Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 430452, 7 pages
doi:10.1155/2009/430452

Research Article

Some Computational Formulas for D-Nörlund Numbers

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Received 30 June 2009; Accepted 11 October 2009

Recommended by Lance Littlejohn

The author establishes some identities involving the D numbers, Bernoulli numbers, and central factorial numbers of the first kind. A generating function and several computational formulas for D-Nörlund numbers are also presented.

1. Introduction and Results

The Bernoulli polynomials $B_n^{(k)}(x)$ of order k, for any integer k, may be defined by (see [1–5])

$$\left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1.1}$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k, $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers (see [2, 6, 7]). By (1.1), we can get (see [4, page 145])

$$\frac{d}{dx}B_n^{(k)}(x) = nB_{n-1}^{(k)}(x), \tag{1.2}$$

$$B_n^{(k+1)}(x) = \frac{k-n}{k}B_n^{(k)}(x) + (x-k)\frac{n}{k}B_{n-1}^{(k)}(x), \tag{1.3}$$

$$B_n^{(k+1)}(x+1) = \frac{nx}{k}B_{n-1}^{(k)}(x) - \frac{n-k}{k}B_n^{(k)}(x), \tag{1.4}$$

where $n \in \mathbb{N}$, with $\mathbb{N}$ being the set of positive integers.

The numbers $B_n^{(n)}$ are called the Nörlund numbers (see [4, 8]). A generating function for the Nörlund numbers $B_n^{(n)}$ is (see [4, page 150])

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}. \tag{1.5}$$

The D numbers $D_{2n}^{(k)}$ may be defined by (see [4, 5])

$$(t\csc t)^k = \sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!}, \quad |t| < \pi. \tag{1.6}$$

By (1.1), (1.6), and note that $\csc t = 2i/(e^{it} - e^{-it})$ (where $i^2 = -1$), we can get

$$D_{2n}^{(k)} = 4^n B_{2n}^{(k)}\left(\frac{k}{2}\right). \tag{1.7}$$

Taking $k = 1, 2$ in (1.7), and note that $B_{2n}^{(1)}(1/2) = (2^{1-2n} - 1)B_{2n}$, $B_{2n}^{(2)}(1) = (1-2n)B_{2n}$ (see [4, pages 22 and 145]), we have

$$D_{2n}^{(1)} = \left(2 - 2^{2n}\right)B_{2n}, \qquad D_{2n}^{(2)} = 4^n(1-2n)B_{2n}. \tag{1.8}$$

The numbers $D_{2n}^{(2n)}$ are called the D-Nörlund numbers. These numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ have many important applications. For example (see [4, page 246])

$$\int_0^{\pi/2} \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}, \qquad \int_0^{\pi/2} \frac{\sin t}{t} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n}(2n-1)(n!)^2}, \tag{1.9}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{(2n-1)(2n)!}. \tag{1.10}$$

We now turn to the central factorial numbers $t(n,k)$ of the first kind, which are usually defined by (see [9–12])

$$x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right)\cdots\left(x + \frac{n}{2} - n + 1\right) = \sum_{k=0}^{n} t(n,k)x^k, \tag{1.11}$$

or by means of the following generating function:

$$\left(2\log\left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}\right)\right)^k = k!\sum_{n=k}^{\infty} t(n,k)\frac{x^n}{n!}. \tag{1.12}$$

It follows from (1.11) or (1.12) that

$$t(n,k) = t(n-2,k-2) - \frac{1}{4}(n-2)^2 t(n-2,k), \tag{1.13}$$

and that

$$\begin{gathered} t(n,0) = \delta_{n,0} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \qquad t(n,n) = 1 \quad (n \in \mathbb{N}), \\ t(n,k) = 0 \quad (n+k \,\text{odd}), \quad t(n,k) = 0 \quad (k > n \text{ or } k < 0), \end{gathered} \tag{1.14}$$

where $\delta_{m,n}$ denotes the Kronecker symbol.

By (1.13), we have

$$t(2n+1,1) = \frac{(-1)^n (2n)!}{4^{2n}} \binom{2n}{n}, \quad t(2n+2,2) = (-1)^n (n!)^2 \quad (n \in \mathbb{N}_0), \tag{1.15}$$

$$t(2n+2,4) = (-1)^{n+1} (n!)^2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) \quad (n \in \mathbb{N}), \tag{1.16}$$

$$t(2n+1,3) = \frac{(-1)^{n-1} (2n)!}{4^{2n-1}} \binom{2n}{n} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2}\right) \quad (n \in \mathbb{N}). \tag{1.17}$$

The main purpose of this paper is to prove some identities involving D numbers, Bernoulli numbers, and central factorial numbers of the first kind and obtain a generating function and several computational formulas for the D-Nörlund numbers. That is, we will prove the following main conclusion.

Theorem 1.1. *Let $n \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$. Then*

$$D_{2n}^{(k)} = \frac{(2n-k+2)(2n-k+1)}{(k-2)(k-1)} D_{2n}^{(k-2)} - \frac{2n(2n-1)(k-2)}{k-1} D_{2n-2}^{(k-2)}. \tag{1.18}$$

Remark 1.2. By (1.18), we may immediately deduce the following (see [4, page 147]:

$$D_{2n}^{(2n+1)} = \frac{(-1)^n (2n)!}{4^n} \binom{2n}{n}, \qquad D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n+1} (n!)^2. \tag{1.19}$$

Theorem 1.3. *Let $n \geq k$ $(n,k \in \mathbb{N}_0)$. Then*

$$D_{2n-2k}^{(2n+1)} = \frac{4^{n-k}}{\binom{2n}{2k}} t(2n+1, 2k+1), \tag{1.20}$$

$$D_{2n-2k}^{(2n)} = \frac{4^{n-k}}{\binom{2n-1}{2k-1}} t(2n, 2k) \quad (k \geq 1). \tag{1.21}$$

Remark 1.4. By (1.20) and (1.17), we may immediately deduce the following:

$$D_{2n}^{(2n+3)} = \frac{(-1)^n (2n)!}{2\cdot 4^{2n}} \binom{2n+2}{n+1} \left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\frac{1}{(2n+1)^2}\right). \tag{1.22}$$

Theorem 1.5. *Let $n \in \mathbb{N}_0$. Then*

$$\sum_{j=0}^{n} \frac{(-1)^j}{4^j (2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!} = \frac{(-1)^n}{4^n} \binom{2n}{n}, \tag{1.23}$$

so one finds $D_0^{(0)} = 1, D_2^{(2)} = -2/3, D_4^{(4)} = 88/15, D_6^{(6)} = -3056/21, D_8^{(8)} = 319616/45, D_{10}^{(10)} = -18940160/33, \ldots$.

By (1.23)*, and note that*

$$\log\left(t+\sqrt{1+t^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (2n+1)} \binom{2n}{n} t^{2n+1} \quad (|t|<1),$$
$$\frac{1}{\sqrt{1+t^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \binom{2n}{n} t^{2n} \quad (|t|<1), \tag{1.24}$$

one may immediately deduce the following Corollary 1.6*.*

Corollary 1.6. *Let $n \in \mathbb{N}_0$. Then*

$$\sum_{n=0}^{\infty} D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!} = \frac{t}{\sqrt{1+t^2}\log\left(t+\sqrt{1+t^2}\right)} \quad (|t|<1). \tag{1.25}$$

Theorem 1.7. *Let $n \in \mathbb{N}$. Then*

(i)

$$D_{2n}^{(2n)} = \frac{(-1)^n (2n)!}{4^n} \binom{2n}{n} + n\cdot 4^n \sum_{j=1}^{n} \frac{\left(1-2^{1-2j}\right)}{j} t(2n,2j) B_{2j} \tag{1.26}$$

(ii)

$$D_{2n}^{(2n)} = \frac{(-1)^n 4^n (n!)^2}{2n+1} + n\cdot 4^n \sum_{j=1}^{n} \frac{2j-1}{j} t(2n,2j) B_{2j} \tag{1.27}$$

(iii)

$$D_{2n}^{(2n)} = \sum_{j=0}^{n} \frac{4^{n-j}}{2j+1} t(2n+1, 2j+1). \tag{1.28}$$

Theorem 1.8. *Let $n \in \mathbb{N}_0$. Then*

(i)

$$\sum_{j=0}^{n} \frac{2-4^j}{4^j} t(2n+1, 2j+1) B_{2j} = \frac{(-1)^n (n!)^2}{2n+1} \tag{1.29}$$

(ii)

$$\sum_{j=0}^{n} (1-2j) t(2n+1, 2j+1) B_{2j} = \frac{(-1)^n (2n)!}{2 \cdot 4^{2n}} \binom{2n+2}{n+1} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2}\right). \tag{1.30}$$

2. Proof of the Theorems

Proof of Theorem 1.1. By (1.4) and (1.3), we have

$$\begin{aligned} B_{2n}^{(k)}(x+1) &= \frac{2nx}{k-1} B_{2n-1}^{(k-1)}(x) - \frac{2n-k+1}{k-1} B_{2n}^{(k-1)}(x) \\ &= \frac{2nx}{k-1} \left(\frac{k-2n-1}{k-2} B_{2n-1}^{(k-2)}(x) + (x-k+2) \frac{2n-1}{k-2} B_{2n-2}^{(k-2)}(x) \right) \\ &\quad - \frac{2n-k+1}{k-1} \left(\frac{k-2n-2}{k-2} B_{2n}^{(k-2)}(x) + (x-k+2) \frac{2n}{k-2} B_{2n-1}^{(k-2)}(x) \right) \\ &= \frac{(2n-k+1)(2n-k+2)}{(k-1)(k-2)} B_{2n}^{(k-2)}(x) + \frac{2n(2n-1)}{(k-1)(k-2)} x(x-k+2) B_{2n-2}^{(k-2)}(x) \\ &\quad - \frac{2n(2n-k+1)}{(k-1)(k-2)} (2x-k+2) B_{2n-1}^{(k-2)}(x). \end{aligned} \tag{2.1}$$

Setting $x = (k-2)/2$ in (2.1), we get

$$B_{2n}^{(k)}\left(\frac{k}{2}\right) = \frac{(2n-k+1)(2n-k+2)}{(k-1)(k-2)} B_{2n}^{(k-2)}\left(\frac{k-2}{2}\right) - \frac{2n(2n-1)(k-2)}{4(k-1)} B_{2n-2}^{(k-2)}\left(\frac{k-2}{2}\right). \tag{2.2}$$

By (2.2) and (1.7), we immediately obtain (1.18). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3. By the usage of Theorem 1.1 and (1.13). □

Proof of Theorem 1.5. Note the identity (see [4, page 203])

$$B_{2n+1}^{(k)}\left(x+\frac{k}{2}\right) = \sum_{j=0}^{n}\binom{2n+1}{2j+1}\frac{D_{2n-2j}^{(k-2j-1)}}{2^{2n-2j}}x\left(x^2-\left(\frac{1}{2}\right)^2\right)\left(x^2-\left(\frac{3}{2}\right)^2\right)\cdots\left(x^2-\left(\frac{2j-1}{2}\right)^2\right), \tag{2.3}$$

we have

$$\begin{aligned}\lim_{x\to 0}\frac{B_{2n+1}^{(2n+1)}(x+(2n+1)/2)}{x} &= \frac{1}{4^n}\sum_{j=0}^{n}\binom{2n+1}{2j+1}D_{2n-2j}^{(2n-2j)}(-1)^j 1^2\cdot 3^2\cdots(2j-1)^2 \\ &= \frac{(2n+1)!}{4^n}\sum_{j=0}^{n}\frac{(-1)^j}{4^j(2j+1)}\binom{2j}{j}\frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!}.\end{aligned} \tag{2.4}$$

By (2.4) and (1.2), we have

$$\lim_{x\to 0}(2n+1)B_{2n}^{(2n+1)}\left(x+\frac{2n+1}{2}\right) = \frac{(2n+1)!}{4^n}\sum_{j=0}^{n}\frac{(-1)^j}{4^j(2j+1)}\binom{2j}{j}\frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!}, \tag{2.5}$$

that is,

$$B_{2n}^{(2n+1)}\left(\frac{2n+1}{2}\right) = \frac{(2n)!}{4^n}\sum_{j=0}^{n}\frac{(-1)^j}{4^j(2j+1)}\binom{2j}{j}\frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!}. \tag{2.6}$$

By (2.6) and (1.7), we have

$$D_{2n}^{(2n+1)} = (2n)!\sum_{j=0}^{n}\frac{(-1)^j}{4^j(2j+1)}\binom{2j}{j}\frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!}. \tag{2.7}$$

By (2.7) and (1.19), we immediately obtain (1.23). This completes the proof of Theorem 1.5. □

Proof of Theorem 1.7. By (1.6), we have

$$D_{2n}^{(k)} = \sum_{j=0}^{n}\binom{2n}{2j}D_{2n-2j}^{(k-l)}D_{2j}^{(l)}, \tag{2.8}$$

where l is an integer.

Setting $k = 2n+1, l = 1$ in (2.8), and note that $D_0^{(1)} = 1$, we have

$$D_{2n}^{(2n+1)} = \sum_{j=0}^{n} \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(1)} = D_{2n}^{(2n)} + \sum_{j=1}^{n} \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(1)}. \tag{2.9}$$

By (2.9), (1.19), (1.8), and (1.21), we immediately obtain (1.26).

Setting $k = 2n+2, l = 2$ in (2.8), and note that $D_0^{(2)} = 1$, we have

$$D_{2n}^{(2n+2)} = \sum_{j=0}^{n} \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(2)} = D_{2n}^{(2n)} + \sum_{j=1}^{n} \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(2)}. \tag{2.10}$$

By (2.10), (1.19), (1.8), and (1.21), we immediately obtain (1.27).

Setting $k = 2n, l = -1$ in (2.8), and note that (1.20) and $D_{2j}^{(-1)} = 1/(2j+1)$, we immediately obtain (1.18). This completes the proof of Theorem 1.7. □

Proof of Theorem 1.8. Setting $k = 2n+2,\ l = 1$ in (2.8), and note (1.19), (1.20), and (1.8), we immediately obtain (1.29).

Setting $k = 2n+3, l = 2$ in (2.8), and note (1.22), (1.20), and (1.8), we immediately obtain (1.30). This completes the proof of Theorem 1.8. □

Acknowledgment

This work was supported by the Guangdong Provincial Natural Science Foundation (no. 8151601501000002).

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