\phi_1,t) - v_0(\phi_1,t)}, \quad \phi_1(t_1^*) = \phi_1^1(t_1^*, 0), \\
\phi_{it} &= \frac{f\left(v_{i-1}(\phi_i,t)\right) - f\left(v_i(\phi_i,t)\right)}{v_{i-1}(\phi_i,t) - v_i(\phi_i,t)}, \quad \phi_i(t_1^*) = \phi_i^1(t_1^*, 0),
\end{aligned} \tag{5.18}$$

where $i = 4, \ldots, n$, $t \in [t_1^*, t_2^* + (t_3^* - t_2^*)/2)$.

If by $\phi_{20}(t)$ and $\phi_{30}(t)$ we denote global solutions of the following Cauchy problems (see corresponding equations (4.7)):

$$\left.\begin{aligned}
\phi_{20t} &= \frac{f\left(v_2(\phi_{20},t)\right) - f\left(v_{1[2]}(\phi_{20},t)\right)}{v_2(\phi_{20},t) - v_{1[2]}(\phi_{20},t)}, \quad \phi_{20}(t_1^*) = \phi_2^1(t_1^*, 0), \\
\phi_{30t} &= \frac{f\left(v_3(\phi_{20},t)\right) - f\left(v_2(\phi_{20},t)\right)}{v_3(\phi_{30},t) - v_2(\phi_{30},t)}, \quad \phi_{30}(t_1^*) = \phi_3^1(t_1^*, 0),
\end{aligned}\right. \quad t \in \left[t_1^*, t_2^* + \frac{(t_3^* - t_2^*)}{2}\right), \tag{5.19}$$

we have the quantity τ_2 defined by

$$\tau_2 = \frac{\phi_{20}(t) - \phi_{30}(t)}{\varepsilon}. \tag{5.20}$$

If we also put

$$\begin{aligned} \phi_{[2]}^{10}(t) &= f'\left(v\left(x_{[2]} + \varepsilon^{\mu}\right)\right)t + x_{[2]} + \varepsilon^{\mu}, \\ \phi_{[2]}^{20}(t) &= f'\left(v\left(x_{[2]} - \varepsilon^{\mu}\right)\right)t + x_{[2]} - \varepsilon^{\mu}, \end{aligned} \quad t \in \left[t_1^*, t_2^* + \frac{(t_3^* - t_2^*)}{2}\right), \tag{5.21}$$

then we have the quantity $\tau_{[2]}$ given by

$$\tau_{[2]} = \frac{\phi_{[2]}^{10}(t) - \phi_{[2]}^{20}(t)}{\varepsilon}. \tag{5.22}$$

Note that $\phi_{[2]}^{i0}$, $i = 1, 2$, are standard characteristics emanating from $x_{[2]} + \varepsilon^{\mu}$ and $x_{[2]} - \varepsilon^{\mu}$.

The functions $v_i(x,t)$, $i = 0, 2, 3, \dots, n$ are classical solutions to Cauchy problems (5.14) in the regions $t \in [t_1^*, t_2^* + (t_3^* - t_2^*)/2)$, $x \in \mathbf{R}$, while $v_{0[2]}(x,t)$ and $v_{1[2]}(x,t)$ satisfy, respectively, the following equations:

$$\begin{gathered} v_{0[2]t} + f'\left(v_{0[2]}\right)v_{0[2]x} = 0, \quad v_{0[2]}(x,0) = v(x), \quad t \in \left[t_1^*, t_2^* + \frac{(t_3^* - t_2^*)}{2}\right), \quad x \in \mathbf{R}, \\ v_{1[2]t} + f'\left(v_{1[2]}\right)v_{1[2]x} = 0, \qquad v_{1[2]}(x,0) = v(x). \end{gathered} \tag{5.23}$$

The function $u_{1[2]}(x_0^1(x,t,\varepsilon))$ is analogical to the function $u_1(x,t,\varepsilon)$ from (5.11) with an obvious difference in indexing.

The functions ϕ_i, $i = 1, 4, 5, \dots, n$, are independent on ε and they are given by Rankine-Hugoniot conditions

$$\begin{aligned} \phi_{it} &= \frac{f\left(v_i(\phi_i,t)\right) - f\left(v_{i-1}(\phi_i,t)\right)}{v_i(\phi_i,t) - v_{i-1}(\phi_i,t)}, \\ \phi_{1t} &= \frac{f\left(v_{0[1]}(\phi_1,t)\right) - f\left(v_0(\phi_1)\right)}{v_{0[1]}(\phi_1,t) - v_0(\phi_1,t)}, \end{aligned} \quad \phi_i(t_1^*) = \phi_i^1(t_1^*,0), \ i = 1, 4, 5, \dots, n. \tag{5.24}$$

Thus, we have described the weak asymptotic solution to (1.1), (5.1) in the intervals $[0, t_1^* + (t_2^* - t_1^*)/2)$ and $[t_1^*, t_2^* + (t_3^* - t_2^*)/2)$ through the functions $u_\varepsilon^{[1]}$ and $u_\varepsilon^{[2]}$, respectively. Notice, then, the partition of unity $\eta(t) \in C_0^1(\mathbf{R}^+)$ of the time axis satisfying

$$\begin{gathered} \eta(t) \equiv 1, \quad t \le t_1^*, \\ \eta(t) \equiv 0, \quad t \ge t_1^* + \frac{(t_3^* - t_2^*)}{2}. \end{gathered} \tag{5.25}$$

If we put

$$u_\varepsilon(x,t) = \eta(t)u_\varepsilon^{[1]}(x,t) + (1-\eta(t))u_\varepsilon^{[2]}(x,t), \quad x \in \mathbf{R},\ t \in \left[0, t_3^*\right), \tag{5.26}$$

and use the fact that in the interval $(t_1^*, t_1^* + (t_2^* - t_1^*)/2)$ we have

$$u_\varepsilon^{[1]} = u_\varepsilon^{[2]} + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{5.27}$$

we see that the function u_ε gives the weak asymptotic solution to (1.1) and (5.1) in the interval $[0, t_3^*)$. Repeating described procedure for $t > t_3^*$, we obtain weak asymptotic solution in an arbitrary interval $[0,T] \subset \mathbf{R}^+$ (we have in mind the assumption on discreteness of set of moments in which interactions of nonlinear waves happen).

This concludes the proof. □

Example 5.2. Consider the Hamilton-Jacobi equation

$$u_t + f(u_x) = 0, \quad x \in \mathbf{R},\ t \in \mathbf{R}^+. \tag{5.28}$$

By introducing the following change of the unknown function $u = \int_{x_0}^{x} w(x')dx'$ and differentiating with respect to $x \in \mathbf{R}$ we get the equation

$$w_t + (f(w))_x = 0, \quad x \in \mathbf{R},\ t \in \mathbf{R}^+. \tag{5.29}$$

Thus, using the weak asymptotic method, we have global in time solution to the Hamilton-Jacobi equation providing approximate analytic description of the "essential" part of the Lagrange manifold (see the Section 1).

Acknowledgments

The work of the first author is supported by RFFI grant 05-01-00912, DFG Project 436 RUS 113/895/0-1. The work of the second author is supported in part by the local government of the municipality Budva.

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